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# Lanchester-Type Models of Warfare, Volume II

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# LANCHESTER-TYPE MODELS OF WARFARE

## VOLUME II.

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This monograph is a comprehensive treatise on Lanchester-type models of warfare, i.e. differential-equation models of attrition in force-on-force combat operations. Its goal is to provide both an introduction to and current-state-of-the-art overview of Lanchester-type models of warfare as well as a comprehensive and unified in-depth treatment of them. Both deterministic as well as stochastic models are considered. Such models have been widely used in the United States and elsewhere for the modelling of force-on-force attrition over the complete spectrum of combat operations, from



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20. ABSTRACT CONTINUED

combat between platoon-sized units through theater-level air-ground combat. This material should be of interest primarily to individuals concerned with defense planning, quantitative aspects of military analysis, military OR, war gaming, or combat modelling, although it may also be of interest to the reader concerned with the modelling and analysis of other dynamic systems. It should also be of interest to the concerned citizen who is interested in the foundations for defense analysis and has the appropriate technical background.

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## Chapter 5. LANCHESTER ATTRITION-RATE COEFFICIENTS

### 5.1. General Considerations.

For applying any kind of LANCHESTER-type combat model to study a particular hypothesized combat engagement in a defense-planning study, one must be able to predict the rates at which weapon systems would inflict and sustain casualties. In other words, one must be able to compute a reliable numerical value for the loss rate of each and every weapon-system type on the battlefield. This capability is essential for utilizing LANCHESTER-type models of warfare in combat analyses. Thus, in this chapter we will consider methods for predicting LANCHESTER attrition rates and, in particular, the coefficients that portray these rates.

Two approaches that have been developed and used to predict loss rates for LANCHESTER-type combat models are based on using

(A1) an analytical submodel of the attrition process for the particular target type[1],

and (A2) a statistical estimate based on "combat" data generated by a detailed Monte Carlo combat simulation[2].

In this chapter we will examine each of these approaches in detail. For now, however, let us say a few general words about each of them.

S. BONDER [15] has called the first approach (A1) the use of a <u>freestanding or independent analytical model</u>, since this type of analytical model can be run independently of any detailed Monte Carlo simulation of the same combat process. The basic conceptual idea is to develop an analytical expression for every required kill rate by considering a single firer engaging a "passive" target (i.e. one that doesn't fire back) and then to "tie all the attrition rates together" with a LANCHESTER-type model. One designs such a model to use the same types of inputs as used

by Monte Carlo simulations of the the same combat process. Hopefully, the freestanding analytical model will predict similar outputs in an efficient and easily interpretable manner. An example of such an independent analytical model is the BONDER/IUA differential model, which was first used in the United States in 1969 [15], and the many subsequently enriched versions of it (see Section 1.3 above). BONDER and FARRELL [17] have reported excellent agreement between outputs from the BONDER/IUA model and a corresponding Monte Carlo simulation.

The second approach (A2) has been called by BONDER [15] the use of a fitted-parameter analytical model. The basic idea here for predicting LANCHESTER attrition-rate coefficients is to statistically estimate the parameters of the loss rate for each type of weapon system from the output of a high-resolution Monte Carlo combat simulation. This idea is apparently due to G. CLARK [24] and is schematically shown in Figure 5.1. Thus, the fitted parameter analytical model must be used in conjunction with a Monte Carlo simulation (or appropriate data from the actual process[3]). The data or outputs of the simulation are used to fit one or more free parameters in the analytical model so that the analytical model will (at least) duplicate and (hopefully) predict results comparable to those obtainable from the simulation model. The COMAN model [24] is an example of such a fitted parameter model. Encouraging results have been reported [36]. Such a model is built on a physical basis with only a minimum number of parameters to be estimated (in contrast to statistical regression functions).

Both the above general approaches (A1) and (A2) for predicting LANCHESTER attrition-rate coefficients, however, in some sense make use of the general principle that the loss rate is equal to the reciprocal of

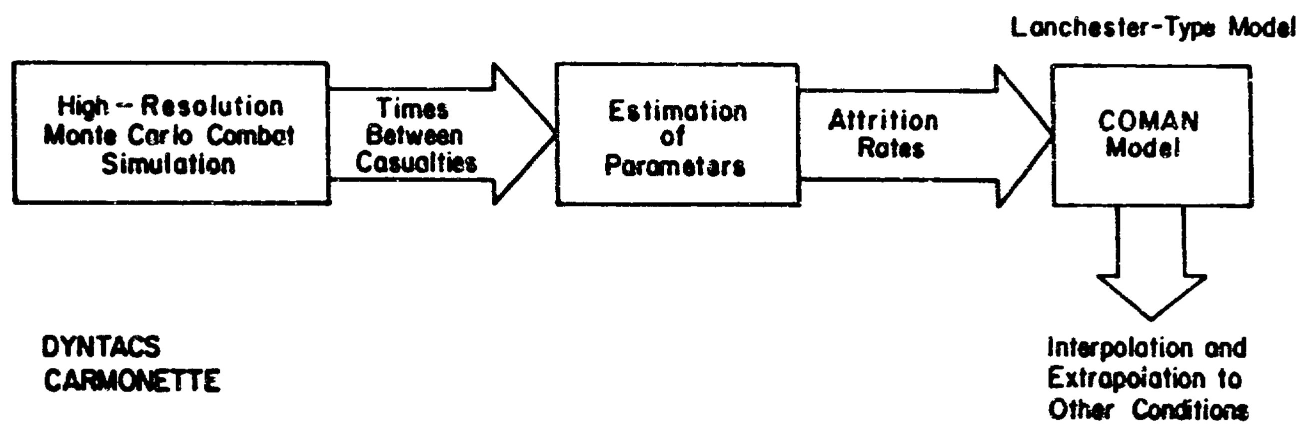


Figure 5.1. Basic idea behind the fitted-parameter analytical model for the complimentary use of Monte Carlo simulation and LANCHESTER-type models.

the expected time for a target to be killed. The details of both approaches should be more readily comprehended if we will keep this principle in mind. Let us therefore provide a motivation for this principle. We start by considering combat between two homogenous forces. Assuming that the loss rates only depend on the numbers of combatants and not time explicitly, we may model the attrition process with the following deterministic LANCHESTER-type equations of warfare

$$\begin{cases} \dfrac{dx}{dt} = -A(x,y) & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -B(x,y) & \text{with } y(0) = y_0, \end{cases} \qquad (5.1.1)$$

where $x(t)$ and $y(t)$ denote, respectively, the X and Y force levels at time $t$. Here we find it convenient to represent, for example, the actual number of X combatants, which is a nonnegative integer, with the real number $x(t)$. Let us assume that there are no replacements and withdrawals, and then A and B are the attrition rates of the X and Y forces, respectively.

If we want to statistically estimate the loss rates in the model (5.1.1) from Monte Carlo simulation output data (i.e. casualty data generated by a (pseudo-) random process), we must consider a stochastic version of (5.1.1) in which casualties occur randomly over time. It is now convenient to consider the restriction that the force levels are really nonnegative integers and to model the combat attrition process as a continuous-parameter MARKOV chain. Letting $M(t)$, a random variable[4], denote the integral number of X combatants alive at time $t$ (with corresponding realization denoted as $m$) and similarly for the Y force, we then have the following so-called forward KOLMOGOROV equations (see Chapter 4) for the evolution of the state probablities for $0 < m \leq m_0$ and $0 < n \leq n_0$

$$\frac{dP}{dt}(t,m,n) = P(t,m+1)\ A(m+1,n) + P(t,m,n+1)\ B(m,n+1)$$

$$- \{A(m,n) + B(m,n)\}\ P(t,m,n), \qquad (5.1.2)$$

where $P(t,m,n) = P[M(t) = m,\ N(t) = n | M(0) = m_0,\ N(0) = n_0]$ and we have adopted the convention that, for example, $A(m,n) = 0$ for $m > m_0$ or $n > n_0$. From this stochastic model, we find that (see Chapter 4 above)

$$E[T_{XY}] = \frac{1}{A(m,n)} , \qquad (5.1.3)$$

where $T_{XY}$, a random variable, denotes the time required for the Y force to kill an X combatant (i.e. the time between two successive X casualties) and $E[T]$ denotes the expected value of $T$. For the case of equal casualty rates that are independent of the numbers of combatants (i.e. $A(m,n) = B(m,n) = \lambda$ = constant), (5.1.3) becomes the well-known result for casualties occurring to a Poisson stream

$$E[T] = \frac{1}{\lambda} ,$$

or

$$\lambda = \frac{1}{\bar{t}} \qquad (5.1.4)$$

where $T$ denotes the time between the occurrences of successive casualty events and $\bar{t} = E[T]$.

The reader may be familiar with this well-known result (5.1.4), and, in any case, the more general version (5.1.3) should provide a heuristic motivation for certain subsequent results in predicting attrition-rate coefficients. Thus, in statistically estimating loss rates from simulation output data, we should expect to use statistics about the times between casualties. Furthermore, BONDER's freestanding analytical model approach

is also conceptually based on (5.1.3): one develops a model for $T_{XY}$, analytically computes $E[T_{XY}]$, and takes $A(x,y) = 1/E\,[T_{XY}]$. Therefore, (5.1.3) should in some sense be taken as a general principle that is essential for understanding subsequent developments in this chapter.

## 5.2. Attrition-Rate Coefficients for LANCHESTER's Equations of Modern Warfare.

Let us now consider the determination of numerical values for the attrition-rate coefficients in a particular combat model. We accordingly consider "aimed-fire" combat between two homogeneous forces and assume that target-acquisition times are constant (independent of the number of enemy targets). This combat situation may be modelled with the following LANCHESTER-type equations for modern warfare[5] (see Section 2.11 for a further discussion of the military circumstances hypothesized to yield them)

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0, \\ \\ \dfrac{dy}{dt} = -bx & \text{with } y(0) = y_0, \end{cases} \tag{5.2.1}$$

where for a particular battle $a$ and $b$ are positive constants called LANCHESTER attrition-rate coefficients (see Figure 5.2). Each of these attrition-rate coefficients in such a combat model represents the fire effectiveness of one side's weapon system against enemy targets. For example, $a$ is the rate at which one $Y$ firer kills $X$ targets. The dimensions of $a$ are (number of $X$ casualties)/(time × number of $Y$ firers). Thus, $a$ is indeed a rate and has the dimensions of reciprocal time.

Before discussing a simple analytical model for determining numerical values for the LANCHESTER attrition-rate coefficient in particular military engagements, let us point out a very important relation between the daily casualty rate (expressed as a fraction of the side's current strength) of a homogeneous force and such a LANCHESTER attrition-rate coefficient. We will show that for the model (5.2.1), for example, the LANCHESTER

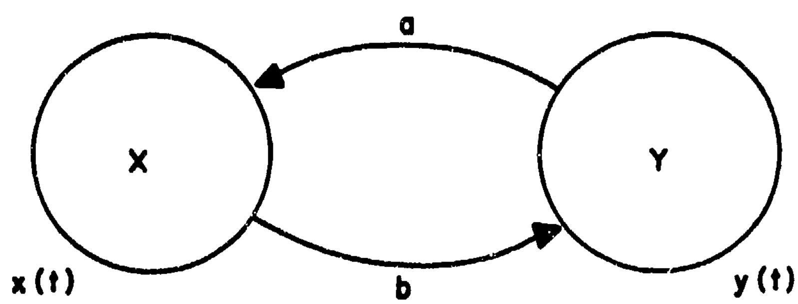


Figure 5.2. LANCHESTER attrition-rate coefficients a and b (here assumed to be constant) for LANCHESTER-type equations of modern warfare. The coefficient a represents the fire effectiveness of the weapon-system type used by the Y force in the operational circumstances of the battle under consideration. More precisely, a is the rate at which one Y firer kills X targets.

attrition-rate coefficient $a$ is the slope of the plot of fractional casualties per unit time versus a certain force ratio. Let us accordingly consider, for example, X's fractional casualties per unit time. From the first of equations (5.2.1), we obtain

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \begin{pmatrix}\text{X's fractional casualties}\\ \text{per unit time}\end{pmatrix} = \frac{a}{u} = av \ , \qquad (5.2.2)$$

where $u$ denotes the force ratio of X to Y, i.e. $u = x/y$, and $v$ denotes its reciprocal, i.e. $v = y/x$.

In Figure 5.3 we have plotted X's fractional casualties per unit time as a function of a certain force ratio. The force ratio that we have used is the quotient of the attacker's strength (here, force level) divided by that of the defender and have denoted it as A/D, since most combat analyses use this ratio A/D and consequently we will be able to more easily relate the simple LANCHESTER-type model (5.2.1) to them. The solid line in Figure 5.3 represents X's fractional casualties per unit time as a function of the force ratio A/D when X defends and Y attacks. It is a straight line through the origin with a slope equal to the value of the LANCHESTER attrition rate coefficient $a$ as the reader can see by referring back to (5.2.2). Thus, we have developed an important relation between fractional casualty rate and the LANCHESTER attrition-rate coefficient. Finally, the dashed line (which is a hyperbola) in Figure 5.3 represents X's fractional casualties per unit time as a function of the force ratio A/D in the other case in which X attacks and Y defends. Similar curves for daily casualty rates are commonly used to assess casualties in currently operational large-scale ground-combat models (<u>see</u> Section 7.13).

Let us now return to our discussion of numerically determining the

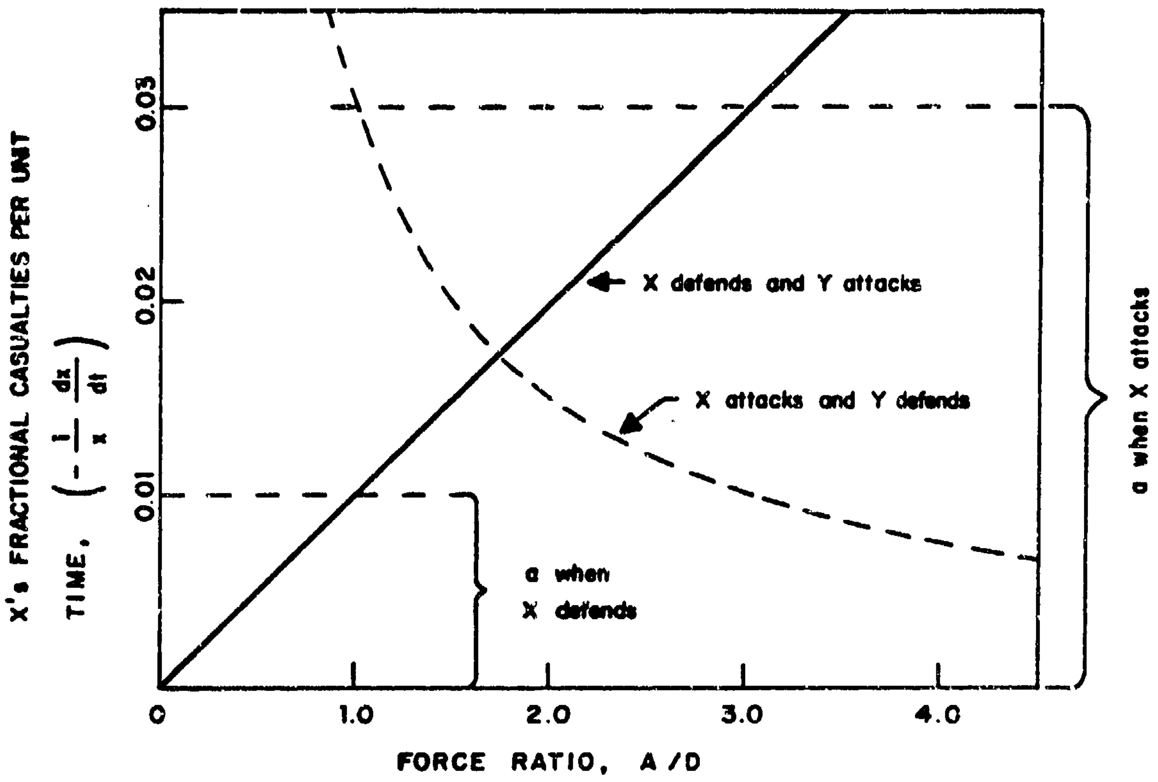


Figure 5.3. Relation between X's casualty rate (expressed as a fraction of his current force level $x(t)$) and the force ratio (expressed as the ratio of the attacker's force level to that of the defender) for LANCHESTER's equations of modern warfare (5.2.1). [NOTE: In the bottom legend of the above figure, A denotes the attacker's force level, and D denotes that of the defender.]

LANCHESTER attrition-rate coefficients $a$ and $b$ for the model (5.2.1). In general, we may think of, for example, the LANCHESTER attrition-rate coefficient $a$ as being given by (cf. (5.1.3) above)

$$a = \frac{1}{E[T_{XY}]} , \tag{5.2.3}$$

where $T_{XY}$ again is a random variable (frequently abbreviated r.v.) and denotes the time for an individual $Y$ firer to kill a single $X$ target. Justification for using (5.2.3) is given in the next section (Section 5.3). As we discussed in general terms in Section 5.1 above, such a LANCHESTER attrition-rate coefficient may be predicted for particular military engagements by using

(W1) an analytical submodel involving physically measurable weapon-system characteristics of the attrition process for an individual friendly firer engaging a single enemy target,

or

(W2) a statistical estimate based on "combat" data generated by a detailed Monte Carlo combat simulation.

In the remainder of this section we will discuss the first way (W1), while the second way (W2) is discussed in Section 5.15 below.

In the simplest case (a more complicated one is considered below), the LANCHESTER attrition-rate coefficient is simply given by, for example,

$$a = \nu_Y P_{SSK_{XY}} , \tag{5.2.4}$$

where $\nu_Y$ denotes Y's firing rate, and $P_{SSK_{XY}}$ denotes Y's single shot kill probability against X. This simple expression (5.2.4) is usually hypothesized to apply to "aimed-fire" combat when the following conditions

hold:

(C1) negligible target-acquisition time,

(C2) statistical independence among firing outcomes,

and (C3) uniform rate of fire.

The reader can probably best appreciate the intuitive plausibility of the expression (5.2.4) by noting that $a$ represents the average number of kills per unit time by a single $Y$ firer, $\nu_Y$ denotes his rate of fire, and (on the average) he kills a given fraction of an $X$ target with each round fired denoted by $P_{SSK_{XY}}$.

As we see from (5.2.3), the LANCHESTER attrition-rate coefficient is the reciprocal of the average time for an individual firer to kill an enemy target. Let us therefore consider a simple model for the time to kill a target. If we let $T$, a r.v., denote this time for a firer to kill an enemy target, then $T$ is given by

$$T = T_a + T_{k|a} , \tag{5.2.5}$$

where $T_a$ denotes the time to acquire a target, and $T_{k|a}$ denotes the time to kill an acquired target.

Again, in the simplest case (as above, assuming: (A1) a uniform rate of fire, and (A2) statistical independence among firing outcomes) we have

$$E[T_{k|a}] = t_{k|a} = \frac{1}{\nu P_{SSK}} , \tag{5.2.6}$$

where $\nu$ denotes the firing rate, and $P_{SSK}$ denotes the single-shot kill probability. The reader may find the following intuitive justification for the average time to kill an acquired target (5.2.6) to be helpful: $1/P_{SSK}$ represents the average number of rounds to kill[6], while $1/\nu$

represents the average time between rounds, and consequently their product is the average time to kill an acquired target $E[T_{k|a}]$.

Thus, if we let

$$E[T_a] = t_a , \tag{5.2.7}$$

then our simple model for the time to kill a target yields

$$E[T] = t_a + \frac{1}{\nu P_{SSK}} , \tag{5.2.8}$$

and consequently, for example,

$$a = \frac{1}{E[T_{XY}]} , \tag{5.2.9}$$

where $E[T_{XY}] = t_{a_{XY}} + 1/(\nu_Y P_{SSK_{XY}})$. Thus, we see that (5.2.4) is just the special case of (5.2.9) in which $t_{a_{XY}} = 0$.

Let us finally note that, strictly speaking, (5.2.8) holds only when (A1) and (A2) are satisfied [i.e. there is (A1) a uniform rate of fire, and (A2) statistical independence among firing outcomes]. There are, however, many weapon systems and engagement circumstances under which these assumptions are not at all appropriate. Consequently, S. BONDER has developed an expression more complicated than (5.2.8) for target engagement modelled by MARKOV-dependent fire. He developed this expression for the analysis of tank operations in which it is very important to consider MARKOV dependence. We will examine BONDER's work in the section following the next one.

## 5.3. Justification of General Expression for Attrition-Rate Coefficients for LANCHESTER's Equations of Modern Warfare.

In this section we present justification for taking an attrition-rate coefficient for LANCHESTER's equations of modern warfare (5.2.1) as the reciprocal of the expected time for an individual firer to kill a target, e.g.

$$a = \frac{1}{E[T_{XY}]}, \tag{5.3.1}$$

where $T_{XY}$ is a random variable (abbreviated r.v.) denoting the time for an individual $Y$ firer to kill an $X$ target and $E[T]$ denotes the expected value of $T$. BONDER and FARRELL [17] (see also [28; 88; 89]) have based their approach for determining attrition-rate coefficients for a wide spectrum of weapon-system types on this definition (5.3.1). It is therefore of considerable interest to inquire as to what justification there is for basing the calculation of LANCHESTER attrition-rate coefficients on (5.3.1). We have already provided heuristic justification of (5.3.1) in Section 5.1 above, and here we will consider several more rigorous justifications.

All justifications of (5.3.1) known to this author are ultimately based on the following basic hypothesis.

> BASIC HYPOTHESIS: Combat is a complex random process, and the LANCHESTER-type equations (5.2.1) are an approximation to the mean course of combat.

If we assume that real-world combat attrition may be modelled as a continuous-parameter MARKOV chain corresponding to (5.2.1), then the probability distribution for the numbers of combatants satisfies (5.1.2) with, for example, $A(m,n) = an$. Here, $m$ is the realization of an integer-valued r.v. $M(t)$ denoting the number of $X$ combatants at time $t$, and similarly for $n$ and $N(t)$.* In this case, the times between casualties for each side are exponentially distributed, and (5.3.1) holds exactly. In other words, <u>(5.3.1) holds exactly for exponentially-distributed times between casualties</u>. Let us finally observe that as long as there is "negligible" probability that either side is annihilated, then the mean course of combat may be taken to be given by (<u>see</u> Section 4.12 above)

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -a\bar{n} & \text{with } \bar{m}(0) = m_0 , \\ \\ \dfrac{d\bar{n}}{dt} = -b\bar{m} & \text{with } \bar{n}(0) = n_0 , \end{cases} \tag{5.3.2}$$

where $\bar{m}(t)$ denotes the average $X$ force level at time $t$, i.e. $\bar{m}(t) = E[M(t)]$, and $\bar{n}(t)$ denotes the average $Y$ force level at time $t$.

Both BONDER [11] and BARFOOT [3] base their determinations of an expression for the LANCHESTER attrition-rate coefficient on considering the mean course of combat corresponding to (5.2.1) to be given by

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -\bar{\alpha}\bar{n} & \text{with } \bar{m}(0) = m_0 , \\ \dfrac{d\bar{n}}{dt} = -\bar{\beta}\bar{m} & \text{with } \bar{n}(0) = n_0 , \end{cases} \tag{5.3.3}$$

where $\bar{\alpha}$ denotes the expected value of the rate at which an individual Y firer kills X targets and similarly for $\bar{\beta}$. This definition of the LANCHESTER attrition-rate coefficient as [cf. (5.3.2)], for example,

$$a = \bar{\alpha} = E\left[\begin{matrix}\text{rate at which a single Y} \\ \text{firer kills X targets}\end{matrix}\right] \tag{5.3.4}$$

implies an underlying distribution for the attrition-rate coefficient (as stressed by BONDER [14; 15]). No particular distribution for the times between casualties has been assumed here, though. In his original paper [11] BONDER took the LANCHESTER attrtion-rate coefficient to be given by $a = E[1/T_{XY}]$ but could not obtain explicit results for it. BARFOOT [3] then pointed out that there are two possibilities for computing $\bar{\alpha}$, the average rate at which a single Y firer ills X targets: namely,

(P1) arithmetic mean, $\bar{\alpha} = E\left[\frac{1}{T_{XY}}\right]$ ;

and (P2) harmonic mean, $\bar{\alpha} = \frac{1}{E[T_{XY}]}$ .

Furthermore, BARFOOT has argued that the harmonic mean is more appropriate, since we should think of the probability distribution function for an

attrition-rate coefficient as representing the fraction of targets killed at each rate. Thus, BARFOOT [3] has justified (5.3.1) for any distribution of the times between casualties.

Following BONDER and FARRELL[7] [17], let us now give a more rigorous justification[8] of (5.3.1). As above, we consider combat in which the initial numbers of X and Y combatants, denoted as $m_0$ and $n_0$, are sufficiently large to insure that there is a "negligible" probability that their side will be annihilated during our examination of the battlefield. Let us now focus on a single Y weapon system. We will make no assumption about the distribution of times between kills, but we will assume that each individual Y weapon system kills enemy targets according to an attrition process in which the times <u>between kills are independent and identically distributed random variables</u> (so-called i.i.d. random variables). In the parlance of the theory of stochastic processes, such an attrition process is called a <u>renewal process</u> (e.g. <u>see</u> PARZEN [58, Chapter 5] for further details). Let $N_c^X(t)$ be a r.v. denoting the number of X casualties produced by a single Y weapon system, and let $\bar{n}_c^X(t)$ denote its expected value, i.e.

$$\bar{n}_c^X(t) = E[N_c^X(t)] , \tag{5.3.5}$$

the expected number of X casualties produced by a single Y weapon system in $[0,T]$. Let us now introduce $\Delta\bar{n}_c^X(\Delta t,t)$ defined by

$$\Delta\bar{n}_c^X(\Delta t,t) = \bar{n}_c^X(t + \Delta t) - \bar{n}_c^X(t) , \tag{5.3.6}$$

which is the expected number of X casualties produced by a single Y weapon system in the time interval[9] $(t, t + \Delta t)$. <u>For exponentially distributed times between kills</u>, we have that (e.g. <u>see</u> PARZEN [58, p. 177])

$$\Delta \bar{n}_c^X(\Delta t, t) = \frac{\Delta t}{\mu_T} , \tag{5.3.7}$$

where $\mu_T$ denotes the average time for a single Y firer to kill an X target, i.e. $\mu_T = E[T_{XY}]$. <u>For any other distribution for the times between kills</u>, (5.3.7) holds only asymptotically in the sense that

$$\lim_{t \to +\infty} \Delta \bar{n}_c^X(\Delta t, t) = \frac{\Delta t}{\mu_T} . \tag{5.3.8}$$

The above result (5.3.8) is usually known as BLACKWELL's theorem (<u>see</u> PARZEN [58, p. 183]). Assuming now that each Y firer acts independently and identically, we find that for the entire Y force

$$E \begin{bmatrix} \text{number of kills by } Y \\ \text{force in } (t, t + \Delta t) \end{bmatrix} = \frac{\bar{n}\Delta t}{\mu_T} , \tag{5.3.9}$$

which holds exactly for exponentially distributed times between kills and only asymptotically in the same sense as (5.3.8) for any other distribution. LANCHESTER's equations for modern warfare (5.2.1) with "large enough" numbers of combatants suggest that [cf. (5.3.2)]

$$-\Delta \bar{m} = E \begin{bmatrix} \text{number of kills by } Y \\ \text{force in } (t, t+\Delta t) \end{bmatrix} = a\bar{n}\Delta t . \tag{5.3.10}$$

Comparison of (5.3.9) and (5.3.10) suggests taking the LANCHESTER attrition-rate coefficient to be the reciprocal of the average time for an individual firer to kill an enemy target, i.e. (5.3.1) has been justified.

More generally, BONDER and FARRELL [17] take an attrition-rate coefficient for a specific range $r$ in heterogeneous-force combat to be given by, for example,

$$a_{ij}(r) = \frac{1}{E[T_{X_i Y_j} | r]} , \qquad (5.3.11)$$

where $E[T_{X_i Y_j} | r]$ denotes the expected time for a single $Y$ firer of type $j$ to kill an enemy target of type $i$, given that the range between the firer-target pair is $r$. Again, this definition of an attrition-rate coefficient for heterogenous-force combat is equivalent to the harmonic mean for the attrition rate of a single combat system when this single-system attrition rate is viewed as a random variable at range $r$.

## 5.4. BONDER's Model for MARKOV-Dependent Fire.

For many weapon systems and engagement circumstances modelled by (5.2.1), the extremely simple analytical model (5.2.4) for prediction of numerical values for the LANCHESTER attrition-rate coefficient is totally inadequate. Ideally one should analyze the engagement process for each particular target type by each particular weapon-system type to predict such attrition-rate coefficients. BONDER and FARRELL [17] have developed general methodology for predicting attrition-rate coefficients for a wide spectrum of weapon-system types. Basically, their approach is founded upon calculation of the LANCHESTER attrition-rate coefficient as the reciprocal of the expected time to kill a single target, e.g. (5.3.1) above. Hence, central to their developments is the analysis and modelling of the time to kill a target.

To facilitate such analysis BONDER and FARRELL [17] have classified the engagement of particular target types by different weapon-system types according to the taxonomy[10] shown in Table 5.I. Weapon-system types are first classified according to the mechanism by which they kill particular target types (i.e. their lethality characteristics) as being either impact-to-kill systems or area-lethality systems[11]. Within each of these two categories BONDER and FARRELL further classify weapon-system types according to how they use firing information to control the system's aim point and their delivery characteristics, i.e. the firing doctrine employed. Expressions have been developed for LANCHESTER attrition-rate coefficients corresponding to the weapon-system classifications tagged with asterisks * in Table 5.I.

TABLE 5.I. Classification of Weapon-System Types for the Development of LANCHESTER Attrition-Rate Coefficients for the Model (5.2.1).

<u>Lethality Mechanism</u>

(1) Impact

(2) Area

<u>Firing Doctrine</u>

(1) Repeated Single Shot

   (a)* Without Feedback Control of Aim Point

   (b)* With Feedback on Immediately Preceding Round (MARKOV-Dependent Fire)

   (c) With Complex Feedback

(2) Burst Fire

   (a)* Without Aim Change or Drift in or Between Bursts

   (b)* With Aim Drift in Bursts, Aim Refixed to Original Aim Point for Each Burst

   (c) With Aim Drift, Re-aim Between Bursts

(3) Multiple Tube Firing: Feedback Situations (1a), (1b), (1c)

   (a)* Salvo or Volley

(4) Mixed-Mode Firing

   (a) Adjustment Followed by Multiple Tube Fire

   (b)* Adjustment Followed by Burst Fire

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*Indicates that analysis of this category has been performed by BONDER and FARRELL [17].

A large class of weapon systems (e.g. tanks firing at tanks, anti-tank weapon systems firing at tanks, etc.) may be classified as MARKOV-dependent-fire weapons, i.e. the outcome of the firing of a round by the weapon system depends on only the outcome of the immediately preceding round. For such weapon systems and an impact-to-kill lethality mechanism[12], BONDER [11; 14] has developed a general expression for the LANCHESTER attrition-rate coefficient[13]. His expression applies when the following assumptions hold:

(A1) MARKOV-dependent fire with parameters $p_1$, $P(h|h)$, and $P(h|m)$ ,

(A2) geometric distribution for the number of hits required for a kill with parameter $P(K|H)$.

Here $p_1$ denotes Prob[hit on first round], $P(h|h)$ denotes the conditional hit probability Prob[hit|previous round hit], $P(h|m)$ denotes the conditional hit probability Prob[hit|previous round miss], and $P(K|H)$ denotes the conditional kill probability Prob[kill target|hit target]. It is well known (e.g. <u>see</u> PARZEN [57, pp. 129-132]) that the three hit probabilities $p_1$, $P(h|h)$, and $P(h|m)$ completely describe MARKOV-dependent fire in contrast to the situation with statistical independence between the outcomes of any two rounds fired in which case only a single hit probability, denoted simply as $p$, completely describes the process.

As above let us denote the time for the firer to kill a target as $T$ (a r.v.). Then, BONDER [11; 14] has developed that

TABLE 5.II. Factors Included in Expression for LANCHESTER Attrition-Rate Coefficient for Single-Shot MARKOV-Dependent-Fire Weapon Systems with a Geometric Distribution for the Number of Hits Required for a Kill.

Time to acquire a target, $t_a$

Time to fire first round after target acquired, $t_1$

Time to fire a round following a hit, $t_h$

Time to fire a round following a miss, $t_m$

Time of flight of the projectile, $t_f$

Probability of a hit on first round, $p_1$

Probability of a hit on a round following a hit, $P(h|h)$

Probability of a hit on a round following a miss, $P(h|m)$

Probability of destroying a target given it is hit, $P(K|H)$

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)}$$

$$+ \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\} , \tag{5.4.1}$$

where all the variables are defined in Table 5.II. The corresponding LANCHESTER attrition-rate coefficient (see Section 5.3 above) is then the reciprocal of (5.4.1)[13], i.e. for the homogeneous-force model (5.2.1) we have, for example,

$$a = \frac{1}{E[T_{XY}]} , \tag{5.4.2}$$

where $T_{XY}$ (a r.v.) denotes the time for an individual Y firer to kill a single X target. (5.4.1) is the general expression[15] for the expected time to kill a target with MARKOV-dependent fire and a geometric distribution for the number of hits required for a kill. It may be developed (see the next section) by considering the time required for an individual firer to engage and kill a single enemy target. We will see in Section 5.10 below how this complex expression reduces to very simple ones in special cases, e.g. $E[T] = 1/(\nu P_{SSK})$ for a uniform rate of fire, statistical independence between rounds, and negligible time of flight and target-acquisition time.

Together (5.4.1) and (5.4.2) allow us to estimate attrition-rate coefficients for a homogeneous-force F|F LANCHESTER-type attrition process [i.e. force-on-force combat attrition modelled by equations (5.2.1) above], and consequently one may consider using such a model to operationally

analayze combat between two homogeneous forces. In such an operational model or its extension to heterogeneous forces (see Section 7.7), we would want to consider variable attrition-rate coefficients to model temporal variations in fire effectiveness when, for example, the range between firers and targets changes appreciably during battle. We will discuss below in Section 5.11 the variables upon which such attrition-rate coefficients (indirectly) depend, with some typical range dependencies being given in Section 5.12. Moreover, this attrition-rate-coefficient model given by (5.4.1) and (5.4.2) is a general one in the sense that it allows a uniform treatment of both area-fire as well as direct-fire weapons (see Section 5.13 below and also BONDER [11, p. 231] for further details). Furthermore, we note that the MARKOV-dependent-fire assumption has been naturally motivated, since BONDER's model for MARKOV-dependent fire arose in the analysis of armored operations (e.g. see BONDER [9; 11], BONDER and FARRELL [17], or KIMBLETON [49] for further details). For example, in the analysis of tank main guns it is usually assumed (e.g. see BONDER [12, p. III-11]) that the result of the previous round is observed before the next one is fired. If the round fired misses the target, the tank gunner will make an appropriate adjustment; if a hit is obtained, the same gun setting will be used again.

Finally, let us briefly discuss data sources for BONDER's model (5.4.1). All the input data for this model is shown in Table 5.II. Data is available for all these inputs from a variety of sources: ballistics-laboratory tests, military field experiments, troop exercises, further submodels, etc. A detailed discussion of such data sources is is given in, for example, [54, pp. 167-168] and [28, pp. 173-174]. We

should add, however, that all such experimental data is for systems under simulated combat conditions and not for actual combat.

## 5.5. Derivation of BONDER's Result for the Expected Time to Kill a Target (Approach Based on the Exact Distribution of Time to Kill).

In this section we will derive BONDER's expression (5.4.1) for the expected time to kill a target, which applies under the following conditions:

(C1) MARKOV-dependent fire,

(C2) geometric distribution for the number of hits to kill,

(C3) deterministic event times (i.e. $t_a$, $t_1$, $t_h$, $t_m$, and $t_f$ are all assumed to be deterministic quantities[16]).

BONDER's result (5.4.1) is particularly significant because it is the basis for estimating weapon-system kill rates in a variety of operational models that are fairly widely used in defense planning today (see Section 7.9 for further details). The combat-modelling approach of S. BONDER and his associates at VECTOR RESEARCH, INC. basically decomposes the battlefield into unit and subunit engagements, which are essentially further decomposed into a series of one-on-one duels between opposing weapon-system types. For each type of firer-target pair, one must perform a detailed analysis of a single firer engaging a passive (i.e. one that does not return fire) target and compute the weapon-system type's kill rate according to (5.4.1) and (5.4.2), e.g. see BONDER and FARRELL [17], TAYLOR [80, Section 5.5; 81, Section 6.6], Section 7.9 of the book at hand, or [28; 88; 89]. Thus, (5.4.1) is a key result in the force-on-force combat-modelling business (see also [84; p. 16-2]).

Before we derive (5.4.1), though, let us briefly examine the shortcomings (i.e. limitations) of BONDER's approach to estimating weapon-system kill rates based on the logical[17] analysis of a single firer engaging a single passive target. Besides assuming that the above stated conditions (C1) through (C3) hold, BONDER's approach possesses the following limitations:

(L1) no consideration of interactions between firer and target,

(L2) cumulative damage assumed to be negligible,

(L3) precludes situations of both group firers and group targets.

The first limitation (L1) is a direct consequence of BONDER's general approach of considering a firer engaging a passive target. In reality, there are interactions between firer and target, e.g. the firer may "duck" and degrade his firing effectiveness when the target returns fire. The second limitation (L2) is due to the assumption of a geometric distribution of hits to kill. In reality, a target may be partially killed by the first hit and "finished off" by a second one. However, BARFOOT [3, pp. 890-892] (see also KIMBLETON [49, pp. 704-705]) has indicated how to overcome this shortcoming. The last limitation (L3) may in some sense be considered to be an elaboration and extension of the first limitation. In particular, the infantry fire fight, for example, has been characterized as being a group-target/group-firer environment (see STOCKFISCH [72; pp. 72-73]; also [83; p. 2-42]),

and it is extremely questionable whether the attendant combat interactions can be captured by any methodology based on consideration of a single firer engaging a passive target.

Thus, we will now derive (5.4.1) by analyzing the process of a single firer engaging a single passive target and following S. BONDER's [11] original analysis path[18], which included determining the probability distribution for the number of rounds necessary to achieve $z$ hits, $p_{N|Z}(n|z)$, where $N$ (a r.v. with realization $n$) is the number of rounds fired, $Z$ (a r.v. with realization $z$) is the number of hits achieved, and $p_{N|Z}(n|z)$ denotes a conditional probability mass function. In some sense, this approach might be called a "brute force" approach, due to the laborious direct computation of the conditional expectation $E[N|Z = z]$ by means of its definition as $\sum_{n=1}^{\infty} np_{N|Z}(n|z)$. We will later (<u>see</u> Section 5.6 below) present a much simpler and more general approach for developing not only $E[N|Z = z]$ but also $E[T]$ (<u>see</u> Section 5.8). Our review here of BONDER's original approach for determining $E[T]$ will let the reader appreciate the simplicity of our new approach. Finally, BONDER's original approach is limited to consideration of only deterministic event times (i.e. $t_a$, $t_1$, $t_h$, $t_m$, and $t_f$ are all assumed to be deterministic quantities), but our new approach will be able to handle stochastic ones (<u>see</u> Section 5.8 below).

Accordingly (following BONDER [11]), we consider the process by which a single firer engages and kills a single passive enemy target. We conceptualize this process as consisting of the following sequence of events from target acquisition to destruction:

(E1) The sequence begins with target acquisition which takes $t_a$ minutes to occur.

(E2) The first round is then fired and arrives in the target area $(t_1 + t_f)$ minutes later.

(E3) If the first round misses, the next round will arrive $(t_m + t_f)$ minutes after the first.

(E4) If the first round hits the target and more than one hit is required (i.e. $z > 1$), the next round will arrive $(t_h + t_f)$ minutes later.

(E5) The above sequence of firing after hits and misses is continued until the final hit, which destroys the target, is obtained.

The above conceptual target-destruction-process model is consistent with the assumption of MARKOV-dependent fire in which the outcome of the previous round is observed before the next one is fired. .

For the above conceptual model of a single firer engaging a single passive target, we will now compute the average time for the firer to kill a target, E[T]. This important result will be obtained by accomplishing the following steps:

(S1) development of mathematical model for the time to obtain $z$ hits $T_z$ (a r.v.),

(S2) computation of the expected value for $T_z$, i.e. $E[T_z] = E[T|Z = z]$ which is the expected time to kill the target given that $z$ hits are required for a kill,

(S3) computation of the unconditional expectation $E[T]$ from the conditional expectation obtained in step (S2), i.e.

$$E[T] = \sum_{z=1}^{\infty} p_Z(z)\ E[T|Z = z]\ . \tag{5.5.1}$$

Here, $p_Z(z)$ denotes the probability mass function for the number of hits to kill (assumed to follow a geometric distribution in BONDER's developments). The reader should note that the conceptual approach taken here for determining the time to kill a target is to decompose the killing process into a hitting process and a process of killing the target with hits[19]. For a geometric distribution of the number of hits to kill, we have

$$p_Z(z) = \{1 - P(K|H)\ ^{z-1}\ P(K|H)\}\ . \tag{5.5.2}$$

Let us now carry out the above three computational steps (S1) through (S3) for obtaining $E[T]$. We will see that this computation will require us to use the expected number of rounds to obtain $z$ hits, $E[N|Z = z]$, which will be subsequently derived below. Turning to the first computational step (S1), we consider the above sequence of events (E1) through (E5) to kill a target and focus on the time to obtain $z$ hits, $T_z$, which is a r.v. In this case, the number of hits $z$ is considered to be a parameter (realization of the r.v. Z). Observing

that there are $(z-1)$ rounds fired after immediately preceding hits and $(N_z - z)$ rounds fired after immediately preceding misses because the target is assumed to be destroyed by the $z^{\underline{th}}$ hit, we may mathematically express our model as

$$T_z = t_a + (t_1 + t_f) + (t_h + t_f)(z-1) + (t_m + t_f)(N_z - z) , \qquad (5.5.3)$$

where the first term on the left $t_a$ corresponds to (E1), the second $(t_1 + t_f)$ corresponds to (E2), the third $(t_h + t_f)(z-1)$ to (E3), and the fourth to (E4). Thus, we have completed step (S1).

Turning now to step (S2), we write (5.5.3) in the more convenient form

$$T_z = t_a + t_1 - t_h + (t_h - t_m)z + (t_m + t_f)N_z , \qquad (5.5.4)$$

and take its expected value to obtain

$$E[T_z] = t_s + t_1 - t_h + (t_h - t_m)z + (t_m + t_f)\ E[N_z], \qquad (5.5.5)$$

or

$$E[T|Z = z] = t_a + t_1 - t_h + (t_h - t_m)z + (t_m + t_f)\ E[N|Z=z], \quad (5.5.6)$$

It should be noted that (5.5.6) has been obtained without our making any assumption about the r.v. N, i.e. (5.5.6) holds in general. We could at this point uncondition the conditional expectation (5.5.6) and obtain

$$E[T] = t_a + t_1 - t_h + (t_h - t_m)\, E[Z] + (t_m + t_f)\, E[N] \ , \qquad (5.5.7)$$

but we will not follow this course of development any further here, since we wish to follow BONDER's original analysis path. Here $E[N]$ denotes the average number of rounds required to kill the target. Thus, (5.5.7) is an important result that relates the expected time to kill a target to the expected number of rounds required to kill the target and the expected number of hits required to kill. Only deterministic event times, cf. condition (C3) above, are required for it to hold. Again, it should be noted that (5.5.7) has been obtained without our making any assumptions about the random variables $N$ and $Z$. Returning now to BONDER's original development path, we again consider (5.5.6) and substitute for $E[N|Z = z]$. It will be shown below that for MARKOV-dependent fire

$$E[N|Z = z] = z + \frac{(1-p_1)}{P(h|m)} + \frac{\{1 - P(h|h)\}}{P(h|m)}\,(z-1) \ , \qquad (5.5.8)$$

Substituting (5.5.8) into (5.5.6), we obtain

$$E[T|Z=z] = t_a + t_1 - t_h + (t_m + t_f)\left\{\frac{P(h|h) - p_1}{P(h|m)}\right\}$$

$$+ \left\{(t_h + t_f) + (t_m + t_f)\,\frac{[1 - P(h|h)]}{P(h|m)}\right\} z \qquad (5.5.9)$$

We are now ready to execute step (S3). Assuming a geometric distribution for the number of hits to kill [i.e. (5.5.2) holds], we may uncondition (5.5.9) by multiplying both sides of it by $p_Z(z)$ and

and summing over $z$ from 1 to $\infty$, whence follows (5.4.1), since

$$\sum_{z=1}^{\infty} p_Z(z) = 1 \quad \text{and} \quad \sum_{z=1}^{\infty} z p_Z(z) = \frac{1}{P(K|H)} . \tag{5.5.10}$$

The reader should observe how the conditions (C1) through (C3) have entered into the above development of (5.4.1).

It remains for us to derive the result (5.5.8) for the conditional expectation $E[N|Z = z]$. To derive this key intermediate expression, we assume MARKOV-dependent fire and execute the following two tasks[20]

(T1) develop expression for the distribution of the number of rounds to obtain $z$ hits $p_{N|Z}(n|z)$ ,

(T2) compute the desired conditional expectation $E[N|Z = z]$ by "brute force," i.e.,

$$E[N|Z = z] = \sum_{n=1}^{\infty} n p_{N|Z}(n|z) \quad . \tag{5.5.11}$$

To develop the distribution for the number of rounds to obtain $z$ hits (with the sequence of firings ending in a hit), it is convenient to split the probability that $N$ rounds are required to obtain $z$ hits into two parts as follows

$$\begin{aligned} p_{N|Z}(n|z) &= P[N = n|Z = z] \\ &= P[N = n|Z = z \text{ with hit on first round}] \\ &\quad + P[N = n|Z = z \text{ with miss on first round}] , \end{aligned} \tag{5.5.12}$$

which holds because the outcome of the first firing is either a hit or a miss. This split will be seen to be convenient in light of subsequent combinatorial arguments. For convenience we will also write (5.5.12) as

$$p_{N|Z}(n|z) = p_z(n|H) + p_z(n|M) , \tag{5.5.13}$$

where $p_z(n|H)$ denotes the first of the two probabilities on the right-hand side of (5.5.12) and $p_z(n|M)$ denotes the second.

We will now focus on the development of the probability $p_z(n|H)$. To develop this probability, we consider the sequence of events, denoted as $S_H$, in which the following occurs:

<u>In the first</u> $r_1$ <u>firings, the event hit occurs everytime</u>;
<u>In the next</u> $s_1$ <u>firings, the event miss occurs everytime</u>;
<u>In the next</u> $r_2$ <u>firings, the event hit occurs everytime</u>;
<u>In the next</u> $s_2$ <u>firings, the event miss occurs everytime</u>;
⋮
<u>In the next</u> $s_{k-1}$ <u>firings, the event miss occurs everytime</u>;
<u>In the last</u> $r_k$ <u>firings, the event hit occurs everytime</u>.

We observe that for the joint occurrence of the above events

$$\sum_{i=1}^{k} r_i = z \quad \text{and} \quad \sum_{i=1}^{k-1} s_i = n - z , \tag{5.5.14}$$

where $r_i$ and $s_i$ are positive integers for all $i \geq 1$. The probability of the joint occurrence of the above events, denoted as

$P[S_H$ occurs], is obtained according to the MARKOV-dependence assumption by multiplying together the probabilities of all the individual firing-outcome events. Hence

$$P[S_H \text{ occurs}] = p_1 u^{r_1-1}(1-u)(1-v)^{s_1-1} vu^{r_2-1}(1-u)(1-v)^{s_2-1} \cdots vu^{r_k-1} , \tag{5.5.15}$$

or

$$P[S_H \text{ occurs}] = p_1 u^{r_1+r_2+\cdots+r_k-k}(1-u)^{k-1}(1-v)^{s_1+s_2+\cdots+s_{k-1}-(k-1)} v^{k-1}, \tag{5.5.16}$$

where for convenience we have introduced

$$u = P(h|h) \quad \text{and} \quad v = P(h|m) . \tag{5.5.17}$$

Using (5.5.14), we may write this latter probability as

$$P[S_H \text{ occurs}] = p_1 u^{z-k}(1-u)^{k-1}(1-v)^{n-z-k+1} v^{k-1} . \tag{5.5.18}$$

Now the above probability holds for <u>any particular sequence of events</u> $S_H$ in which there are $z$ hits and $(n-z)$ misses. Furthermore, the $z$ hits occur in $k$ strings of one or more hits between which there are sandwiched $(k-1)$ strings of one or more misses. Thus, to compute the probability $p_z(n|H)$ we must consider the number of ways in which such an $S_H$ can occur with $z$ hits and $(n-z)$ misses. Now

$$\begin{pmatrix}\text{number of ways} \\ \text{in which such} \\ \text{an } S_H \text{ can occur}\end{pmatrix}$$

$$= \begin{pmatrix}\text{number of ways in} \\ \text{which } k \text{ strings} \\ \text{of one or more hits} \\ \text{can contain exactly} \\ z \text{ hits}\end{pmatrix} \times \begin{pmatrix}\text{number of ways in which} \\ (k-1) \text{ strings of one} \\ \text{or more misses can} \\ \text{contain exactly } (n-z) \\ \text{misses}\end{pmatrix} . \qquad (5.5.19)$$

Also (cf. Lemma 5.5.2 below)

$$\begin{pmatrix}\text{number of ways in} \\ \text{which } k \text{ strings} \\ \text{of one or more hits} \\ \text{can contain exactly} \\ z \text{ hits}\end{pmatrix} = \binom{z-1}{k-1} , \qquad (5.5.20)$$

where $\binom{z}{k}$ denotes the binomial coefficient $z!/\{(z-k)!\,k!\}$ and $k!$ denotes "$k$ factorial" $= \prod_{i=1}^{k} i$ for $k \geq 1$. Similarly

$$\begin{pmatrix}\text{number of ways in} \\ \text{which } (k-1) \text{ strings} \\ \text{of one or more misses} \\ \text{can contain exactly} \\ (n-z) \text{ misses}\end{pmatrix} = \binom{n-z-1}{k-2} . \qquad (5.5.21)$$

Hence

$$\begin{pmatrix}\text{number of ways in} \\ \text{which such an } S_H \\ \text{can occur}\end{pmatrix} = \binom{z-1}{k-1} \binom{n-z-1}{k-2} , \qquad (5.5.22)$$

and thus

P[N = n|Z = z  with hit on first round]

$$= \binom{z-1}{k-1}\binom{n-z-1}{k-2} \; P[S_H \text{ occurs}] \,, \tag{5.5.23}$$

or

P[N = n|Z = z  with hit on first round]

$$= p_1\binom{z-1}{k-1}\binom{n-z-1}{k-2} \; u^{z-k}(1-u)^{k-1}(1-v)^{n-z-k+1}v^{k-1} \,. \tag{5.5.24}$$

Such an outcome can occur for all values of $k$ such that $1 \leq k \leq z$. It follows that

$$p_z(n|H) = \begin{cases} p_1 u^{z-1} & \text{for } n = z \,, \\ p_1 \displaystyle\sum_{k=2}^{z} \binom{z-1}{k-1}\binom{n-z-1}{k-2} u^{z-k}(1-u)^{k-1}v^{k-1}(1-v)^{n-z-k+1} & \\ & \text{for } n > z \,, \end{cases} \tag{5.5.25}$$

since $\binom{n-z-1}{k-2} = 0$ for $k = 1$ and $n > z$ (i.e. it is impossible to have (k-1) strings of one or more misses sandwiched between $k$ strings of one or more hits when $n > z$ and $k = 1$). In a similar fashion it may be shown that

$$p_z(n|M) = (1-p_1)\sum_{k=1}^{z}\binom{z-1}{k-1}\binom{n-z-1}{k-1} u^{z-k}(1-u)^{k-1}v^{k}(1-v)^{n-z-k} \,. \tag{5.5.26}$$

Substituting (5.5.25) and (5.5.26) into (5.5.13), we obtain the desired distribution $p_{N|Z}(n|z)$ for the number of rounds to obtain z hits

$$p_{N|Z}(n|z) = \begin{cases} 0 & \text{for } n < z, \\ p_1 u^{z-1} & \text{for } n = z, \\ p_1 \sum_{k=2}^{z} \binom{z-1}{k-1}\binom{n-z-1}{k-2} u^{z-k}(1-u)^{k-1} v^{k-1}(1-v)^{n-z-k+1} & \\ \quad + (1-p_1) \sum_{k=1}^{z} \binom{z-1}{k-1}\binom{n-z-1}{k-1} u^{z-k}(1-u)^{k-1} v^{k}(1-v)^{n-z-k} & \\ & \text{for } n > z, \end{cases} \tag{5.5.27}$$

where the reader should recall that $u$ and $v$ are conditional hit probabilities defined by (5.5.17). Thus, we have completed the first task (T1) for deriving $E[N|Z = z]$.

For accomplishing the second task (T2), it is more convenient to consider the characteristic function for $p_{N|Z}(n|z)$, denoted as $\phi_{N|Z}(s)$, i.e.

$$\phi_{N|Z}(s) = \sum_{n=0}^{\infty} e^{isn} \, p_{N|Z}(n|z) , \tag{5.5.28}$$

where $i = \sqrt{-1}$, than it is to compute $E[N|Z = z]$ directly by (5.5.11). The desired conditional expectation $E[N|Z = z]$ is then given by

$$E[N|Z = z] = \left(\frac{1}{i}\right) \frac{d}{ds}\phi_{N|Z}(0) . \tag{5.5.29}$$

to compute $\phi_{N|Z}(s)$ we begin by splitting it into two summations $\Sigma_1$ and $\Sigma_2$, i.e.

$$\phi_{N|Z}(s) = \Sigma_1 + \Sigma_2 , \tag{5.5.30}$$

where

$$\Sigma_1 = p_1\left\{e^{isz}u^{z-1} + \sum_{n=z+1}^{\infty}\sum_{k=2}^{z}\binom{z-1}{k-1}\binom{n-z-1}{k-2}e^{isn}u^{z-k}(1-u)^{k-1}v^{k-1}(1-v)^{n-z-k+1}\right\} , \tag{5.5.31}$$

and

$$\Sigma_2 = (1-p_1)\sum_{n=z+1}^{\infty}\sum_{k=1}^{z}\binom{z-1}{k-1}\binom{n-z-1}{k-1}e^{isn}u^{z-k}(1-u)^{k-1}v^{k}(1-v)^{n-z-k} \tag{5.5.32}$$

We will now concentrate on simplifying the expression (5.5.31) for $\Sigma_1$. Interchanging the order of summation in (5.5.31), we obtain

$$\Sigma_1 = p_1\left\{e^{isz}u^{z-1} + \sum_{k=2}^{z}\binom{z-1}{k-1}u^{z-k}\{v(1-u)\}^{k-1}\sum_{n=z+1}^{\infty}\binom{n-z-1}{k-2}e^{isn}(1-v)^{n-z-k+1}\right\} . \tag{5.5.33}$$

We will now concentrate on evaluating the last summation in (5.5.33). To this end, let us denote this summation as $S_k$, i.e. for $k = 2, 3, \ldots$

$$S_k = \sum_{n=z+1}^{\infty}\binom{n-z-1}{k-2}e^{isn}(1-v)^{n-z-k+1} . \tag{5.5.34}$$

For subsequent manipulations, it is convenient to introduce

$$m = n - z - 1 \quad \text{and} \quad j = k - 2 , \tag{5.5.35}$$

and then write (5.5.34) as

$$S_k = T_j = \sum_{m=0}^{\infty} \binom{m}{j} e^{is(m+z+1)} (1-v)^{m-j}, \qquad (5.5.36)$$

or, simplifying, for $j = 0,1,2,\ldots$

$$T_j = e^{is(z+k-1)} \sum_{m=j}^{\infty} \binom{m}{j} [e^{is}(1-v)]^{m-j} , \qquad (5.5.37)$$

since $\binom{m}{j} = 0$ when $m < j$. It is then convenient to further introduce $\ell = m-j$ and rearrange (5.5.37) into

$$T_j = e^{is(z+k-1)} \sum_{\ell=0}^{\infty} \binom{j+\ell}{j} [e^{is}(1-v)]^{\ell} . \qquad (5.5.38)$$

Let us now recall that the binomial theorem says that for $|x| < 1$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2} x^2 + \cdots,$$

or

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} x^k . \qquad (5.5.39)$$

Let us now temporarily assume that $P(h|m) > 0$. It follows that $|e^{is}(1-v)| < 1$, and consequently (5.5.38) may be written as

$$T_j = \frac{e^{is(z+k-1)}}{\{1 - e^{is}(1-v)\}^{j+1}} , \qquad (5.5.40)$$

or, equivalently,

$$S_k = \frac{e^{is(z+k-1)}}{\{1 - e^{is}(1-v)^{k-1}\}} . \qquad (5.5.41)$$

Using (5.5.34), we may write (5.5.33) as

$$\Sigma_1 = p_1 u^{z-1} \left\{ e^{isz} + \sum_{k=2}^{z} \binom{z-1}{k-1} \left[\frac{v(1-u)}{u}\right]^{k-1} S_k \right\},$$

whereupon substitution of (5.5.41), for $S_k$ yields

$$\Sigma_1 = p_1 e^{isz} u^{z-1} \left\{ 1 + \sum_{k=2}^{z} \binom{z-1}{k-1} \left[\frac{e^{is}\, u(1-u)}{u\{1 - e^{is}(1-v)\}}\right]^{k-1} \right\},$$

which by introduction of $\ell = k-1$ may be more conveniently written as

$$\Sigma_1 = p_1 e^{isz} u^{z-1} \sum_{\ell=0}^{z-1} \binom{z-\ell}{\ell} \left[\frac{e^{is} v(1-u)}{u\{1 - e^{is}(1-v)\}}\right]^{\ell}. \qquad (5.5.42)$$

Again recalling the binomial theorem, i.e. for integer $n$ we have $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, we may rewrite (5.5.42) to obtain $\Sigma_1$ in its final form

$$\Sigma_1 = p_1 e^{isz} \left\{ u + \frac{e^{is} v(1-u)}{\{1 - e^{is}(1-v)\}} \right\}^{z-1}. \qquad (5.5.43)$$

It may be similarly shown that

$$\Sigma_2 = \frac{(1-p_1) e^{is(z+1)} v}{\{1 - e^{is}(1-v)\}} \left\{ u + \frac{e^{is} v(1-u)}{\{1 - e^{is}(1-v)\}} \right\}^{z-1}. \qquad (5.5.44)$$

Substituting (5.5.43) and (5.5.44) into (5.5.30), we obtain our desired result for $\phi_{N|Z}(s)$, namely

$$\phi_{N|Z}(s) = e^{isz} \left\{ \frac{p_1 - e^{is}(p_1 - v)}{1 - e^{is}(1-v)} \right\} \left\{ \frac{u - e^{is}(u-v)}{1 - e^{is}(1-v)} \right\}^{z-1} . \quad (5.5.45)$$

Let us observe that (as it should) $\phi_{N|Z}(0) = 1$, since $p_{N|Z}(n|z)$ is a probability mass function and consequently $\sum_{n=0}^{\infty} p_{N|Z}(n|z) = 1$.

For the computation of the conditional expectation $E[N|Z = z]$ by (5.5.29), it is convenient to split $\phi_{N|Z}(s)$ into three multiplicative factors $e^{isz}$, $F_1(s)$, and $F_2(s)$ as follows

$$\phi_{N|Z}(s) = e^{isz} F_1(s) F_2(s) , \quad (5.5.46)$$

where

$$F_1(s) = \frac{p_1 - e^{is}(p_1 - v)}{1 - e^{is}(1-v)} , \quad (5.5.47)$$

and

$$F_2(s) = \left\{ \frac{u - e^{is}(u-v)}{1 - e^{is}(1-v)} \right\}^{z-1} . \quad (5.5.48)$$

For future purposes, we observe that

$$\phi_{N|Z}(0) = F_1(0) = F_2(0) = 1 . \quad (5.5.49)$$

Because of the multiplicative representation of $\phi_{N|Z}(s)$ (5.5.46), it is convenient to obtain $d\phi_{N|Z}/ds$ from its logarithmic derivative $d\{\ell n\ \phi_{N|Z}(s)\}/ds$, which is given by

$$\frac{d}{ds} \ell n\ \phi_{N|Z}(s) = iz + \frac{1}{F_1(s)} \frac{dF_1}{ds}(s) + \frac{1}{F_2(s)} \frac{dF_2}{ds}(s) . \quad (5.5.50)$$

Consequently, we find that

$$\frac{d}{ds}\phi_{N|Z}(s) = iz\,\phi_{N|Z}(s) + \frac{\phi_{N|Z}(s)}{F_1(s)}\,\frac{dF_1}{ds}(s) + \frac{\phi_{N|Z}(s)}{F_2(s)}\,\frac{dF_2}{ds}(s) ,$$

where

$$\frac{dF_1}{ds}(s) = \frac{ie^{is}v(1-p)}{\{1 - e^{is}(1-v)\}^2} \tag{5.5.52}$$

and

$$\frac{dF_2}{ds}(s) = (z-1)\left\{\frac{ie^{is}v(1-u)}{[1-e^{is}(1-v)]^2}\right\}\left\{\frac{u - e^{is}(u-v)}{1 - e^{is}(1-v)}\right\}^{z-2} . \tag{5.5.53}$$

It follows from (5.5.49) that

$$\frac{d}{ds}\phi_{N|Z}(s) = iz + i\,\frac{(1-p_1)}{v} + i\,\frac{(z-1)(1-u)}{v} , \tag{5.5.54}$$

since

$$\frac{d}{ds}F_1(0) = i\,\frac{(1-p_1)}{v}, \tag{5.5.55}$$

and

$$\frac{d}{ds}F_2(0) = i\,\frac{(z-1)(1-u)}{v} , \tag{5.5.56}$$

Recalling (5.5.29), we see that

$$E[N|Z=z] = z + \frac{(1-p_1)}{v} + (z-1)\,\frac{(1-u)}{v} , \tag{5.5.57}$$

and thus by (5.5.17) we have proved (5.5.8) for $P(h|m) > 0$. It should be clear, however, that (5.5.8) holds for $P(h|m) \geq 0$.

Finally, it remains to justify (5.5.20). Thus, we consider the <u>number of ways in which $k$ strings of one or more hits can contain exactly $z$ hits</u>. It is obvious that this number is the same as the number of ways to obtain $z$ hits on $k$ targets with each target being hit at least once. Moreover, the problem of determining this number has exactly the same mathematical structure as the classic <u>occupany problem</u> of probability theory (<u>see</u> FELLER [35, pp. 36-37]), when we agree to treat the hits as indistinguishable. To set the stage for proving (5.5.20), let us consider the somewhat simpler problem of determining the number of ways to obtain $z$ hits on $k$ targets without requiring that each target be hit at least once. To this end, we state and prove the following lemma.

LEMMA 5.5.1: The number of ways to obtain $z$ hits on $k$ targets (without requiring that each target be hit at least once) is given by $\binom{z+k-1}{k-1}$.

<u>PROOF</u>. Consider $z$ hits distributed among $k$ targets. Use the symbol * (star) to represent a hit and the symbol | (bar) to represent a target's boundary. Any stars contained within two bars between which no further bars lie represent the hits on a target. Thus, |**||***|*| would represent 6 hits on 4 targets with the first target having 2 hits, the second 0 hits, the third 3 hits, and the fourth 1 hit. In general, $(k+1)$ bars are required to represent $k$ targets. The desired number of ways for obtaining hits is determined by considering the number of possible arrangements for the above symbols. In all such arrangements, however, the first and last

symbols must be bars, and accordingly there are $z$ stars and $(k-1)$ bars remaining to be arranged. Thus, the desired number of arrangements is determined by considering the number of ways to select $(k-1)$ places out of $(z+k-1)$, which is well known (e.g. <u>see</u> FELLER [35, pp. 32-35]) to be given by the binomial coefficient

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n} .$$

<u>Q.E.D.</u>

We are now ready to prove (5.5.20) in the following equivalent form.

LEMMA 5.5.2: The number of ways to obtain $z$ hits on $k$ targets with each target being hit at least once is given by

$$\binom{z-1}{k-1} .$$

<u>PROOF</u>. Introducing the star and bar symbols as used above in the proof of Lemma 5.5.1, we consider the number of possible arrangements for these symbols. Again, the first and last symbols must always be bars, and consequently there are $z$ stars and $(k-1)$ bars remaining to be arranged. However, this time the requirement that each target must receive at least one hit imposes the additional condition that no two bars can ever be adjacent to each other in such arrangements. We may conceptualize this situation by moving and placing each of the $(k-1)$ arrangeable bars above the star to its left. In other words, we would consider |**|*|***|*| as

||   |
*******.  Since the last star receives no bar [recall that the first and last of the original $(k+1)$ bars have been omitted from further consideration because they are fixed and consequently not arrangeable], there will be $(k-1)$ stars with bars over them out of a total of $(z-1)$ stars available for such arrangements. Thus, the desired number of arrangements is determined by considering the number of ways to select $(k-1)$ places out of $(z-1)$, which is given by the binomal coefficient

$$\binom{z-1}{k-1} .$$

<u>Q.E.D.</u>

## 5.6. A Simple Derivation of the Expected Number of Rounds Necessary to Obtain $z$ Hits.

In this section we will present a very simple derivation of a general expression for the expected number of rounds to obtain $z$ hits, denoted above as the conditional expectation $E[N|Z = z]$. In the special case of MARKOV-dependent fire, our general expression reduces to BONDER's result (5.5.8), which was a key result in the development of the expected time to kill a target with MARKOV-dependent fire in Section 5.5 above. The approach that we will use here is particularly significant, since it readily leads to other important more general results [e.g. see (5.8.1) below].

Let $N_1$ (a r.v.) denote the number of rounds fired to obtain the first hit, and let $N_i$ (a r.v.) for $i \geq 2$ denote the number of rounds fired after the $(i-1)^{\underline{st}}$ hit to obtain the $i^{\underline{th}}$ hit. We then have the following very simple model for the number of rounds to obtain $z$ hits $N_z$ (also a r.v.)

$$N_z = N_1 + \sum_{i=2}^{z} N_i . \tag{5.6.1}$$

The above result (5.6.1) is a particularly transparent model for $N_z$. It follows that

$$E[N_z] = E[N_1] + \sum_{i=2}^{z} E[N_i] . \tag{5.6.2}$$

Let us again denote $E[N_z]$ as $E[N|Z = z]$ and assume that the random variables $N_i$, $i = 2,3,\ldots,z$, are identically distributed. Let us also introduce $N_s$ as a random variable having the same distribution as the random variables $N_i$ for $i \geq 2$. It follows then that

$$E[N|Z = z] = E[N_1] + (z-1)\ E[N_s] \quad . \tag{5.6.3}$$

We have therefore proved the following important lemma.

> LEMMA 5.6.1: Let the random variables $N_i$, $i = 2,3,\ldots,z$, be identically distributed. The conditional expectation for the number of rounds to achieve $z$ hits, $E[N|Z = z]$, is then given by (5.6.3), where $N_s$ denotes a random variable having the same distribution as the random variables $N_i$ for $i \geq 2$.

It should be noted that there is no assumption about MARKOV dependence for (5.6.3) to hold, only that the random variables $N_i$, $i = 2,3,\ldots,z$, be identically distributed.

For the case of MARKOV-dependent fire, it may be shown (and we will do so below) that

$$E[N_1] = 1 + \frac{(1-p_1)}{P(h|m)} \ , \tag{5.6.4}$$

and

$$E[N_s] = 1 + \frac{[1 - P(h|h)]}{P(h|m)} \ . \tag{5.6.5}$$

Substituting (5.6.4) and (5.6.5) into (5.6.3), we obtain BONDER's expression for MARKOV-dependent fire (5.5.8).

It remains for us to develop the expressions (5.6.4) and (5.6.5). We begin by observing that the random variable $N_1$ has the distribution

$$p_{N_1}(n) = \begin{cases} p_1 & \text{for } n = 1 \ , \\ (1-p_1)\{1-P(h|m)\}^{n-2}\ P(h|m) & \text{for } n \geq 2 \end{cases} \tag{5.6.7}$$

and similarly the random variable $N_s$ has the distribution

$$P_{N_s}(n) = \begin{cases} P(h|h) & \text{for } n = 1, \\ \{1-P(h|h)\}\{1-P(h|m)\}^{n-2}\, P(h|m) & \text{for } n \geq 2. \end{cases} \tag{5.6.8}$$

Direct computation now yields

$$E[N_1] = p_1 + \left\{ \frac{(1-p_1)\, P(h|m)}{1 - P(h|m)} \right\} \sum_{n=2}^{\infty} n\{1 - P(h|m)\}^{n-1} . \tag{5.6.9}$$

Let us now observe that for $0 < |x|$ differentiation of the <u>geometric series</u>

$$\sum_{n=0}^{\infty} (1-x)^n = \frac{1}{x} \tag{5.6.10}$$

yields

$$\sum_{n=1}^{\infty} n(1-x)^{n-1} = \frac{1}{x^2} , \tag{5.6.11}$$

and (for future purposes)

$$\sum_{n=2}^{\infty} n(n-1)(1-x)^{n-2} = \frac{2}{x^3} . \tag{5.6.12}$$

It follows that for $0 < |x|$

$$\sum_{n=2}^{\infty} n(1-x)^{n-1} = \frac{(1-x)(1+x)}{x^2} . \tag{5.6.13}$$

Let us now temporarily assume that $P(h|m) > 0$. Our desired result (5.6.4) for $E[N_1]$ now follows by using (5.6.13) to simplify (5.6.9). It should be clear that (5.6.4) holds for $P(h|m) \geq 0$. The expression (5.6.5) for $E[N_s]$ follows similarly.

5.7. The Number of Rounds Necessary to Kill a Target (General Derivation).

It is of considerable interest to also compute the expected number of rounds necessary to kill a target $E[N]$. Our development here is particularly significant because it suggests a way to compute both the mean and the variance of the time to kill a target under very general conditions. These important new results are given in the next section.

Assuming that the random variable $Z$ is independent of $N_i$ for all $i \geq 1$ and then taking the expected value of (5.6.3), we accordingly obtain

$$E[N] = E[N_1] + \{E[Z] - 1\} E[N_s] \tag{5.7.1}$$

where $Z$ denotes the random variable that the $z^{\underline{th}}$ hit kills the target. We have therefore proved the following important lemma.

> LEMMA 5.7.1: Let the random variables $N_i$, $i = 2,3,\ldots$ be identically distributed and assume that the number of hits required to kill the target, a random variable denoted as $Z$, is independent of the random variables $N_i$ for all $i \geq 1$. The expected number of rounds to kill a target, $E[N]$, is then given by (5.7.1), where $Z$ denotes the random variable that the zth hit kills the target and $N_s$ denotes a random variable having the same distribution as the random variables $N_i$ for $i \geq 2$.

It should be noted here that no assumption has been made about the specific nature of the distribution of the number of hits to kill a target. In other words, (5.7.1) applies under much more general circumstances than just for a geometric distribution of the number of hits to kill a target. However, if we do assume MARKOV-dependent fire and a geometric distribution for the number of hits to kill, then we may substitute (5.6.4) and (5.6.5) into (5.7.1) to obtain

$$E[N] = \frac{1}{P(h|m)} \left\{ P(h|h) - p_1 + \left[ \frac{1 + P(h|m) - P(h|h)}{P(K|H)} \right] \right\} , \tag{5.7.2}$$

since we have for a geometric distribution of the number of hits to kill

$$E[Z] = \frac{1}{P(K|H)} . \tag{5.7.3}$$

Finally, it should be noted that (5.7.2) and (5.7.3) may be substituted into (5.5.7) to yield BONDER's result for the expected time to kill a target.

The above approach of considering $N_z$ as a sum of random variables (5.6.1) is particularly significant, since it allows us to also compute higher moments for $N_z$ (and consequently also for $N$). We will accordingly now compute the variance of the number of rounds to kill a target, denoted as $\text{var}[N]$, which gives us some idea of the variability in the average number of rounds to kill a target $E[N]$. We will begin by computing the conditional variance $\text{var}[N| Z = z]$. Here we will assume

(AI) the random variables $N_i$, $i = 1,2,3,\ldots$, are not only independent of one another, but they are also independent of the random variable $Z$ representing the number of hits required to kill the target,

and

(AII) the random variables $N_i$, $i = 2,3,4,\ldots$, are identically distributed.

It then follows from (5.6.1) (e.g. <u>see</u> PARZEN [57, pp. 405-407]) that

$$\mathrm{var}[N|Z=z] = \mathrm{var}[N_1] + (z-1)\,\mathrm{var}[N_s]\ . \tag{5.7.4}$$

We have therefore proved the following companion result to Lemma 5.6.1

LEMMA 5.7.2: Assume that (AI) and (AII) hold. The conditional variance for the number of rounds to achieve $z$ hits, $\mathrm{var}[N|Z=z]$, is then given by (5.7.4), where $N_s$ is as defined in Lemma 5.6.1.

<u>For the case of MARKOV-dependent fire</u>, it may be shown (and we will do so below) that

$$\mathrm{var}[N_1] = \frac{\{1-p_1\}\{1 + p_1 - P(h|m)\}}{P^2(h|m)}\ , \tag{5.7.5}$$

and

$$\mathrm{var}[N_s] = \frac{\{1 - P(h|h)\}\{1 + P(h|h) - P(h|m)\}}{P^2(h|m)}\ . \tag{5.7.6}$$

It should be noted that for independent fire, i.e. $p_1 = P(h|h) = P(h|m)$, (5.7.5) and (5.7.6) both reduce to the well-known result for the geometric distribution, namely var[number of rounds for first hit] $= (1-p_1)/p_1^2$. Substituting (5.7.5) and (5.7.6) into (5.7.4), we find that for MARKOV-dependent fire the conditional variance for the number of rounds to achieve z hits is given by

$$\mathrm{var}[N|Z=z] = \frac{\{P(h|h) - p_1\}\{P(h|h) + p_1 - P(h|m)\}}{P^2(h|m)}$$

$$+ \frac{z\{1 - P(h|h)\}\{1 + P(h|h) - P(h|m)\}}{P^2(h|m)}, \tag{5.7.7}$$

which for independent fire reduces to $\mathrm{var}[N|Z=z] = z(1-p_1)/p_1^2$

It remains for us to develop the expressions (5.7.5) and (5.7.6). We begin by computing $E[N_1^2]$. Direct computation yields $E[N_1^2] = \sum_{n=1}^{\infty} n^2 p_{N_1}(n)$, or by (5.6.7)

$$E[N_1^2] = p_1 + (1-p_1)P(h|m)\left[\sum_{n=2}^{\infty} n(n-1)\ \{1 - P(h|m)\}^{n-2} + \frac{1}{\{1 - P(h|m)\}}\sum_{n=2}^{\infty} n\{1 - P(h|m)\}^{n-1}\right], \tag{5.7.8}$$

whence substitution of (5.6.12) and (5.6.13) into (5.7.8) and some algebraic manipulation yields

$$E[N_1^2] = \frac{P^2(h|m) + \{1-p_1\}\{2 + P(h|m)\}}{P^2(h|m)}. \tag{5.7.9}$$

Substituting (5.6.4) and (5.7.9) into $var[N_1] = E[N_1^2] - E^2[N_1]$, we easily obtain our desired result (5.7.5). The expression (5.7.6) for $var[N_s]$ may be developed in a similar way.

To compute the unconditional variance $var[N]$ from (5.7.4), we observe that there is an important formula (e.g. <u>see</u> PARZEN [58, p. 55]) expressing the unconditional variance in terms of the conditional variance, namely

$$var[N] = E_Z[var[N|Z]] + var_Z[E[N|Z]] , \qquad (5.7.10)$$

where $E_Z[\cdot]$ explicitly denotes that the expected value is being computed with respect to the r.v. $Z$ and similarly for $var_Z[\cdot]$. Again we will assume that assumptions (AI) and (AII) hold. From (5.7.4), we see that the expected value of the conditional variance $E_Z[var[N|Z]]$ is given by

$$E_Z[var[N|Z]] = var[N_1] + \{E[Z]-1\} var[N_s] . \qquad (5.7.11)$$

From (5.6.3), we see that the variance of the conditional expectation $var_Z[E[N|Z]]$ is given by

$$var_Z[E[N|Z]] = var[Z] E^2[N_s] . \qquad (5.7.12)$$

Substituting (5.7.11) and (5.7.12) into (5.7.10), we obtain the following expression for the variance of the number of rounds to kill a target

$$\mathrm{var}[N] = \mathrm{var}[N_1] + \{E[Z]-1\}\ \mathrm{var}[N_s] + \mathrm{var}[Z]\ E^2[N_s] \ . \tag{5.7.13}$$

We have therefore proved the following important lemma.

> LEMMA 5.7.3: Assume that (AI) and (AII) hold. The variance of the number of rounds to kill a target, var[N], is then given by (5.7.13), where $Z$ and $N_s$ are as defined in Lemma 5.7.1.

For the special case of MARKOV-dependent fire and a geometric distribution for the number of hits to kill, (5.7.13) becomes

$$\mathrm{var}[N] = \frac{(u-p_1)(u + p_1 - v)}{v^2} + \frac{\{(1-u+v)^2 + 2\omega(u-v)(1-u+v/2) - \omega v\}}{(\omega v)^2} , \tag{5.7.14}$$

where $u = P(h|h)$, $v = P(h|m)$, and $\omega = P(K|H)$. This important result (5.7.14) is equivalent to one obtained by KIMBLETON [49] by other means in a much less explicit form.

## 5.8. General Results for Time to Kill a Target.

In this section we will extend the approach used in the previous section (for developing the mean and the variance for the number of rounds to kill a target) to develop new important results for the time for a single firer to kill a single passive enemy target. Specifically, we will use a very transparent, simple model to obtain very general expressions for the mean and variance of the time to kill a target. As the reader undoubtedly knows by now, such results are very significant because they provide a basis for estimating weapon-system kill rates in detailed operational LANCHESTER-type models of combat attrition, and our new results allow such kill rates to be estimated under more general conditions than before. Additionally, the simple direct approach used to obtain these new important results is significant in its own right, since it appears to be applicable in other related cases of interest.

Thus, the main result of the section at hand is to show that under fairly general circumstances the expected time to kill a target, $E[T]$, is given by

$$E[T] = E[T_a] + E[T_{fr}] - E[T_h] + \{E[T_h] + E[T_f]\}\, E[Z]$$
$$+ \{E[T_m] + E[T_f]\} \Big( E[Z]\, \{E[N_s] - 1\} + E[N_1] - E[N_s] \Big) , \qquad (5.8.1)$$

where

$T_a$ (a r.v.) denotes the time to acquire a target.

$T_{fr}$ (a r.v.) denotes the time to fire the first round after the target has been acquired,

$T_h$ (a r.v.) denotes the time to fire a round following a hit,

$T_m$ (a r.v.) denotes the time to fire a round following a miss,

$T_f$ (a r.v.) denotes the time of flight of the projectile,

$N_1$ (a r.v.) denotes the number of rounds fired to obtain the first hit,

$N_s$ (a r.v.) denotes the number of rounds fired to obtain any hit subsequent to the first one (and measured from the occurrence of the last hit),

and

$Z$ (a r.v.) denotes the number of hits required to kill the target.

Also, a somewhat less explicit and more complicated result for the variance of the time to kill a target is given by (5.8.11), (5.8.20), and (5.8.28) below. For the special case of <u>MARKOV-dependent fire</u> and a <u>geometric distribution of the number of hits to kill</u>, the above general result for the expected time to kill a target reduces to[21]

$$E[t] = E[T_a] + E[T_{fr}] - E[T_h] + \frac{\{E[T_h] + E[T_f]\}}{P(K|H)} + \frac{\{E[T_m] + E[T_f]\}}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \tag{5.8.2}$$

which the reader will easily recognize as (5.4.1) with the deterministic event times $t_a$, $t_1$, $t_h$, $t_m$, and $t_f$ replaced by the expected values of the corresponding random variables.

Let us now turn to the development of (5.8.1) for the expected value of the time for a single firer to kill a single passive enemy target and the variance of this time. We will again consider the conceptual model (given in Section 5.5) of the process by which a single firer engages and kills a

single passive enemy target. It consists of the sequence of events (E1) through (E5) given above in Section 5.5. For this model we will compute the average time for the firer to kill a target, E[T], by executing the two following steps:

(S1) relate expected time to kill a target to the expected times to obtain the first and subsequent hits and to the expected number of hits to kill [<u>see</u> (5.8.6) below],

(S2) develop submodel for the expected times to obtain the first and subsequent hits [<u>see</u> (5.8.15) and (5.8.23) below].

The variance of the time to kill, var[T], will be obtained in a similar (but much less explicit and more complicated) manner. The basic idea behind developing these results is to decompose an event time of interest into the sum of a random number of component event times and to compute the appropriate moments along the lines as done in Section 5.7 above. For the development of these results, we will let $T_1$ (a r.v.) denote the length of the time interval from the time at which the last target was killed until the first hit is obtained on the target at hand, and $T_i$ (a r.v. for $i = 2, 3, 4, \ldots$) denote the length of the time interval from the time at which the $(i-1)^{\underline{st}}$ hit was achieved until the $i^{\underline{th}}$ hit is obtained on the target. We will then assume that

(A1) the random variables $T_i$, $i = 1,2,3, \ldots$, are all independent of the random variable $Z$ representing the number of hits required to kill the target,

(A2) the random variables $T_i$, $i = 2,3,4, \ldots$, are all identically distributed,

and (A3) the random variables $T_i$, $i = 1,2,3, \ldots$, are all independent of one another.

Let us now carry out the above two computational steps (S1) and (S2) for obtaining $E[T]$ and $var[T]$. Accordingly, we turn to the first computational step (S1) and consider [cf. (5.6.1) above] the following model for the time to obtain $z$ hits, $T_z$ (a r.v.),

$$T_z = T_1 + \sum_{i=2}^{z} T_i , \tag{5.8.3}$$

where $z$ denotes a previously-specified positive-integer number (i.e. it is a positive-integer-valued deterministic parameter upon which the r.v. is conditioned). Here (as elsewhere) we have adopted the convention that $\sum_{i=2}^{z} T_i = 0$ for $z < 2$. The above result (5.8.3) is a particularly transparent model for $T_z$. It follows that

$$E[T_z] = E[T_1] + \sum_{i=2}^{z} E[T_i] , \tag{5.8.4}$$

Denoting $E[T_z]$ as $E[T|Z = z]$ and recalling assumption (A2) above, we may then write

$$E[T|Z = z] = E[T_1] + (z-1)\ E[T_s]\ , \tag{5.8.5}$$

where $T_s$ denotes a r.v. having the same distribution as the random variables $T_i$ for $i \geq 2$. Recalling assumption (A1), we multiply both sides of (5.8.5) by $p_Z(z)$ and sum from 1 to $\infty$ to obtain the expected value for the time to kill a target

$$E[T] = E[T_1] + \{E[Z] - 1\}\ E[T_s]\ . \tag{5.8.6}$$

To compute var[T], we observe that (cf. Section 5.7 above or PARZEN [58, p. 55])

$$\text{var}[T] = E_Z[\text{var}[T|Z]] + \text{var}_Z[E[T|Z]]\ . \tag{5.8.7}$$

Now it follows by arguments similar to those used for the development of (5.7.4) above that

$$\text{var}[T|Z] = \text{var}[T_1] + (z-1)\ \text{var}[T_s]\ , \tag{5.8.8}$$

whence

$$E_Z[\text{var}[T|Z]] = \text{var}[T_1] + \{E[Z] - 1\}\ \text{var}[T_s]\ . \tag{5.8.9}$$

Here, assumption (A3) is needed for (5.8.8) to hold. We also observe that (5.8.5) yields [cf. the development of (5.7.12) above]

$$\mathrm{var}_Z[E[T|Z]] = \mathrm{var}[Z]\ E^2[T_s] \ . \qquad (5.8.10)$$

Substituting (5.8.9) and (5.8.10) into (5.8.7), we obtain the following expression for the variance of the time to kill a target in terms of the variance for the time to obtain the first hit $T_1$ and that for the time to obtain any subsequent hit $T_s$

$$\mathrm{var}[T] = \mathrm{var}[T_1] + \{E[Z] - 1\}\ \mathrm{var}[T_s] + \mathrm{var}[Z]\ E^2[T_s] \ . \qquad (5.8.11)$$

We have therefore proved the following important lemma.

LEMMA 5.8.1: Assume that (A1) and (A2) hold. The expected time to kill a target, $E[T]$, is then given by (5.8.6), where $T_1$ (a r.v.) denotes the time to obtain the first hit, $T_s$ (a r.v.) denotes the time between any two subsequent consecutive hits, and $Z$ (a r.v.) denotes the number of hits required to kill the target. If we additionally assume that (A3) holds, then the variance of the time to kill a target, $\mathrm{var}[T]$, is given by (5.8.11).

The reader should note that the above results for the time to kill a target are expressed in terms of the moments for the time to obtain the first hit and the time between any two subsequent consecutive hits, and not in terms of the basic event times for the sequence of events (E1) through (E5) in the conceputal model of Section 5.5 (i.e. the random variables $T_a$, $T_{fr}$, $T_h$, $T_m$, and $T_f$). Accordingly, we now turn to the second computational step

(S2) mentioned above and consider the following model for the time to obtain the first hit, $T_1$,

$$T_1 = T_a + (T_{fr} + T_f) + (N_1 - 1)(T_m + T_f) , \tag{5.8.12}$$

where $N_1$ (a r.v.) denotes the number of rounds fired to obtain the first hit. We will now assume that

($\bar{A}$1) the random variables $T_a$, $T_f$, $T_{fr}$, and $T_m$ are all independent of the random variable $N_1$ representing the number of rounds fired to obtain the first hit,

and ($\bar{A}$2) the random variables $T_a$, $T_f$, $T_{fr}$, and $T_m$ are all independent of one another.

To compute the expected value of $T_1$, we consider the time required to fire $n$ rounds $T_1^n$ (here $n$ may be considered to be a realization of $N_1$) and obtain from (5.8.12)

$$T_1^n = T_a + (T_{fr} + T_f) + (n-1)(T_m + T_f) , \tag{5.8.13}$$

and hence

$$E[T_1|N_1 = n] = E[T_a] + E[T_{fr}] + E[T_f] + (n-1)\{E[T_m] + E[T_f]\} , \tag{5.8.14}$$

where $E[T_1^n]$ has been denoted as $E[T_1|N_1 = n]$. Using arguments similar to those used above, we may uncondition $E[T_1|N_1 = n]$ to obtain

$$E[T_1] = E[T_a] + E[T_{fr}] + E[T_f] + \{E[N_1] - 1\} \{E[T_m] + E[T_f]\} . \tag{5.8.15}$$

To compute $var[T_1]$, we first observe that

$$var[T_1] = E_{N_1}[var[T_1|N_1]] + var_{N_1}[E[T_1|N_1]] . \tag{5.8.16}$$

From (5.8.13) and assumption ($\bar{A}2$) it follows that

$$var[T_1|N_1 = n] = var[T_a] + var[T_{fr}] + var[T_f] + (n-1) \{var[T_m] + var[T_f]\}, \tag{5.8.17}$$

whence

$$E_{N_1}[var[T_1|N_1]] = var[T_a] + var[T_{fr}] + var[T_f] + \{E[N_1]-1\}\{var[T_m] + var[T_f]\}. \tag{5.8.18}$$

Here, assumption ($\bar{A}1$) is needed to justify obtaining (5.8.18) from (5.8.17). Also, (5.8.14) yields

$$var_{N_1}[E[T_1|N_1]] = var[N_1] \{E[T_m] + E[T_f]\}^2 . \tag{5.8.19}$$

Substituting (5.8.18) and (5.8.19) into (5.8.16), we obtain the following expression for the variance of the time to obtain the first hit

$$\begin{aligned} var[T_1] = {} & var[T_a] + var[T_{fr}] + var[T_f] + \{E[N_1] - 1\}\{var[T_m] + var[T_f]\} \\ & + var[N_1] \{E[T_m] + E[T_f]\}^2 . \end{aligned} \tag{5.8.20}$$

We have therefore proved the following important lemma.

LEMMA 5.8.2: Assume that ($\bar{A}1$) holds. The expected time to obtain the first hit on a target, $E[T_1]$, is then given by (5.8.15). If we additionally assume that ($\bar{A}2$) holds, then the variance of the time to obtain the first hit on a target, $\mathrm{var}[T_1]$, is given by (5.8.20).

We have now completed the first half of step (S2). This computational step is completed by repeating the above calculation procedure for the time between any two subsequent consecutive hits on the target $T_s$, which has the same distribution as $T_i$ for $i \geq 2$. Here we will merely sketch developments, since the details are completely analogous to those given above for $T_1$. We will now assume that

($\bar{\bar{A}}1$) the random variables $T_f$, $T_h$, and $T_m$ are all independent of the random variable $N_i$ (for $i \geq 2$) representing the number of rounds fired after the $(i-1)^{\underline{st}}$ hit to obtain the $i^{\underline{th}}$ hit,

and ($\bar{\bar{A}}2$) the random variables $T_f$, $T_h$, and $T_m$ are all independent of one another.

Similar to the above, it may be shown that the following model (for $i \geq 2$)

$$T_i = T_h + T_f + (N_i - 1)(T_m + T_f) \tag{5.8.21}$$

leads to

$$E[T_i | N_i = n] = E[T_h] + E[T_f] + (n-1)\ \{E[T_m] + E[T_f]\}, \tag{5.8.22}$$

and consequently

$$E[T_s] = E[T_h] + E[T_f] + \{E[N_s] - 1\}\{E[T_m] + E[T_f]\}, \tag{5.8.23}$$

where we have taken the liberty of replacing $T_i$ and $N_i$ by their equivalents $T_s$ and $N_s$. We now turn to the variance. In general, we have for $i \geq 2$

$$\mathrm{var}[T_i] = E_{N_i}[\mathrm{var}[T_i | N_i]] + \mathrm{var}_{N_i}[E[T_i | N_i]] \ . \tag{5.8.24}$$

It is easily shown that

$$\mathrm{var}[T_i | N_i = n] = \mathrm{var}[T_h] + \mathrm{var}[T_f] + (n-1)\ \{\mathrm{var}[T_m] + \mathrm{var}[T_f]\}, \tag{5.8.25}$$

$$E_{N_i}[\mathrm{var}[T_i | N_i]] = \mathrm{var}[T_h] + \mathrm{var}[T_f] + \{E[E[N_i]-1\}\{\mathrm{var}[T_m] + \mathrm{var}[T_f]\}\ , \tag{5.8.26}$$

and

$$\mathrm{var}_{N_i}[E[T_i | N_i]] = \mathrm{var}[N_i]\ \{E[T_m] + E[T_f]\}^2\ , \tag{5.8.27}$$

whence (again, replacing $T_i$ by $T_s$ and $N_i$ by $N_s$) follows

$$\text{var}[T_s] = \text{var}[T_h] + \text{var}[T_f] + \{E[N_s] - 1\}\{\text{var}[T_m] + \text{var}[T_f]\}$$
$$+ \text{var}[N_s]\ \{E[T_m] + E[T_f]\}^2 \tag{5.8.28}$$

and the following important lemma.

LEMMA 5.8.3: Assume that ($\bar{\bar{A}}$1) holds. The expected time to obtain any subsequent hit on a target (where this time interval is measured from the occurrence of the last previous hit), $E[T_s]$, is then given by (5.8.23). If we additionally assume that ($\bar{\bar{A}}$2) holds, then the variance of the time to obtain any subsequent hit on a target, $\text{var}[T_s]$, is given by (5.8.28).

We are now ready to develop our final results for $E[T]$ and $\text{var}[T]$. Substituting (5.8.15) and (5.8.23) into (5.8.6), we obtain the desired final result (5.8.1) for the expected time to kill a target. Because of the complexity of corresponding terms for the variance of the time to kill a target, we will not present here one final expression for $\text{var}[T]$ in terms of the fundamental operational variables appearing in (5.8.1), but we will let $\text{var}[T]$ be given by (5.8.11) in terms of $\text{var}[T_1]$ and $\text{var}[T_s]$, which in turn are expressed in terms of the fundamental operational variables by (5.8.20) and (5.8.28). Thus, to compute $\text{var}[T]$ one must first use (5.8.20) to compute $\text{var}[T_1]$ and (5.8.28) to compute $\text{var}[T_s]$ and then use (5.8.11) to combine these intermediate results into the final desired result for $\text{var}[T]$. It remains for us to reconcile the three different sets of assumptions used to develop Lemmas 5.8.1, 5.8.2, and 5.8.3, upon which the final results for $E[T]$ and $\text{var}[T]$ are based. In particular, if we assume that the random

variables $N_i$ for $i = 1,2,3,\ldots$ are independent of one another, then assumption $(\bar{A}1)$, $(\bar{A}2)$, $(\bar{\bar{A}}1)$, and $(\bar{\bar{A}}2)$ imply that assumption (A3) holds (i.e. the random variables $T_i$ for $i = 1,2,3,\ldots$ are independent of one another). Thus, all these above assumptions may be merged into the following consolidated set:

$(\tilde{A}1)$ the random variables $T_a$, $T_f$, $T_{fr}$, and $T_m$ are all independent of the random variable $N_1$ representing the number of rounds fired to obtain the first hit,

$(\tilde{A}2)$ the random variables $T_f$, $T_h$, and $T_m$ are all independent of the random variable $N_i$ (for $i \geq 2$) representing the number of rounds fired after the $(i-1)^{\underline{st}}$ hit to obtain the $i^{\underline{th}}$ hit,

$(\tilde{A}3)$ the random variables $N_i$ for $i = 1,2,3,\ldots$ are all independent of the random variable $Z$ representing the number of hits required to kill the target,

$(\tilde{A}4)$ the random variables $N_i$ for $i = 2,3,4,\ldots$ are all identically distributed (let $N_s$ denote a random variable having the same distribution as these random variables),

$(\tilde{A}5)$ the random variables $N_i$ for $i = 1,2,3,\ldots$ are all independent of one another,

and $(\tilde{A}6)$ the random variables $T_a$, $T_f$, $T_{fr}$, $T_h$, and $T_m$ are all independent of one another.

We are now ready to summarize the final results of this section for the mean $E[T]$ and the variance $var[T]$ of the time to kill a target. We do this with the following theorem.

THEOREM 5.8.1: Assume that $(\tilde{A}1)$ through $(\tilde{A}4)$ hold. The expected time to kill a target, $E[T]$, is then given by (5.8.1). If we additionally assume that $(\tilde{A}5)$ and $(\tilde{A}6)$ hold, then the variance of the time to kill a target, $var[T]$, is given by (5.8.11), with (in turn) $var[T_1]$ given by (5.8.20) and $var[T_s]$ given by (5.8.28).

The above result (5.8.1) for the expected time to kill a target holds under the very general conditions described by assumptions $(\tilde{A}1)$ through $(\tilde{A}4)$. Moreover, there are some special cases of particular interest to the combat modeller. In particular, for MARKOV-dependent fire (with stationary transition probabilities), we have shown that (see Section 5.7)

$$\{E[N_1] - 1\} = \frac{1 - p_1}{P(h|m)} , \tag{5.8.29}$$

and

$$\{E[N_s] - 1\} = \frac{1 - P(h|h)}{P(h|m)} . \tag{5.8.30}$$

For a geometric distribution of the number of hits to kill, we have

$$E[Z] = \frac{1}{P(K|H)} . \tag{5.8.31}$$

Thus, for MARKOV-dependent fire and a geometric distribution of the number of hits to kill, (5.8.2) then follows from (5.8.1). We leave it as an exercise

for the reader to verify that assumptions ($\tilde{A}1$) through ($\tilde{A}6$) are satisfied in this case. Finally, we could also use in this special case (5.7.5) and (5.7.6) to compute var[T] by means of (5.8.11), (5.8.20), and (5.8.28).

## 5.9. Development of Expected Time to Kill a Target as Mean State-Recurrence Time in Continuous-Time Semi-MARKOV Process.

In this section we present a third approach for developing the expected time to kill a target. It is based on conceputalizing the process by which a single firer engages a single passive target as a so-called continuous-time semi-MARKOV (or MARKOV-renewal) process and invoking a result by BARLOW [4, p. 53] for the mean recurrence time for a state in such a stochastic process with an imbedded ergodic MARKOV chain (i.e. the system can be in any one of a finite number of states after a sufficiently long lapse of time). Although our approach based on considering the expected value of the sum of a random number of random variables is undoubtedly the simplest and most transparent one for deriving attrition-rate-coefficient results for homogeneous-force combat, the state-recurrence-time approach may have greater applicability for heterogeneous-force combat, and it does form the basis for determining numerical values for attrition-rate coefficients in the VECTOR series of combat models[22] of VECTOR RESEARCH, INC. [28; 54; 89; 90] (<u>see</u> also Section 5.16 below).

The state-recurrence-time approach may be considered to have received its impetus from BARFOOT [3], who in 1969 (besides first proposing that an attrition-rate coefficient be defined as the reciprocal of the expected time to kill a target) presented an alternative (to BONDER's [11]) method for deriving an expression for the expected time for a single firer to kill a target. BARFOOT considered that the target could be in one of, in general, m states [to obtain a result like BONDER's [11] for the time to kill a target, one of three states: killed, hit (but not killed), and missed (and not killed)], transitions between these states would

occur from the impacts of rounds in the target area, and this target-destruction process formed a MARKOV chain. FARRELL [17, pp. 136-137] then observed that if the target-destruction process could be conceptualized in such a way that every state has some probability of eventually occurring, then one can invoke a known result on mean state-recurrence time from the theory of semi-MARKOV processes to determine the expected time to kill a target.

Loosely speaking, a semi-MARKOV process (SMP) is completely described by a matrix of transition probabilities for an imbedded MARKOV chain (MC) and a matrix of distribution functions for the "wait" in a state before going to another state. For a continuous-time MC, the "wait" in a state is exponentially distributed, while the SMP considers more general distributions for waiting times (e.g. see BARLOW [4], ÇINLAR [22], COX and MILLER [30, p. 352], or ROSS [59; 69]). For such a SMP, BARLOW [4, p. 53] (see also ÇINLAR [22, Theorem 6.12] or ROSS [59, Theorem 5.16]) proved the following important result.

THEOREM 3.9.1 (BARLOW [4]): Consider a semi-MARKOV process (with $J$ states $S_1, S_2, \ldots, S_J$) in which all states communicate. The mean recurrence time for state $S_i$, denoted as $\ell_{ii}$, is then given by

$$\ell_{ii} = \frac{1}{\pi_i} \sum_{j=1}^{J} \pi_j \mu_j \, , \tag{5.9.1}$$

where $\mu_j$ denotes the unconditional mean wait in state $S_j$ and $\pi_j$ is an element (corresponding to state $S_j$) of the stationary distribution of the imbedded MARKOV chain. It follows that

$$\pi_j = \sum_{i=1}^{J} \pi_i p_{ij} , \tag{5.9.2}$$

and

$$\mu_j = \sum_{k=1}^{J} p_{jk} \mu_{jk} , \tag{5.9.3}$$

where $p_{ij}$ is the transition probability that the system goes from state $S_i$ to state $S_j$ when such a change does occur, and $\mu_{jk}$ denotes the mean time that the system remains in state $S_j$ before it transitions to state $S_k$.

It should be noted that no assumption at all is made here about the distribution of waiting time in state $S_j$ before the system transitions to state $S_k$.

Let us now show how BARLOW's result (Theorem 5.9.1) may be used to develop the general result (5.8.2) for the expected time for an individual firer to kill a single passive enemy-target type with MARKOV-dependent fire [a special case of which is BONDER's result (5.4.1)]. After developing results for this important special case, we will outline how this approach may be used to determine the expected time to kill a target under more general circumstances (e.g. under conditions of several target types with different priorities for their engagement).

To develop (5.8.2), we consider a single firer trying to engage and kill a single type of target. We assume that all the assumptions required for (5.8.2) (and given in Section 5.8) hold. Let us focus on the target. It can be

(1) undetected,

(2) hit,

(3) missed,

or (4) killed.

When a target has been killed, search immediately begins for a new target. We now seek to define the system states so that the conditions requisite for invoking BARLOW's theorem (i.e. Theorem (5.9.1) are met (in particular, given any starting state, after sufficient lapse of time the system could be in any state). Thus, the "killed" state cannot be absorbing. To accomplish such a defining of system states, we observe that the following two situations are mathematically treated the same: (I) a new target immediately appearing upon the destruction of the currently engaged target, and (II) the same target being repeatedly killed. Thus, we will define the following three system states:

$S_1$ = killed state (which lasts from the destruction of the previous target until the first round has been fired at a new target),

$S_2$ = hit state (in which the target has been hit but not killed by the last round fired),

and $S_3$ = missed state (in which the target has been missed and not killed by the last round fired).

These states and the corresponding transition probabilities for changes in system state are shown in Figure 5.4. The transition probabilities for the imbedded MARKOV chain are given by

$$\begin{array}{lll} p_{11} = p_1 P(K|H), & p_{21} = P(h|h)\ P(K|H), & p_{31} = P(h|m)\ P(K|H), \\ p_{12} = p_1\{1-P(K|H)\}, & p_{22} = P(h|h)\{1-P(K|H)\}, & p_{32} = P(h|m)\{1-P(K|H)\}, \\ p_{13} = 1-p_1, & p_{23} = 1-P(h|h), & p_{33} = 1-P(h|m), \end{array} \tag{5.9.4}$$

Furthermore, the expected wait in each state is independent of the next state visited and given by

$$\mu_1 = E[T_a] + E[T_{fr}] + E[T_f] ,$$

$$\mu_2 = E[T_h] + E[T_f] , \tag{5.9.5}$$

and

$$\mu_3 = E[T_m] + E[T_f] ,$$

where all the subscripted $T$'s are as defined in Section 5.8.

With the above definitions, all states communicate, and the expected time to kill a target is just the expected time between visits to state $S_1$, i.e. the mean recurrence time $\ell_{11}$ of state $S_1$. Hence, the expected time to kill a target $E[T]$ is given by

$$E[T] = \ell_{11} = \frac{1}{\pi_1} \sum_{j=1}^{3} \pi_j \mu_j , \tag{5.9.6}$$

where the stationary probabilities are given by the system of equations

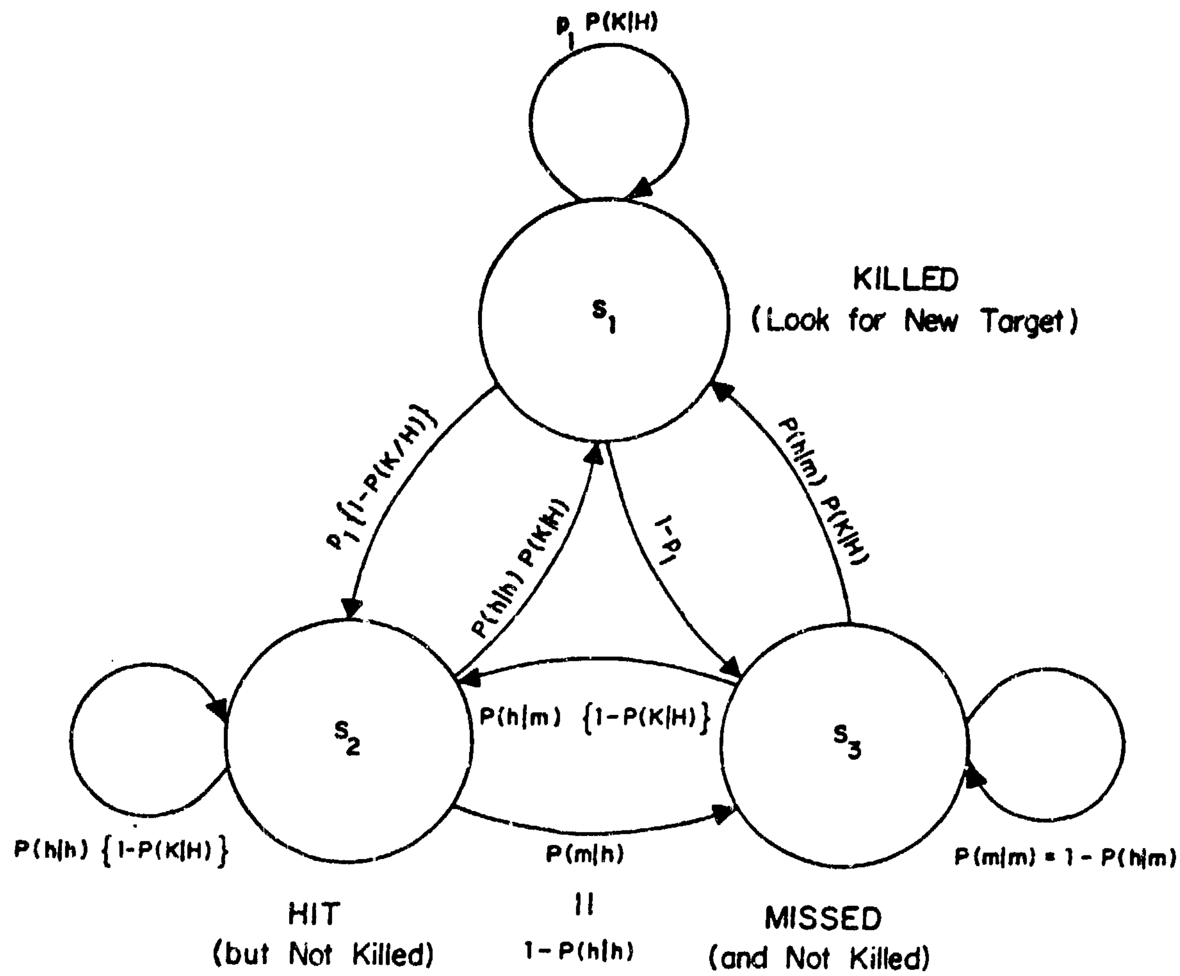


Figure 5.4. System states and transition probabilities used in alternate derivation of expected time to kill a target by invoking BARLOW's [4] result for mean recurrence time of semi-MARKOV process with imbedded ergodic MARKOV chain.

$$\pi_j = \sum_{i=1}^{3} \pi_i p_{ij} \qquad \text{for} \quad j = 1,2,3. \tag{5.9.7}$$

From (5.9.6) we see that what we need for computing the mean recurrence time for a target being killed $\ell_{11}$ is not the stationary probabilities $\pi_j$ for $j = 1,2,3$ themselves but the ratios $\pi_j/\pi_1$ for $j = 1,2,3$. Accordingly, let us define

$$r_j = \frac{\pi_j}{\pi_1}. \tag{5.9.8}$$

We may then write

$$E[T] = \ell_{11} = \mu_1 + r_2\mu_2 + r_3\mu_3 , \tag{5.9.9}$$

where $r_2$ and $r_3$ are determined by the linear system of equations

$$\begin{cases} (p_{22} - 1)r_2 + p_{32}r_3 = -p_{12} , \\ p_{23}r_2 + (p_{33}-1)r_3 = -p_{13} . \end{cases} \tag{5.9.10}$$

The reader should recall here that only two of the three equations (5.9.7) are linearly independent, since $\sum_{j=1}^{3} p_{ij} = 1$. Solving (5.9.10), we find that

$$r_2 = \frac{p_{12}(1 - p_{33}) = p_{12}p_{32}}{(1-p_{22})(1-p_{33})-p_{23}p_{32}} ,$$

and (5.9.11)

$$r_3 = \frac{p_{13}(1 - p_{22}) + p_{12}p_{23}}{(1-p_{22})(1-p_{33}) - p_{23}p_{32}} .$$

Substituting (5.9.4) into (5.9.11), we find that

$$r_2 = \frac{\{1 - P(K|H)\}}{P(K|H)},$$

and

$$r_3 = \frac{1}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\},$$

whence follows (5.8.2) from substitution of (5.9.5) and (5.9.12) into (5.9.6).

In general, the above approach may be used to develop an expression for the expected time to kill a target $E[T]$ in any firing process with a set of $J$ distinguishable states $S_1, S_2, \ldots, S_J$ as long as the following assumptions hold:

(A1) the process makes transitions at distinct points in time,

(A2) given that one is in state $S_i$, the probability of transition to state $S_j$ does not depend on any history of the process; we let $p_{ij}$ denote the probability of transition to state $S_j$ from state $S_i$, i.e.

$$p_{ij} = P\left[\begin{array}{c|c} \text{system in state} & \text{system in state} \\ S_j \text{ after transition} & S_i \text{ before transition} \end{array}\right]$$

(A3) given that one is in state $S_i$, the mean wait before a transition to state $S_j$ depends only on the specification of these two states; we let $\mu_{ij}$ denote the mean wait in state $S_i$ before a transition to state $S_j$,

(A4) no matter where the system starts, every state has some probability of eventually occurring,

and (A5) the states are so defined that the expected time interval between successive entries into state $S_1$ corresponds to the expected time between casualties.

In essence, this approach may be applied to any target-destruction process that can be modelled as a semi-MARKOV process[23]. Let us now introduce the ratio $r_j = \pi_j/\pi_1$. The expected time to kill a target $E[T]$ is then simply the expected time between the occurrences of two successive casualties $\ell_{11}$ and is given by

$$E[T] = \mu_1 + \sum_{j=2}^{J} r_j \mu_j \ , \tag{5.9.13}$$

where $r_2, \ldots, r_J$ are determined by the linear system of equations

$$\sum_{i=2}^{J} (p_{ij} - \delta_{ij}) r_i = -p_{1j} \qquad \text{for } j = 2,\ldots,J, \tag{5.9.14}$$

and $\delta_{ij}$ denotes the KRONECKER delta defined as $= 1$ for $i = j$ and $= 0$ otherwise. Here we should recall that assumption (A4) guarantees that we can always solve the linear system of equations (5.9.14) (e.g. <u>see</u> FELLER [35, pp. 356-362] or PARZEN [57, p. 265]). If the $\mu_j$ are not directly available, they may be obtained from the $\mu_{ij}$ by using (5.9.3).

## 5.10. Special Cases of BONDER's General Expression for the LANCHESTER-Attrition-Rate Coefficient.

We began our examination of the analytical modelling of a LANCHESTER attrition-rate coefficient [i.e. approach (A1) of Section 5.1] by considering in Section 5.2 some very simple models for such coefficients in the case of aimed fire and an impact lethality mechanism, and then we progressed to much more complicated models for the time to kill a target [namely, BONDER's result (5.4.1) for MARKOV-dependent fire and our more general ones (5.8.1) and (5.8.2)]. Thus, we started by presenting without justification results for a couple of very simple analytical submodels for a LANCHESTER attrition-rate coefficient under conditions of "aimed" fire, and we subsequently developed a fairly general model for the expected time to kill a target and obtained a general result for this model. At this juncture it now seems appropriate for us to show how the earlier-obtained simple results may be viewed as special cases of these later-obtained, more general results. In particular, we will show how BONDER's result for the expected time to kill a target with MARKOV-dependent fire (5.4.1) simplifies and yields (under the appropriate circumstances) a simple result like (5.2.4) for the LANCHESTER attrition-rate coefficient. We will also examine an analogous simplification that yields that "aimed" fire can lead to an FT target-type-attrition process[24] when a model proposed for target-acquisition times by H. BRACKNEY [20] is considered. In preparation for developing these results, though, let us briefly review how the different results that we have developed for varying degrees of generality are related to one another.

The <u>most general result</u> that we have developed to the expected time for an individual firer to kill a single enemy passive target <u>is</u>

given by (5.8.1), which holds for assumptions ($\tilde{A}1$) through ($\tilde{A}6$) of Section 5.8. The operational conditions corresponding to these assumptions are more general than MARKOV-dependent fire and a geometric distribution of the number of hits required for a kill with random event times. When we do assume MARKOV-dependent fire and a geometric distribution for the number of hits to kill, however, our most general result (5.8.1) simplifies and we obtain (5.8.2), which still contains random event times. BONDER's result (5.4.1) is a special case of (5.8.2), i.e. it is the special case in which all event times are deterministic. In turn, (5.2.8) is a special case of BONDER's result (5.4.1), and (5.2.4) corresponds to a special case of (5.2.8), i.e. the special case in which the time to acquire a target is negligible with respect to the time required to destroy an acquired target and is taken to be equal to zero.

Let us now consider more systematically the simplification of BONDER's general result (5.4.1) in some special cases of tactical interest. Other such special cases (and ones that we will not examine here) are to be found in BONDER and FARRELL [17, pp. 106-107] and also [88, p. 28]. We begin by listing assumptions that are more restrictive than those used to develop (5.4.1) but are yet of tactical interest (see [88, p. 28] for a further discussion):

(A1) statistical independence among firing outcomes, i.e.

$$p_1 = P(h|h) = P(h|m) = p_{SSH};$$

(A2) "uniform" rate of fire, i.e. $t_1 = t_h = t_m = t_\nu = 1/\nu$;

(A3) negligible time of flight for projectile, i.e. assume that $t_f = 0$;

and (A4) target-acquisition time negligible, i.e. assume that $t_a = 0$.

If we take assumption (A1) to hold, i.e. independent fire instead of MARKOV-dependent fire, then BONDER's general expression reduces to

$$E[T] = t_a + t_1 - t_h + \frac{(t_m + t_f)}{P_{SSK}} + \frac{(t_h - t_m)}{P(H|K)} , \quad (5.10.1)$$

where $P_{SSK} = P_{SSH} P(K|H)$ denotes the single-shot kill probability. If we additionally take assumption (A2) to hold, i.e. uniform firing rate, then this last result further reduces to

$$E[T] = t_a + \frac{(t_\nu + t_f)}{P_{SSK}} , \quad (5.10.2)$$

which may also be written as

$$E[T] = t_a + \frac{(1 + \nu t_f)}{\nu P_{SSK}} , \quad (5.10.3)$$

where $\nu$ denotes the firing rate (assumed uniform). If we additionally take assumption (A3) to hold, i.e. negligible projectile flight time, then this last result further reduces to

$$E[T] = t_a + \frac{1}{\nu P_{SSK}} , \quad (5.10.4)$$

which is the same as (5.2.8) above. If we additionally take assumption (A4) to hold, i.e. negligible target-acquisition time, then we finally obtain

$$E[T] = \frac{1}{\nu P_{SSK}} , \tag{5.10.5}$$

which is equvalent to the LANCHESTER attrition-rate coefficient being given by, for example, (5.2.4), i.e. the kill rate of a single weapon system is equal to the product of its firing rate times the (single-shot) kill probability of each round. We summarize the above results with the following lemma.

LEMMA 5.10.1: Assume that assumptions (A1) through (A3) above hold. BONDER's general expression for the expected time to kill a target (5.4.1) then reduces to (5.10.4), with the LANCHESTER attrition-rate coefficient being given by, for example, (5.3.1) [i.e. $a = 1/\{t_{a_{XY}} + 1/(\nu_Y P_{SSK_{XY}})\}$]. If we additionally take assumption (A4) to hold, i.e. $t_a = 0$, then (5.10.4) reduces to (5.10.5) and the LANCHESTER attrition-rate coefficient is given, for example, by (5.2.4).

Thus, we have shown that the simple models that we initially considered may be viewed as special cases of much more general ones.

Along the same lines, let us now consider a target-acquisition-time model proposed by H. BRACKNEY [20] and see how "aimed" fire can lead to an FT target-type-attrition process when target-acquisition times are

target-type-force-level dependent and are the constraining factor in the attrition process. Following BRACKNEY [20, p. 32], let us accordingly replace assumption (A4) above by ($\bar{A}4$).

($\bar{A}4$) the mean time to acquire a target is inversely proportional (let $k$ denote the constant of proportionality) to the target density, i.e. $t_a = k/\rho$ where $\rho$ denotes the density of targets in the target area $A$ that is searched.

In analytical terms, assumption ($\bar{A}4$) yields that, for example,

$$E[T_{a_{XY}}] = \frac{k_Y A_X}{x}, \tag{5.10.6}$$

where $T_{a_{XY}}$ (a r.v.) denotes the time required for a $Y$ firer to acquire an $X$ target, $A_X$ denotes the area occupied by $X$ targets (and searched by $Y$ firers), $x$ denotes the $X$ force level within this region, and $k_Y$ denotes a constant of proportionality for this model of the time for a $Y$ firer to acquire an $X$ target. The above considerations lead to the following interesting result.

LEMMA 5.10.2: Assume that assumptions (A1) through (A3) and ($\bar{A}4$) hold. The expected time for a, for example, $Y$ firer to kill an $X$ target is then given by

$$E[T_{XY}] = \frac{k_Y A_X}{x} + \frac{1}{\nu_Y P_{SSK_{XY}}} \tag{5.10.7}$$

This last lemma has the following important consequence: if the time to acquire targets is the constraining factor in the target-attrition process, then one has approximately, for example,

$$E[T_{XY}] \approx \frac{k_Y A_X}{x} \gg \frac{1}{\nu_Y P_{SSK_{XY}}} , \tag{5.10.8}$$

which yields that the LANCHESTER attrition-rate coefficient may be taken under such circumstances to be given by

$$a = \tilde{a}x , \tag{5.10.9}$$

where $\tilde{a} = 1/(k_Y A_X)$. Consequently, the rate of change of the $X$ force level under these circumstances would be given by

$$\frac{dx}{dt} = -\tilde{a}xy . \tag{5.10.10}$$

Thus, we have shown that when BRACKNEY's target-acquisition-time model is used and target acquisition is the constraining factor on the rate of attrition, "aimed" fire yields an FT target-type-attrition process. Thus, both "area" fire against a target type and also the above situation for "aimed" fire may be hypothesized to yield the same target-type-attrition-rate equation, and this situation was the reason why we introduced in Section 2.12 our classification scheme for homogeneous-force LANCHESTER-type attrition processes (and which we have adapted just above to a single target type's attrition).

One can use BRACKNEY's above target-acquisition-time model (5.10.6) with the general expression for the expected time to kill a target (5.8.1) and its various derivatives which we have discussed above to develop some interesting consequences. In particular, the assault of an X force against a Y force's defensive position may be hypothesized to yield F|FT LANCHESTER-type attrition equations. A convenient place to begin this development is to observe that the conditions of Lemma 5.10.2 [i.e. assumptions (A1) through (A3) and ($\bar{A}4$) being satisfied] yield the following LANCHESTER-type equations

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{y}{\{k_Y A_X/x + 1/(\nu_Y P_{SSK_{XY}})\}} & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -\dfrac{x}{\{k_Y A_Y/y + 1/(\nu_X P_{SSK_{YX}})\}} & \text{with } y(0) = y_0 . \end{cases} \tag{5.10.11}$$

Limiting cases of these equations provide some important insights into the dynamics of combat. Such limiting cases may be generated by considering the relative size of the time to acquire a target in relation to the time required to kill an acquired target. BRACKNEY [20, pp. 32-33] considered the two limiting cases of (I) when the time to acquire is negligible, and (II) when it is the dominating term. He further reasoned that a combatant's search time (i.e. the time to acquire an enemy target) is negligible when the enemy rushes through an open area and assaults his position. Furthermore, he postulated that a combatant's search time is the dominating term in the expression for the time to kill an enemy target when the enemy remains under cover in their defensive positions.

Consequently, BRACKNEY [20, p. 33] argued that force-on-force attrition for the assault of an $X$ force against a $Y$ force's defensive position could be modelled by

$$
\begin{cases}
\dfrac{dx}{dt} = -\nu_Y P_{SSK_{XY}} y & \text{with } x(0) = x_0 , \\[2ex]
\dfrac{dy}{dt} = -\dfrac{xy}{k_X A_Y} & \text{with } y(0) = y_0 ,
\end{cases}
\tag{5.10.12}
$$

which are readily recognized by the reader as the equations for an F|FT LANCHESTER-type attrition process. This model (5.10.12) was proposed by BRACKENY [20, pp. 32-33] and used, for example, by SCHAFFER [65, p. 488] to study sieges in guerrilla-warfare operations (see Section 7.6 below). Furthermore, when both sides remain in their (covered) defensive positions (a situation that BRACKNEY [20, p. 36] termed a fire duel), BRACKNEY argued that force-on-force attrition could then be modelled by

$$
\begin{cases}
\dfrac{dx}{dt} = -\dfrac{xy}{k_Y A_X} & \text{with } x(0) = x_0 , \\[2ex]
\dfrac{dy}{dt} = -\dfrac{xy}{k_X A_Y} & \text{with } y(0) = y_0 .
\end{cases}
\tag{5.10.13}
$$

5.11. Variables Upon Which Attrition-Rate Coefficients Depend.

It is intuitively obvious (and born out by empirical evidence) that, in general terms, the fire effectiveness of a weapon system depends on the target type engaged and the environmental circumstances of the engagement[25]. Thus, a numerical value for a LANCHESTER attrition-rate coefficient depends on both the characteristics of the firer's weapon system and also those of the target. However, this dependence of a LANCHESTER attrition-rate coefficient on firer-weapon-system-type and target characteristics is not direct but indirect through the operational variables (e.g. time to acquire a target, hit probabilities, etc.) upon which such an attrition-rate coefficient directly depends. Consequently, it seems appropriate for us to consider that an attrition-rate coefficient depends on two types of factors:

(T1) direct factors,

and (T2) indirect factors.

Let us now examine more closely this distinction between direct and indirect factors by considering the special case of the LANCHESTER attrition-rate coefficient for an impact-to-kill system under conditions of MARKOV-dependent fire and a geometric distribution for the number of hits required for a kill. Similar remarks will, of course, apply to a LANCHESTER attrition-rate coefficient corresponding to other circumstances. To return to the case at hand, we again focus on an impact-to-kill system with MARKOV-dependent fire and a geometric distribution for the number of hits to kill.

As we have seen above in Section 5.4, the direct factors upon which the LANCHESTER attrition-rate coefficient depends correspond to the variables appearing in (5.4.1) (<u>see</u> also Table 5.II). However, each of these variables, e.g. $p$ or $P(h|h)$, themselves in turn depend on other operational factors in the tactical environment. For example, the hit probabilities depend on such variables as range (i.e. distance) between target and firer, tactical posture of the target and/or firer, etc. We will refer to such variables as the <u>indirect factors</u> upon which a LANCHESTER attrition-rate coefficient depends. Table 5.III lists some indirect factors upon which the LANCHESTER attrition-rate coefficient may depend. This list is not meant to be exhaustive, but it should be considered to be suggestive of functional dependencies that should be considered in modelling force-on-force combat interactions.

For many weapon systems, the range (i.e. distance) between firer and target has a very significant effect on weapon-system fire effectiveness. In such cases (as stressed by BONDER [9-11; 13]), if the range between firers and targets changes appreciably during the course of an engagement, then use of constant attrition-rate coefficients in a LANCHESTER-type model can yield quite misleading results (<u>see</u> Section 6.2 for further details). BONDER has consequently emphasized the importance of explicitly considering in LANCHESTER-type combat analyses such range dependence of weapon-system fire effectiveness, especially for mobile weapon-system types. Thus, in many tactical situations of interest we should consider, for example, that for the model (5.2.1) the LANCHESTER attrtion-rate coefficients $a$ and $b$ explicitly depend on range[26], i.e

TABLE 5.III. Indirect Factors Upon Which LANCHESTER Attrition-Rate Coefficients Depend.

1. Range Between Firer and Target

2. Effects of the Battlefield Environment (e.g. Visibility Conditions, Target-Background Contrast, etc.)

3. Target Posture

4. Firer Posture

5. Terrain

6. Target Movement

7. Firer Movement

$$a = \alpha(r) \qquad \text{and} \qquad b = \beta(r) \ , \tag{5.11.1}$$

where $r$ denotes the range (i.e. distance) between firers and targets. Thus, we should consider LANCHESTER attrition-rate coefficients to be at least (and probably primarily) dependent on the range between firers and targets.

## 5.12. Some Typical Range Dependencies for the LANCHESTER Attrition-Rate Coefficient.

As we have just discussed above, the range (i.e. distance) between firers and targets is one of the principal indirect factors upon which a LANCHESTER attrition-rate coefficient depends. It is intuitively obvious (and born out by empirical evidence) that the fire effectiveness of a weapon system is strongly dependent on the range between firer and target. Based on their examining predicted numerical values of the LANCHESTER attrition-rate coefficient for specific weapon systems with widely differing characteristics and how these values varied with range, BONDER and FARRELL [17, pp. 196-200] have considered a number of functional forms for range-dependent attrition-rate coefficients in "aimed-fire" combat, e.g. for combat modelled by (5.2.1). The functional forms considered by BONDER and FARRELL may be classified as:

(F1) power dependence

(F2) exponential dependence upon range,

(F3) cosine dependence upon range,

(F4) piecewise-constant dependence upon range.

We will accordingly call such attrition-rate coefficients as follows:

(C1) power attrition-rate coefficient

$$\alpha_P(r) = \begin{cases} \alpha_0\left(1 - \frac{r}{r_e}\right)^{\mu} & \text{for } 0 \le r \le r_e , \\ 0 & \text{for } r_e \le r , \end{cases} \tag{5.12.1}$$

(C2) exponential attrition-rate coefficient

$$\alpha_E(r) = \begin{cases} \alpha_0\left[1 - e^{-\alpha_1(r_e - r)}\right] & \text{for } 0 \le r \le r_e , \\ 0 & \text{for } r_e \le r , \end{cases} \tag{5.12.2}$$

(C3) cosine attrition-rate coefficient

$$\alpha_C(r) = \begin{cases} \frac{\alpha_0}{2}\left[1 + \cos\left(\frac{\pi r}{r_e}\right)\right] & \text{for } 0 \le r \le r_e , \\ 0 & \text{for } r_e \le r, \end{cases} \tag{5.12.3}$$

(C4) piecewise-constant attrition-rate coefficient

$$\alpha_{PC}(r) = \begin{cases} \alpha_0 & \text{for } 0 \le r \le r_e , \\ 0 & \text{for } r_e \le r \end{cases} \tag{5.12.4}$$

Here $r_e$ denotes the maximum effective range of the firer's weapon system, $\alpha_0$ and $\alpha_1$ are positive constants, and $\mu$ is a nonnegative constant.

The first two above functional forms for range-dependent attrition-rate coefficients are shown in Figures 5.5 and 5.6. In Figure 5.5 we have plotted the value of the power attrition-rate coefficient $\alpha_P(r)$ given by (5.12.1) versus the range between firers and targets. As we can see from Figure 5.5, the constant $\mu$ is used to model the range dependence of the attrition-rate coefficient $\alpha_P(r)$. For values of $\mu > 1$, the attrition-rate coefficient $\alpha_P(r)$ is a convex function of $r$ on $[0,r_e]$, i.e. the plot of $\alpha_P(r)$ versus $r$ "flexes downward." We have accordingly chosen to call $\mu$ the "shape" parameter, since it controls the shape of the plot of $\alpha_P(r)$. In Figure 5.6 we have similarly plotted the exponential attrition-rate coefficient $\alpha_E(r)$ given by (5.12.2) versus range. In this case, the constant $\alpha_1$ is used to model the range dependence of $\alpha_E(r)$. However, this attrition-rate coefficient $\alpha_E(r)$ is a concave function of $r$ on $[0,r_e]$, i.e., the plot of $\alpha_E(r)$ versus $r$ "flexes upward." Also, we observe that $\alpha_E(r) \rightarrow$ linear dependence on $r$ as $\alpha_1 \rightarrow 0$, and we have similarly chosen to call $\alpha_1$ the "shape" parameter.

Still another model for range dependence of such an attrition-rate coefficient is an exponential fall off in fire effectiveness of the form

$$\alpha_{ED}(r) = \alpha_0 e^{-\alpha_1 r} \qquad (5.12.5)$$

where $\alpha_1 > 0$. We call call the attrition-rate coefficient $\alpha_{ED}(r)$ given by (5.12.5) the <u>exponentially-decaying attrition-rate coefficient</u>. It is plotted versus range in Figure 5.7. As Figure 5.7 shows, it has a range dependence somewhat similar to the attrition-rate coefficient $\alpha_p(r)$. In other words, $\alpha_{ED}(r)$ is a convex function on $[0,r_e]$ as $\alpha_p(r)$ is for

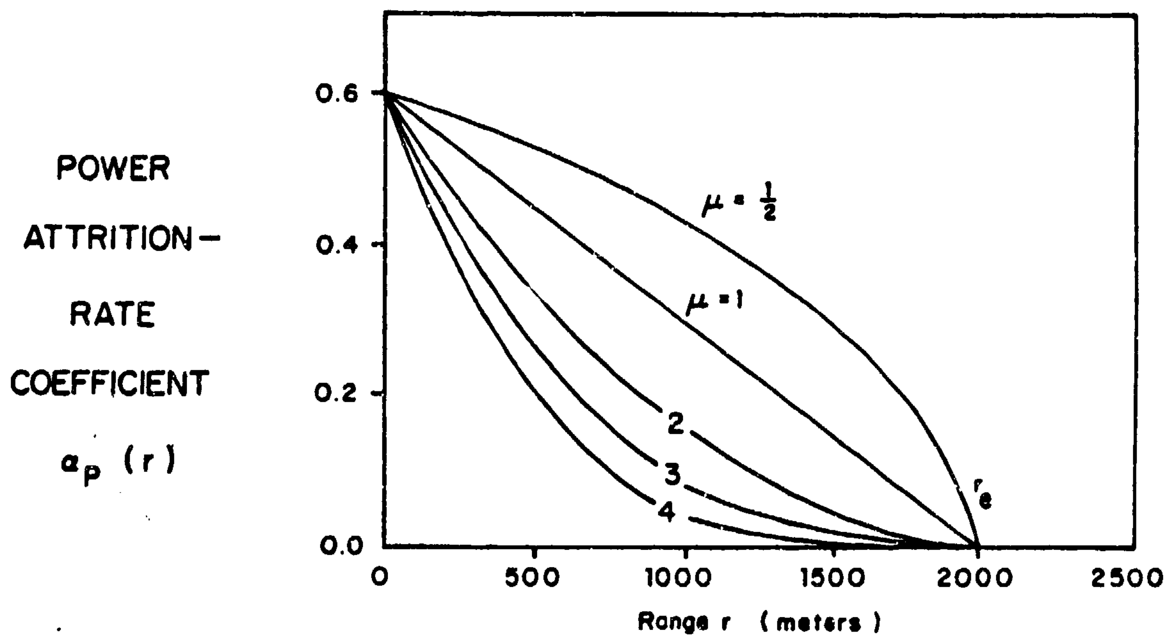


Figure 5.5. Variation in fire effectiveness (measured in kills/minute per firer) with range for the power attrition-rate coefficient $\alpha_p(r)$, which is analytically given by (5.12.1), for several different values of the "shape" parameter $\mu$. The maximum effective range of the weapon-system type is denoted as $r_e$ and for this example $r_e = 2000$ meters. Also, in this example the weapon-system kill rate at zero force separation (range) $\alpha_p(0) = \alpha_0 = 0.6$ X casualties/(unit time × number of Y firers) has been held constant, and the "shape" parameter $\mu$ has been varied (i.e. curves plotted for $\mu = 1/2$, 1, 2, 3, and 4).

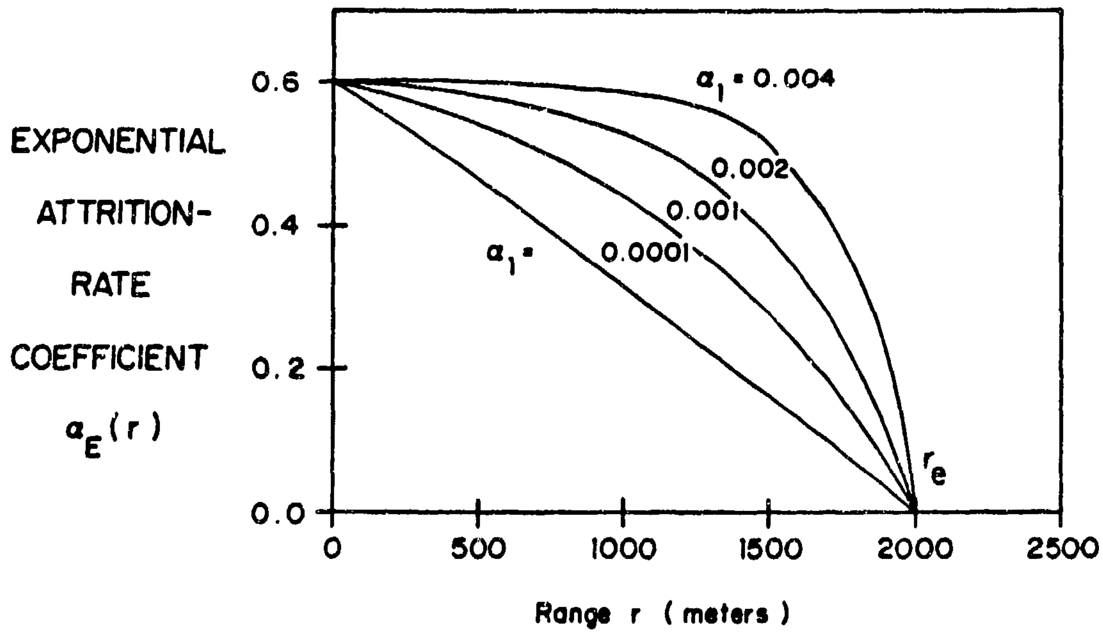


Figure 5.6. Similar to Figure 5.5, variation in fire effectivenss with range for the <u>exponential attrition-rate coefficient</u> $\alpha_E(r)$, which is analytically given by (5.12.2), for several different values of the "shape" parameter $\alpha_1$. Again, the maximum effective range of the weapon system is given by $r_e = 2000$ meters. Also, the weapon-system kill rate at zero force separation (range) $\alpha_E(0) = \alpha_0$ has again been held constant, and the "shape" parameter $\alpha_1$ has been varied.

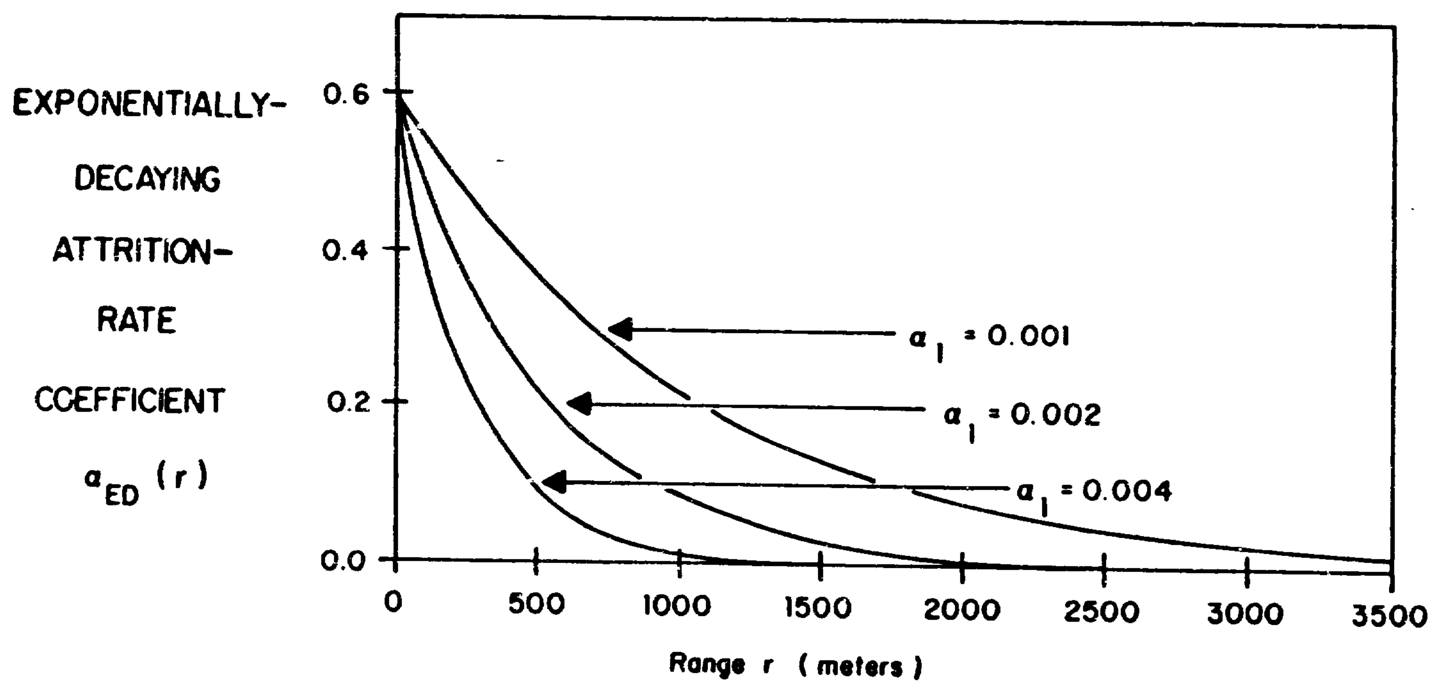


Figure 5.7. Similar to Figures 5.5 and 5.6, variation in fire effectiveness with range for the <u>exponentially-decaying attrition-rate coefficient</u> $\alpha_{ED}(r)$, which is analytically given by (5.12.5), for several different values of the "shape" parameter $\alpha_1$. The weapon-system type theoretically has an infinite maximum effective range, but for all practical purposes the weapon-system type is ineffective when $\alpha_1 r \geq 12$. As in the previous examples, the weapon-system kill rate at zero force separation (range) $\alpha_{ED}(0) = \alpha_0$ has been held constant, and the "shape" parameter $\alpha_1$ has been varied.

$\mu > 1$. Although (5.12.5) implies that the wapon system theoretically has an infinite maximum effective range, for all practical purposes the weapon system becomes "ineffective" (i.e. it ceases to kill) when $\alpha_1 r \geq 12$, since then $\alpha_{ED}(r)$ is less than $10^{-5}$ times its value at $r = 0$ (cf. the curve labeled $\alpha_1 = 0.004$ in Figure 4.7 for ranges greater than 1500 meters).

## 5.13. Attrition-Rate Coefficients for Area-Fire Weapons.

The above attrition-rate-coefficient results [in particular, (5.4.1) and its generalizations (5.8.1) and (5.8.2)] apply to weapon-system types that direct their fire at individual targets that are vulnerable to only the impact of a projectile fired by the weapon system[27]. Let us refer to this situation as "aimed" fire against an impact-sensitive target. Many times, however, a weapon system will engage a target or complex of targets not by aiming its fire at an individual target but by directing its fire into only the general area thought to be occupied by the target or targets. Let us refer to this latter situation as "area" fire (cf. Section 2.11 above). It is for this type of firing mode that we will now consider the determination of LANCHESTER attrition-rate coefficients. Furthermore, such "area" fire may be directed at both fragment-sensitive and also impact-sensitive targets[28]. As far as combat modelling is concerned, the former is far more the important case, since it may be considered to conceptually model artillery engaging enemy dismounted-infantry troops (i.e. those not in protective vehicles) dispersed in tactical formations. An example of the second case (i.e. "area" fire against impact-sensitive targets) would be small-arms fire against poorly located enemy dismounted-infantry troops. This latter tactical situation has been considered in guerrilla-warfare settings by DEITCHMAN [31] and SCHAFFER [65] (see Chapter 7 for further details). Thus, a number of important tactical situations may be modelled by area fire.

Let us accordingly consider combat between two homogeneous forces (denoted as $X$ and $Y$) in which force-on-force attrition occurs at a rate proportional to the number of enemy firers (at least on the surface

it appears to do so) but in which each side uses "area" fire. For the sake of placing something concrete before the eyes of the reader, we will focus on the attrition of the $X$ force caused by the $Y$ firers. According to the assumptions just made, we may write

$$\frac{dx}{dt} = -ay , \tag{5.13.1}$$

with

$$a = \frac{1}{E[T_{XY}]} , \tag{5.13.2}$$

where $T_{XY}$ (a r.v.) denotes the time required for a $Y$ firer to kill an $X$ target. For "area" fire, however, the expression for the expected time to kill a target takes a different form than that for "aimed" fire, i.e. $E[T]$ is no longer given by (5.4.1).

The simplest model for $E[T]$ in the case of "area" fire involves adapting (5.4.1) to this case[29]. This adaptation may be accomplished by conceptualizing the target-destruction process in the following manner: an "area" target is acquired, and "area" fire is directed at it; if a round lands in the target area, the target may be killed; otherwise it is not damaged. Thus, $t_a$ would represent the time to acquire the "area" target, and other quantities in (5.4.1) would be analogously redefined. However, since an area target is usually not reacquired after every kill of one of its elements, we should replace $t_a$ by $t_a/n_K$, where $n_K$ denotes the number of elements killed per acquisition of such an area target. Thus, we would have

$$E[T] = \frac{t_a}{n_K} + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H_{area})}$$

$$+ \frac{(t_m + t_f)}{P_{area}(h|m)} \left\{ \frac{[1 - P_{area}(h|h)]}{P(K|H_{area})} + P_{area}(h|h) - p^1_{area} \right\} , \qquad (5.13.3)$$

where

$t_a$ denotes the time to acquire an area target,

$n_K$ denotes the number of kills per acquisition,

$t_1$, $t_f$, $t_h$, and $t_m$ are defined similarly as for "aimed" fire in Section 5.4,

$p^1_{area}$, $P_{area}(h|h)$, and $P_{area}(h|m)$ denote MARKOV-dependent probabilities for hitting the area target,

and $P(K|H_{area})$ denotes the probability that we kill a target element given that we "hit" the area target.

Here, $P(K|H_{area})$ depends on the lethal area (see [84, Chapter 15]) of the weapon system's projectile[30].

Moreover, there is a special case of the model discussed in the previous paragraph that merits further examination and discussion. To this end, let us make the following assumptions (cf. those made in Section 5.10) concerning the above adaptation of (5.4.1), namely (5.13.3):

(A1) statistical independence among firing outcomes, i.e. $p^1_{area} = P_{area}(h|h) = P_{area}(h|m) = P^{area}_{SSH}$;

(A2) "uniform" rate of fire, i.e. $t_l = t_h = t_m$ and we will denote this common value as $t_\nu = 1/\nu$;

and (A3) negligible time of flight for projectile, i.e. assume that $t_f = 0$.

In this case, (5.13.3) reduces to

$$E[T] = \frac{t_a}{n_K} + \frac{1}{\nu P_{SSK}^{area}} , \tag{5.13.4}$$

where $\nu = 1/t_\nu$ denotes the operational firing rate of the weapon system and $P_{SSK}^{area} = P_{SSH}^{area} P(K|H_{area})$ denotes the single-shot-kill probability for destroying a target element with one round. It is implicitly assumed here that multiple kills are impossible (i.e. at most only one target element can be killed with any one round). Furthermore, when $t_a/n_K$ is negligible compared to $1/(\nu P_{SSK}^{area})$, then Y's attrition-rate coefficient in (5.13.1) may be approximated by [cf. (5.2.4) above]

$$a = \nu_Y P_{SSK_{XY}}^{area} , \tag{5.13.5}$$

where $P_{SSK_{XY}}^{area}$ denotes the single-shot-kill probability for a Y firer engaging an X area target.

Moreover, there are a couple of special cases for the LANCHESTER attrition-rate coefficient (5.13.5) that we should consider. When a weapon system employs "area" fire and enemy targets defend a constant area (see

Table 2.XIX for a more precise list of the associated assumptions), the expression for the LANCHESTER attrition-rate coefficient may be given in an even more explicit form (i.e. one depending on more basic measurable operational quantities) and depends (among other things) on the vulnerable area of the target (denoted as $a_V$) and the lethal area of the projectile fired by the firer's weapon system (denoted as $a_L$). In general, a rather complicated expression is obtained for such an attrition-rate coefficient (e.g. <u>see</u> BONDER and FARRELL [17, pp. 141-162]), but this expression may be stated in a particularly simple form in special cases under the appropriate simplifying assumptions, e.g. for "small-arms fire" when $a_V >> a_L$ and for "fire from a weapon of great lethality" when $a_L >> a_V$. Thus, two cases in which a simple expression is obtained for an attrition-rate coefficient <u>for "area" fire and a constant-area defense are as follows</u>:

(C1) small-arms fire (i.e. $a_V >> a_L$),

and (C2) fire from weapons of large lethality (i.e. $a_L >> a_V$).

A more precise description of the operational conditions that we have in mind is given in the first five assumptions listed in Table 2.XIX. Assuming that $t_a/n_K$ is negligible, we may take, for example, the attrition-rate coefficient $a$ to be given by (5.13.5) if we assume that the attrition-rate of the $X$ force is given by (5.13.1).

For <u>small-arms fire</u> (i.e. $a_V >> a_L$), we may calculate $P^{area}_{SSK_{XY}}$ for use in (5.13.5) by considering a "lethal dot" being randomly placed

into a large region (of area $A_X$) that contains $x$ "vulnerable circles" (each of area $a_{V_X}$). Under these circumstances and the assumptions[31] that a $Y$ firer directs his fire into the region actually occupied by the $X$ targets and that his fire is uniformly distributed over the region into which it is directed, the probability that a target is hit, denoted as $P_{SSH}^{area}$, is given by the ratio of the total vulnerable area of all the targets divided by the area of the region into which fire is directed (see Figure 5.8), i.e.

$$P_{SSH}^{area} = \frac{x a_{V_X}}{A_X} . \tag{5.13.6}$$

It follows that

$$P_{SSK}^{area} = \frac{x a_{V_X} \; P(K|H)_{XY}}{A_X} , \tag{5.13.7}$$

where $P(K|H)_{XY}$ denotes the probability that an $X$ target is killed by a $Y$ projectile when it is hit. Thus, when $P(K|H)_{XY} = 1.0$, the attrition rate of the $X$ force is given by

$$\frac{dx}{dt} = - \frac{a_{V_X}}{A_X} \; \nu_Y \; xy, \tag{5.13.8}$$

which is the result [with $P(K|H)_{XY}$ included] given in Table 2.XIX.

For <u>fire from weapons of large lethality</u> (i.e. $a_L >> a_V$), a slightly different analysis is required. In this case, we may calculate $P_{SSK_{XY}}^{area}$ by considering a "lethal circle" being randomly placed into a region that contains

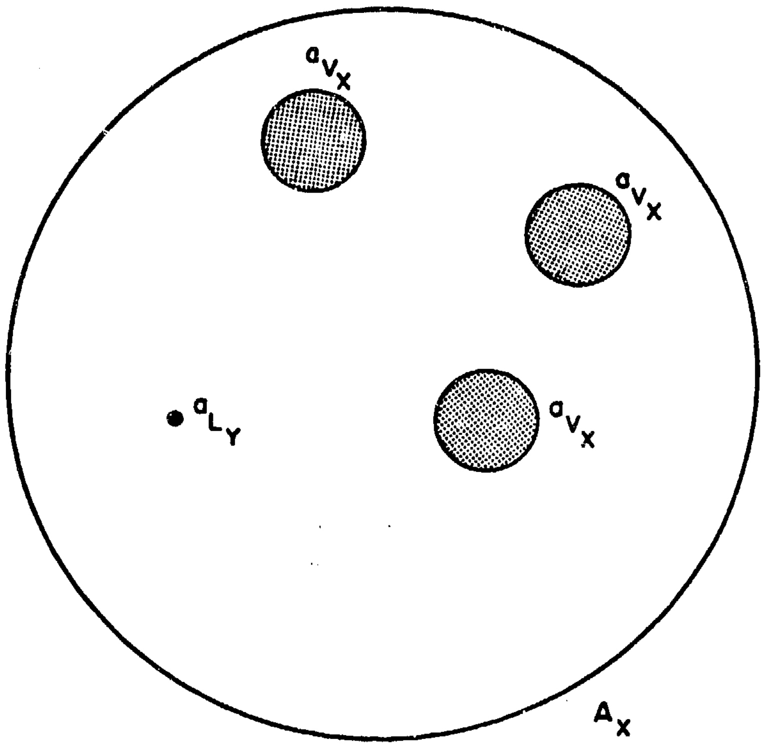


Figure 5.8. Conceptualization of target-destruction process for "area fire" by small arms. In this case $a_V >> a_L$, i.e. the vulnerable area of a target is much larger than the lethal area of a round. The above diagram considers X to be the target and Y the firer.

$x$ randomly placed "vulnerable dots." We assume that these dots are so placed that the "lethal circle" covers at most one of them per throw. Furthermore, the probability of covering one of these $x$ "vulnerable dots" in the region of area $A_X$ is the same as the probability of covering one such dot randomly placed in a region of area $A_X/x$. This latter probability is simply given by the ratio of the total lethal area to the total area of this equivalent region (see Figure 5.9), and hence

$$P_{SSK}^{area} = \frac{x a_{L_Y}}{A_X} . \tag{5.13.9}$$

In the above formula, it is assumed that a "hit" on a target will kill the target. The formula is easily modified to model the case in which each such "hit" (i.e. the covering of a "vulnerable dot" by a "lethal circle") has a probability less than one of killing such a target. Finally, for the above case of fire from weapons of large lethality, the attrition rate of the $X$ force is given by

$$\frac{dx}{dt} = -\frac{a_{L_Y}}{A_X} \nu_Y xy , \tag{5.13.10}$$

which is a result first apparently given by WEISS [91, p. 83] and later used by both DEITCHMAN [31, pp. 821-822] and SCHAFFER [65, p. 470] in the modelling of guerrilla warfare (see Chapter 7). The small-arms-fire result (5.13.8) may be considered to be a particularization of (5.13.10) in which the lethal area of a $Y$ round is taken to be the vulnerable area of an $X$ target (see DEITCHMAN [31, p. 822]).

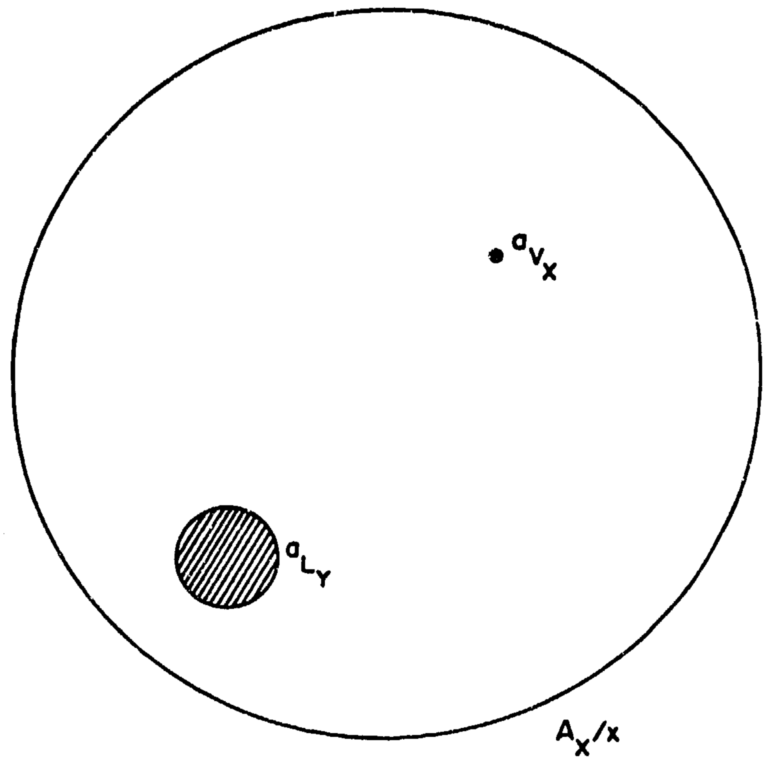


Figure 5.9. Conceptualization of target-destruction process for "area fire" by weapons of large lethality. In this case $a_L \gg a_V$, i.e. the lethal area of a round is much larger than the vulnerable area of a target, and the target density is reflected by considering an equivalent process taking place in a region of area $A_X/x$. The above diagram considers X to be the target and Y the firer.

There is, however, another (more general) approach for developing the above kill-rate result for "area" fire (5.13.10). This other approach is based on the equivalence of expected target coverage to kill probability, and it considers the expected number of survivors by conceptually replacing all the targets by a single equivalent target and computing the probability of destroying this equivalent target [i.e. see (5.13.14) below]. This approach is particularly significant, since it is essentially the one used by BONDER and FARRELL [17, pp. 141-162] to develop attrition rates for multiple-tube-firing cases (for both volley and salvo fire). We will now present this important alternate development of attrition rates for area-fire weapon systems.

A fundamental precept upon which target-coverage analysis (i.e. the theoretical analysis of damage to targets by indirect-fire weapons {e.g. see HESS [43]}) is based on the equivalence of expected target coverage to kill probability[32]. It is simply stated as follows.

> FUNDAMENTAL PRECEPT OF TARGET COVERAGE: The probability of killing a randomly located point target is equal to the expected coverage of a population of objects when the population density is distributed in the same manner as the point target.

If we let $\bar{F}$ denote the average fraction of targets killed and $P_K$ denote the probability of killing the point target, then the fundamental precept of target coverage may be stated in analytical terms as

$$\bar{F} = P_K . \tag{5.13.11}$$

This result may be considered to be equivalent to thinking of the status of the point target as a BERNOULLI random variable and scaling up the expected-fraction-killed result for this single target to that for the entire target population. Implicit in this fundamental premise is the assumption that the exact locations of individual targets in the target area are not known. In this sense, we may take (5.13.11) to be a static mathematical statement of "area" fire which we will now convert into the dynamic result (5.13.10) by a series of logical arguments.

We begin by considering a homogeneous $X$ force receiving area fire from a homogeneous $Y$ force and computing the expected number of survivors. By the fundamental precept of target coverage, this number is given by

$$x(t) = \{1 - P_K^{XY}(t)\}x_0 , \tag{5.13.12}$$

where $P_K^{XY}(t)$ denotes the cumulative kill probability of the entire $Y$ force engaging a single randomly placed $X$ target for a period of time $t$. Taking the logarithmic derivative of (5.13.12), we find that

$$\frac{dx}{dt} = x \frac{d}{dt} \ln\{1 - P_K^{XY}(t)\}. \tag{5.13.13}$$

Assuming independence between the outcomes of any two rounds [<u>recall</u> Assumption (A3) of Table 2.XIX], we also have that

$$P_K^{XY}(t) = 1 - (1 - P_{SSK}^{XY})^{\nu_Y yt} , \tag{5.13.14}$$

where $\nu_Y$ denotes the firing rate of a single $Y$ firer and $P_{SSK}^{XY}$ denotes the single-shot kill probability for a single $Y$ firer engaging a single $X$ point target. From (5.13.14), we readily deduce that

$$\frac{d}{dt} \ell n\{1 - P_K^{XY}(t)\} = \nu_Y y \, \ell n(1 - P_{SSK}^{XY}) , \tag{5.13.15}$$

whence follows

$$\frac{dx}{dt} = \nu_Y\{\ell n(1 - P_{SSK}^{XY})\}xy , \tag{5.13.16}$$

by substitution of (5.13.15) into (5.13.13). The reader should regard (5.13.16) as the <u>fundamental attrition-rate equation for area fire</u>. Comparison of (5.13.16) with, for example, (5.13.1) reveals that we may consider the LANCHESTER attrition-rate coefficient for such area-fire weapons to be given by

$$a = -\nu_Y\{\ell n(1 - P_{SSK}^{XY})\}x , \tag{5.13.16}$$

which should be compared with BONDER and FARRELL's [17, pp. 150-154] result for area-fire weapons (<u>see</u> also [54, p. 170] or [28, p. 176]). Furthermore, $-P_{SSK}^{XY}$ is a good approximation[33] to $\ell n(1 - P_{SSK}^{XY})$ when $P_{SSK}^{XY} \in [0, 0.2]$, and in this case we approximately have

$$a = \nu_Y \, P_{SSK}^{XY} \, x. \tag{5.13.17}$$

Returning to our original problem of modelling the force-on-force attrition of a homogeneous X force receiving "area" fire from an opposing homogeneous Y force, we observe that the probability that a single Y firer kills a single randomly-placed target is equal to the probability that a "lethal circle" of area $a_{L_Y}$ covers a "vulnerable dot" randomly placed within the region of area $A_X$ (under, of course, the assumption that $a_{L_Y} >> a_{V_X}$). Hence

$$P_{SSK}^{XY} = \frac{a_{L_Y}}{A_X}, \tag{5.13.18}$$

and (5.13.10) follows from (5.13.16) when $P_{SSK}^{XY} \leq 0.2$.

BONDER and FARRELL [17, pp. 141-162] have used the basic idea of the above approach[34] based on the fundamental precept of target coverage to develop an expression for the attrition-rate coefficient corresponding to firer by indirect, area-fire weapons. Their expression includes all the factors shown in Table 5.IV. It holds under the following set of assumptions[55].

(A1) no delivery bias exists--no aiming error, target-location error, or intentional offset,

(A2) centers of impact $(p,q)$ of the damage patterns are distributed about a mean center of impact $(\bar{p},\bar{q})$ according to a circular-normal distribution; for convenience, let $(\bar{p},\bar{q}) = (0,0)$ and the standard deviation be normalized to unity; the probability density function for the delivery error is then

$$b(p,q) = \frac{1}{2\pi} \exp\{-(p^2 + q^2)/2\},$$

TABLE 5.IV. Factors Considered in Attrition-Rate Coefficients for Indirect, Area-Fire Weapons by BONDER and FARRELL [17].

| |
|---|
| Weapon aiming and ballistic errors |
| Target location errors |
| Weapon firing rate |
| Volley damage-pattern radius |
| Target distribution |
| Target radius |
| Target posture |
| Probability that the target is destroyed given it is covered by damage pattern |

(A3) the target is a circle of radius $R_t$ centered at the origin; two mathematically equivalent types of targets are considered:

(T1) a circular, homogeneous, area target centered at $(0,0)$ with radius $R_t$,

(T2) a point target $(\xi,\eta)$ of uniformly uncertain location in the area of radius $R_t$; the target density function $W(\xi,\eta)$ is then $1/(\pi R_t^2)$ over the target area and zero elsewhere,

(A4) the damage pattern is a circular cookie-cutter of radius $R_p$; let $d(\xi,\eta;p,q)$ denote the damage function, which is then given by

$$d(\xi,\eta;p,q) = \begin{cases} \lambda & \text{for } (p-\xi)^2 + (q-\eta)^2 \leq R_p^2 , \\ 0 & \text{elsewhere,} \end{cases}$$

where $d(\xi,\eta;p,q)$ is the probability that a point target at $(\xi,\eta)$ will be killed by a damage pattern with center of impact at $(p,q)$; damage is either all or nothing (killed or not killed)--no cumulative damage is considered,

and (A5) the weapon system employs a constant firing rate $\nu$.

BONDER and FARRELL [17] (see also [54, p. 170] or [28, p. 176]) have stated that when the above assumptions hold, an approximation to the attrition-rate coefficient for $a$, for example, $Y$ firer engaging an opposing $X$ force with such an area-fire-weapon-system type is given by

$$a = \nu_Y \{\ell n(1 - \lambda S_1)\} x \ , \qquad (5.13.19)$$

where

$$S_1 = \frac{1}{R_t^2} \int_0^{R_t} P(R_p, r) r \, dr \ , \qquad (5.13.20)$$

$$P(R_p, r) = \iint_{(p-\xi)^2 + (q-\eta)^2 \leq R_p^2} \frac{1}{2\pi} \exp\left\{ - \left( \frac{p^2 + q^2}{2} \right) \right\} dp \, dq \ , \qquad (5.13.21)$$

and $r$ denotes the distance from the point target located at $(\xi, \eta)$ to the mean center of impact at $(0,0)$, i.e. $r^2 = \xi^2 + \eta^2$. The function $P(R_p, r)$ is called the <u>circular coverage function</u> and plays a prominent role in target-coverage analysis (e.g. see SNOW [70], HESS [43], ECKLER [33], or ECKLER and BURR [34]). It is well-known to be also given by

$$P(R_p, r) = e^{-r^2/2} \int_0^{R_p} x e^{-x^2/2} I_0(xr) dx \ , \qquad (5.13.22)$$

where $I_0(x)$ denotes the modified BESSEL function of the first kind of zero order (see HESS [43] or ECKLER and BURR [34] for further details).

BONDER and FARRELL (e.g. [28, p. 176]) have stated that in general the expression (5.13.18) is a good approximation to the attrition rate of a single weapon system "if $R_p >> R_t$, or when $R_t$ is less than the standard deviation of the center of impact of the damage pattern, or when the number of volleys is small." Further details are to be found in [28; 54].

## 5.14. Results for Other Related Weapon-System Types.

We have developed above expressions for the LANCHESTER attrition-rate coefficient under the following two different sets of circumstances:

(S1) MARKOV-dependent fire with an impact-lethality mechanism,

and (S2) an area-lethality mechanism.

In the first case we have developed our results under fairly general circumstances [*see* (5.8.1) and assumptions ($\tilde{A}1$) through ($\tilde{A}6$) in Section 5.8 above]. There are, however, a number of additional operational circumstances and weapon-system types for which it is convenient to have other LANCHESTER-attrition-rate-coefficient results available, especially for building and exercising a complex operational combat model in which a wide spectrum of weapon-system types is to be played. For example, three different types of weapon-system fire (*cf*. BONDER and FARRELL's taxonomy of weapon-system types reproduced here as Table 5.I) are permitted in VECTOR-2 [28, p. 170] (*see* also [86; 87])

(1) MARKOV-dependent fire at a specific target,

(2) repeated-burst fire at a specific target,

and (3) area fire (not directed at any specific target).

Consequently, we will present in this section LANCHESTER-attrition-rate-coefficient results for some other related weapon-system types of tactical interest. Complete derivations of these results will not be given, however, since results previously derived above may be invoked for their development.

Thus, we will give results for the following additional weapon-system types/operational circumstances of tactical interest:

(T1) MARKOV-dependent fire with chance of killing target on a miss,

(T2) burst fire-

(a) one long burst,

(b) mixed-mode firing doctrine [repeated-single-shot-MARKOV-dependent fire until first hit obtained after which there is an immediate switch to burst fire (one long burst)],

(c) repeated-burst fire [multiple (short) bursts independently fired].

In each of the above cases, we will give the appropriate expression for the expected time to kill a target, with the LANCHESTER attrition-rate coefficient (as usual) being obtained as the reciprocal of this quantity (recall Section 5.3 above). The first type of weapon-system fire (T1), i.e. MARKOV-dependent fire with kills on misses, applies to weapon-system types that fire rounds with fragmentation effects at targets with exposed personnel. In such cases it is quite possible to achieve a system kill when a projectile misses the target weapon system but detonates and kills the personnel by fragmentation effects. Thus, a miss may cause a kill, and the usual expression for MARKOV-dependent fire (5.8.2) (which only allows a target to be killed by being hit) must be modified to accommodate this fact. The second type of weapon-system fire (T2), i.e. burst fire, is characteristic of automatic weapons used by infantry and sometimes mounted on armored-personnel carriers

or other vehicles [e.g. the vehicle rapid-fire weapon system (VRFWS) or the secondary armament on a tank]. In particular, infantry doctrine calls for automatic weapons to be fired in repeated short bursts, and the LANCHESTER attrition-rate coefficient must again be modified for automatic weapons to accommodate this fact.

We will first consider the case of <u>MARKOV-dependent fire with chance of killing on a miss</u>, which is a further generalization of MARKOV-dependent fire considered above in Section 5.8. Let us assume that assumptions ($\tilde{A}1$) through ($\tilde{A}6$) of Section 5.8 hold, and we will additionally assume that there is a constant probability, denoted as $P(K|M)$, that a miss kills the target. Then the expected time to kill a target is given by[36]

$$E[T] = E[T_a] + E[T_{fr}] + E[T_f]$$

$$+ \frac{\{E[T_h] + E[T_f]\}\{1 - P(K|H)\}\{[1 - P(K|M)][P(h|m) - p_1] + p_1\}}{P(h|m)\ P(K|H)\{1-P(K|M)\} + P(K|M)\ \{1 - P(h|h)[1 - P(K|H)]\}}$$

$$+ \frac{\{E[T_m] + E[T_f]\}\{1 - P(K|M)\}\{1 - P(h|h) + [P(h|h) - p_1]\ P(K|H)\}}{P(h|m)\ P(K|H)\ \{1 - P(K|M)\} + P(K|M)\{1 - P(h|h)[1 - P(K|H)]\}}, \qquad (5.14.1)$$

which is a generalization of (5.8.2) given above and consequently is the most general result given in this monograph for MARKOV-dependent fire. The above expression (5.14.1) is readily developed by invoking Section 5.9's approach of considering the mean first-passage time for the killed state in a continuous-time semi-MARKOV process: one simply replaces the transition probabilities (5.9.4) by the following

$$p_{11} = p_1 P(K|H),$$

$$p_{12} = p_1\{1 - P(K|H)\},$$

$$p_{13} = (1-p_1)\ \{1 - P(K|M)\},$$

$$p_{21} = P(h|h)\ P(K|H)\ ,$$

$$p_{22} = P(h|h)\ \{1 - P(K|H)\},$$

$$p_{23} = \{1 - P(h|h)\}\{1 - P(K|M)\},$$

$$p_{31} = P(h|m)\ P(K|H)\ ,$$

$$p_{32} = P(h|m)\ \{1 - P(K|H)\}\ ,$$

$$p_{33} = \{1 - P(h|m)\}\{1 - P(K|M)\}, \tag{5.14.2}$$

and substitute (5.9.5), (5.9.11), and (5.14.2) into (5.9.9) to obtain the desired result for the expected time to kill a target.

Let us now turn to the case of burst fire. We will consider weapon-system types that employ impact-lethality projectiles and have the capability of burst fire. BONDER and FARRELL [17, pp. 107-108] have pointed out that such weapon-system types can fire in a number of modes[37]:

(M1) repeated-single-shot-independent fire,

(M2) repeated-single-shot-MARKOV-dependent fire,

(M3) burst fire (one long burst),

(M4) mixed-mode fire [repeated-single-shot-MARKOV-dependent fire until first hit after which there is an immediate switch to burst fire (one long burst)],

and (M5) repeated-burst fire [multiple (short) bursts independently fired].

Modes (M1) and (M2) are special cases of BONDER's model of MARKOV-dependent fire discussed in Sections 5.4 and 5.5 above, while mode (M5) is conceputally the same as mode (M1), and consequently results for the expected time to kill a target may be obtained for them by involing, for example[38], (5.4.1). In particular, VECTOR-2 [28, pp. 174-175] uses the following result for <u>repeated-burst fire [multiple (short) bursts independently fired]</u>

$$E[T] = t_a + t_1^B + t_s^B \left\{ \frac{1 - P_{SBK}^1}{P_{SBK}^s} \right\} , \tag{5.14.3}$$

where

$t_a$ is as previously defined,

$t_1^B$ denotes the time to fire the first burst after the decision to engage the target has been made,

$t_s^B$ denotes the time between the firings of any two successive bursts,

$P_{SBK}^1$ denotes the probability of killing the target with the first burst,

and $P_{SBK}^s$ denotes the probability of killing the target with any subsequent burst.

The simplest model for $P_{SBK}$ is to assume that all rounds within the burst are independently fired, and then

$$P_{SBK} = 1 - (1 - P^B_{SSK})^n , \qquad (5.14.4)$$

where $n$ denotes the number of rounds in the burst and $P^B_{SSK}$ denotes the single-shot hit probability for any round in the burst (and is assumed to be the same whether the round follows a hit or a miss).

For the mixed-firing mode (M4), using arguments similar to those employed in Section 5.5, BONDER and FARRELL [17, pp. 108-113] have derived the following expression for the expected time to kill a target

$$E[T] = t_a + t_1 + t_f + (t_m + t_f) \left\{ \frac{1 - p_1}{P(h_1|m)} \right\}$$
$$+ \; 1 - P(K|H) \left[ t_h + t_f + t_b \left\{ \frac{1 - P^B_{SSK}}{P^B_{SSK}} \right\} \right] , \qquad (5.14.5)$$

where

$t_a$, $t_1$, $t_f$, $t_h$, $t_m$, $p_1$, and $P(K|H)$ are all as previously defined in Table 5.II,

$P(h_1|m)$ denotes the conditional probability of a hit following a miss <u>before</u> the first hit has been obtained,

$t_b$ denotes the time between the firings of any two successive rounds in the burst-fire model,

and $P^B_{SSK} = P^B_{SSH} \, P(K|H)$ denotes the probability of killing the target with any one round in the burst-firing mode and $P^B_{SSH}$ denotes the corresponding hit probability.

Here BONDER and FARRELL [17, p. 109] have assumed that the hit probability for any round in the burst is the same whether it follows a hit or a miss. Mode (M3), firing one long burst, may be obtained as a special case of mode (M4) by assuming

(A1) the time to fire every round except the first is $t_b$, i.e. $t_h = t_m = t_b$;

(A2) after the first round, the hit probability is constant, i.e. $P(h_1|m) = P_{SSH}^B$;

and (A3) only the time of flight for one round need be considered.

It follows that under these conditions the expected time to kill a target with one long burst is given by, i.e. (5.14.5) reduces to

$$E[T] = t_a + t_1 + t_f + t_b \left\{ \frac{1 - p_1 P(K|H)}{P_{SSH}^B \; P(K|H)} \right\} , \tag{5.14.6}$$

which, if the first-round hit probability is the same as that for any subsequent round, further reduces to

$$E[T] = t_a + t_1 + t_f + t_b \left\{ \frac{1 - P_{SSK}}{P_{SSK}} \right\} , \tag{5.14.7}$$

where $P_{SSK} = P_{SSH} \; P(K|H)$ and $P_{SSH} = p_1 = P_{SSH}^B$.

Let us finally note here that data sources for not only all the attrition-rate-coefficient expressions given in this section but also all those given elsewhere in this chapter have to be discussed in the documentation on, for example, VECTOR-2 [28, pp. 173-175]. The interested reader is directed to such places for further information about data sources for computing numerical values for LANCHESTER attrition-rate coefficients.

## 5.15. Maximum-Likelihood Estimation of Attrition-Rate Coefficients.

In the introductory section of this chapter we saw that there are two general approaches for determining numerical values for LANCHESTER attrition-rate coefficients:

(A1) use an analytical submodel of the attrition process to compute the desired numerical value,

and (A2) use "combat" data to compute a statistical estimate of it.

In the previous sections of this chapter we have considered in detail the former approach based on using an analytical submodel, and in this section we will briefly consider the statistical-estimation approach, which presupposes the availability of (either actual[39] or simulated) combat data (recall Figure 5.1). In actual applications some type of "simulated-combat" data (generated, for example, by a high-resolution Monte Carlo combat simulation) is invariably used.

In this latter quasi-empirical approach, one uses the "combat" data to compute statistical estimates of the attrition-rate coefficients (and sometimes parameters contained in the coefficients). In general, there are four principal statistical methods for computing such point estimates (e.g. see BHAT [7, pp. 370-371] for further details): (a) maximum-likelihood estimation, (b) method of moments, (c) BAYES estimation, and (d) method of least squares. Of these four methods, however, only the first one has had any significant application in combat analysis (e.g. see CLARK [24],

[36, pp. 3-1 through 3-10], ANDRIGHETTI [2], STOCKTON [73], or GRAHAM [39]). Accordingly, we will consider only the maximum-likelihood-estimation approach, which determines attrition-rate-coefficient parameters from an appropriate set of "combat" data by selecting their values to maximize the so-called likelihood function corresponding to this data. Our approach here will be to consider a simple example first, before examining more general (and complicated) cases.

Consider now that we have run a Monte Carlo combat simulation and have recorded the times at which casualties have occurred (and also the type of each casualty). Let us run this stochastic simulation until a total of $K$ casualties have occurred. The total time that the simulation will have been run is a random variable that we will denote as $T_K$ (with realization $t_K$). Let us also denote (for $k = 1,2,\dots,K$) the time (a r.v.) at which the $k^{\underline{th}}$ casualty occurs as $T_k$ (with realization $t_k$). We will start the battle at $t = 0$ by setting $t_0 = 0$. Our main assumption is that we will consider that our "battle" data represents a sample from the MARKOV-chain analogue of the deterministic LANCHESTER-type equations

$$\begin{cases} \dfrac{dx}{dt} = -a & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b & \text{with } y(0) = y_0, \end{cases} \tag{5.15.1}$$

i.e. in the corresponding continuous-parameter MARKOV chain the transition (casualty) probabilities are given by Prob[$X$ casualty in small interval of length $\Delta t$] = $a\Delta t$ and Prob[$Y$ casualty in $\Delta t$] = $b\Delta t$.

Let $M(t)$ (a r.v. with realization $m$) denote the number of $X$ combatants at time $t$ in the above stochastic combat model, and let $N(t)$

(a r.v. with realization $n$) denote the number of $Y$ combatants at time $t$ (see Figure 5.10). Furthermore, let us introduce the r.v.'s $C_k^X$ and $C_k^Y$ (with realizations $c_k^X$ and $c_k^Y$) defined by

$$C_k^X = \begin{cases} 1 & \text{if the } k^{th} \text{ casualty is an } X \\ & \text{combatant,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C_k^Y = \begin{cases} 1 & \text{if the } k^{th} \text{ casualty is a } Y \\ & \text{combatant,} \\ 0 & \text{otherwise.} \end{cases}$$

Focussing now on the realizations $c_k^X$ and $c_k^Y$, we have $c_k^X \cdot c_k^Y = 0$ with $c_k^X + c_k^Y = 1$. For future purposes, we will let $c_T^X$ denote the total number of $X$ casualties, i.e.

$$c_T^X = \sum_{k=1}^{K} c_k^X , \tag{5.15.2}$$

and, similarly,

$$c_T^Y = \sum_{k=1}^{K} c_k^Y , \tag{5.15.3}$$

with (of course)

$$c_T^X + c_T^Y = K . \tag{5.15.4}$$

Furthermore, although we will not need them right now, let us denote $m(t_k)$

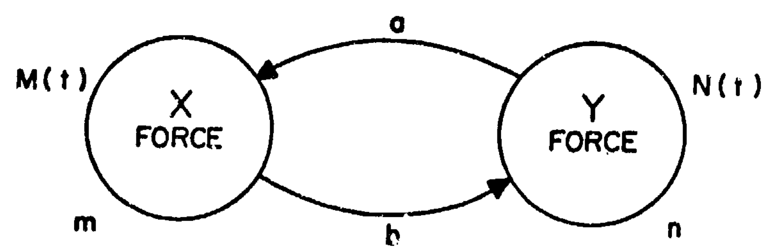


Figure 5.10. Schematic of combat interactions for stochastic battle corresponding to the deterministic LANCHESTER-type equations (5.15.1) for C|C attrition process. Here $a$ denotes the casualty rate of the $X$ force caused by the entire $Y$ force.

as $m_k$ (i.e. $m_k$ is the realization of the number of X combatants just after the occurrence of the $k^{th}$ casualty) and $n(t_k)$ as $n_k$. In other words, there are $m_k$ X combatants and $n_k$ Y combatants "alive" during the interval $[t_k, t_{k+1})$ for $k = 0,1,\ldots, K-1$.

Using the data $t_1, \ldots, t_K, c_1^X, \ldots, c_K^X, c_1^Y, \ldots, c_K^Y$, we will now develop statistical estimates, denoted as $\hat{a}$ and $\hat{b}$, for the continuous-time MARKOV-chain analogue of the LANCHESTER-type model (5.15.1) by the so-called method of <u>maximum-likelihood estimation</u>. The observant reader will notice that in this case the casualty streams are nothing more than two superimposed POISSON processes, and consequently $\hat{a}$ and $\hat{b}$ will turn out to be given by expressions equivalent to well-known results for the maximum-likelihood estimator of a POISSON parameter. In very general terms, the maximum-likelihood-estimation approach choses (based on the available data) the formulas for the computation of $\hat{a}$ and $\hat{b}$ so that they give the greatest probability to the observed combat outcome (<u>see</u> KENDALL [48, p. 178]). This maximization is effected by considering the so-called likelihood function, which (in simple terms) gives the probability of the observed realization of the stochastic attrition process. The likelihood function, in turn, is constructed out of the density functions for the times between casualties, since we should consider the above combat data to be a random sample from these times. For our stochastic attrition process, we may summarize the above maximum-likelihood method as follows:

(S1) determine the probability density function (p.d.f.) for the time to an X casualty (also that for the time to a Y casualty),

(S2) construct the likelihood function (i.e. the density function for the observed sequence of events),

(S3) determine the values of the parameters $a$ and $b$ that maximize the likelihood function (denote these maximizing values as $\hat{a}$ and $\hat{b}$).

We will now carry out the above three steps (S1) through (S3) to determine maximum-likelihood estimators $\hat{a}$ and $\hat{b}$ for the LANCHESTER attrition-rate coefficients for the continuous-time MARKOV-chain analogue of (5.15.1). For step (S1), we consider the time to an $X$ casualty from the occurrence of the last casualty and develop its p.d.f. For our constant-attrition-rate coefficient continuous-time MARKOV-chain attrition model, the times between casualties are exponentially distributed (see Section 4.7 above). Thus, if we let $S$ denote the time between any two consecutive casualties, then the p.d.f. for this nonnegative random variable is given by

$$f_S(s) = (a + b)\, e^{-(a+b)s} . \tag{5.15.5}$$

We now need to convert this p.d.f. for $S$ into one for the time to the occurrence of an $X$ casualty from the occurrence of the last casualty (a r.v. denoted as $S_X$). This may be accomplished by multiplying (5.15.5) by

$$P[X \text{ casualty} | \text{casualty occurs}] = \frac{a}{a + b} , \tag{5.15.6}$$

which is just the probability that an X casualty occurs before a Y one (see Section 4.7 above). Thus

$$f_{S_X}(s) = P[X \text{ casualty} | \text{casualty occurs}]\ f_S(s) ,$$

or

$$f_{S_X}(s) = ae^{-(a+b)s} . \tag{5.15.7}$$

Similarly

$$f_{S_Y}(s) = be^{-(a+b)s} . \tag{5.15.8}$$

We now turn to step (S2). To construct the likelihood function, we observe that casualties have occurred at times $t_1, t_2, \ldots, t_K$, there being a total of $c_T^X$ X casualties and $c_T^Y$ Y casualties with $c_T^X + c_T^Y = K$. Consider now the occurrence of the $k^{\underline{th}}$ casualty, which represents a transition from battle state $(m_{k-1}, n_{k-1})$ to $(m_k, n_k)$. If it is an X casualty, there would be a contribution to the likelihood function of (i.e. the p.d.f. of the population from which the $k^{\underline{th}}$ sample of the time between casualties is drawn would be)

$$a \exp[-(a+b)\ \{t_k - t_{k-1}\}] ; \tag{5.15.9}$$

while if it is a Y casualty, there would be a contribution to the likelihood function of

$$b \exp[-(a+b)\ \{t_k - t_{k-1}\}] . \tag{5.15.10}$$

Introducing the variables $c_k^X$ and $c_k^X$, however, we may write the

contribution from the occurrence of the $k^{\underline{th}}$ casualty to the likelihood function in both the above cases simply as

$$a^{c_k^X} b^{c_k^Y} \exp[-(a+b)\{t_k - t_{k-1}\}] , \tag{5.15.11}$$

since (5.15.11) reduces to (5.15.9) when $c_k^X = 1$ and to (5.15.10) when $c_k^X = 0$ (i.e. when $c_k^Y = 1$). By the memoryless property of our continuous-time MARKOV-chain attrition model, the times between casualties are indep-pendent random variables, and hence the likelihood function for the observed sequence of events is simply the product of all the independent contributions (5.15.11), i.e.

$$L(a,b) = \prod_{k=1}^{K} a^{c_k^X} b^{c_k^Y} \exp[-(a+b)\{t_k - t_{k-1}\}] ,$$

or [from (5.15.2), (5.15.3), and a little manipulation]

$$L(a,b) = a^{c_T^X} b^{c_T^Y} \exp[-(a+b)t_k] , \tag{5.15.12}$$

where $L(a,b)$ denotes the likelihood function depending on the parameters $a$ and $b$.

Finally, we reach step (S3), the determination of the estimates $\hat{a}$ and $\hat{b}$ from maximization of the likelihood function (5.15.12). However, instead of maximizing the likelihood function $L(a,b)$ itself, one usually maximizes its logarithm, since both maximum values occur at the same point and the logarithm form is more tractable. Hence, we consider

$$\ell n\ L(a,b) = c_T^X\ \ell n\ a + c_T^Y\ \ell n\ b - (a+b)t_K\ . \tag{5.15.13}$$

The maximum-likelihood estimates $\hat{a}$ and $\hat{b}$ are then the values of $a$ and $b$ that solve the problems

$$\underset{a,b}{\text{maximize}}\ \ \ell n\ L(a,b)\ , \tag{5.15.14}$$

where

$$c_T^X + c_T^Y = K\ .$$

From (5.15.13) we see that the two-dimensional maximization problem (5.15.14) [with (5.15.13)] factors into two one-dimensional maximization problems. Let us now focus on determining the maximizing value for $a$. Computing

$$\frac{\partial}{\partial a}\ \ell n\ L = \frac{c_T^X}{a} - t_K\ , \tag{5.15.15}$$

we find from $\partial L/\partial a = 0$ that

$$\frac{c_T^X}{a} - t_K = 0\ , \tag{5.15.16}$$

yielding

$$\hat{a} = \frac{c_T^X}{t_K}\ , \tag{5.15.17}$$

which is the desired maximizing value for $a$, since $\partial^2\ \ell n\ L/\partial a^2(\hat{a}) < 0$. Similarly, differentiating (5.15.13) with respect to $b$ and equating to zero, we obtain

$$\hat{b} = \frac{c_T^Y}{t_K} . \tag{5.15.18}$$

The estimates given by (5.15.17) and (5.15.18) are the maximum-likelihood estimates for the LANCHESTER attrition-rate coefficients $a$ and $b$ in the continuous-time MARKOV-chain analogue of (5.15.1). They are also intuitively appealing, since the casualty process can be considered as being composed of two POISSON processes, the X-force casualty process and the Y-force casualty process. The equations (5.15.17) and (5.15.18) then give the estimates of the LANCHESTER attrition-rate coefficients $a$ and $b$ from $c_T^X$ occurrences of an $X$ casualty and $c_T^Y$ occurrences of a $Y$ casualty in time $t_K$, which is the time for $K$ total casualties to occur.

Let us now consider the same maximum-likelihood-estimation problem for the MARKOV-chain analogue of deterministic $F|F$ LANCHESTER-type equations, i.e.

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with} \quad x(0) = x_0 , \\ \dfrac{dy}{dt} = -bx & \text{with} \quad y(0) = y_0 . \end{cases} \tag{5.15.19}$$

Here the transition probabilities for the continuous-time MARKOV-chain attrition process are given by $P[X \text{ casualty in } \Delta t] = an\Delta t$ and $P[Y \text{ casualty in } \Delta t] = bm\Delta t$, where $m$ and $n$ denote realizations of the random variables $M(t)$ and $N(t)$, the numbers of $X$ and $Y$ combatants at time $t$. In this case, for step (S1) we find that

$$f_{S_X}(s) = ane^{-(an+bm)s} , \tag{5.15.20}$$

and

$$f_{S_Y}(s) = bme^{-(an+bm)s} \quad . \tag{5.15.21}$$

Step (S2) then yields that the occurrence of the $k^{\underline{th}}$ casualty at $t_k$ makes a contribution to the likelihood function of

$$(an_{k-1})^{c_k^X} (bm_{k-1})^{c_k^Y} \cdot \exp[-(an_{k-1} + bm_{k-1}) \{t_k - t_{k-1}\}] \ ,$$

whence the likelihood function itself is given by

$$L(a,b) = \prod_{k=1}^{K} (an_{k-1})^{c_k^X} (bm_{k-1})^{c_k^Y} \exp[-(am_{k-1} + bm_{k-1}) \{t_k - t_{k-1}\}]. \tag{5.15.22}$$

Computing the natural logarithm of the likelihood function

$$\ell n\ L(a,b) = \sum_{k=1}^{K} c_k^X \ \ell n(an_{k-1}) + \sum_{k=1}^{K} c_k^Y \ \ell n(bm_{k-1})$$

$$- \sum_{k=1}^{K} (an_{k-1} + bm_{k-1}) \{t_k - t_{k-1}\} \ , \tag{5.15.23}$$

we find in step (S3) that

$$\frac{\partial\ \ell n\ L}{\partial a} = \frac{c_T^X}{a} - \sum_{k=1}^{K} n_{k-1}\{t_k - t_{k-1}\} \ , \tag{5.15.24}$$

whence, setting the above derivative equal to zero, we obtain the maximum likelihood estimate

$$\hat{a} = \frac{c_T^X}{\sum_{k=1}^{k} n_{k-1}\{t_k - t_{k-1}\}} \quad . \tag{5.15.25}$$

Similarly,

$$\hat{b} = \frac{c_Y^Y}{\sum_{k=1}^{K} m_{k-1}\{t_k - t_{k-1}\}} \qquad (5.15.26)$$

The above results for maximum-likelihood estimates of attrition-rate coefficients are characterized by their simplicity, i.e. explicit results are easily written down. Let us now show that for nonautonomous LANCHESTER-type combat, this will always be true when the attrition-rate parameters appear linearly. To see this, let us consider the continuous-time MARKOV-chain analogue of the nonautonomous LANCHESTER-type equations

$$\begin{cases} \dfrac{dx}{dt} = -A(x,y) & \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -B(x,y) & \text{with} \quad y(0) = y_0 . \end{cases} \qquad (5.15.27)$$

In this case, the forward KOLMOGOROV equations for the stochastic evolution of combat are given by (5.1.2), and the infinitesimal transition probabilities are given by $P[X \text{ casualty in } \Delta t] = A(m,n)\Delta t$ and $P[Y \text{ casualty in } \Delta t] = B(m,n)\Delta t$. As usual, $m$ and $n$ are realizations of $M(t)$ and $N(t)$, the numbers of $X$ and $Y$ at time t in the stochastic battle (see Figure 5.11). We will now consider the special case in which the attrition-rate parameters appear linearly in $A(m,n)$ and $B(m,n)$. When the attrition-rate coefficients $a$ and $b$ appear linearly in the attrition rates $A$ and $B$, we may write

$$A(m,n) = ag_a(m,n), \quad \text{and} \quad B(m,n) = bg_b(m,n) . \qquad (5.15.28)$$

In this special case of interest, calculations similar to those given above yield that

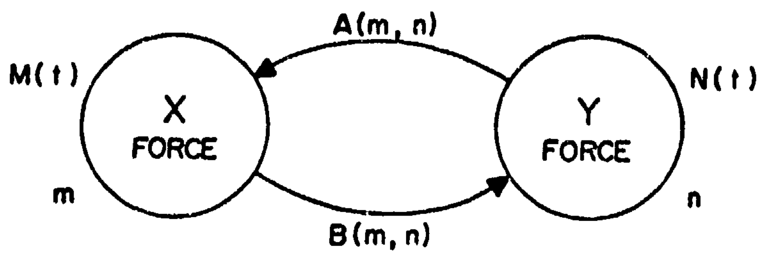


Figure 5.11. Schematic of combat interactions for stochastic battle corresponding to the deterministic nonautonomous LANCHESTER-type equations (5.15.27). Here $A(m,n)$ denotes the casualty rate of the entire $X$ force with $m$ combatants caused by the entire $Y$ force with $n$ combatants.

$$\hat{a} = \frac{c_T^X}{\sum_{k=1}^K g_a(m_{k-1}, n_{k-1}) \{t_k - t_{k-1}\}} , \tag{5.15.29}$$

and

$$\hat{b} = \frac{c_T^Y}{\sum_{k=1}^K g_b(m_{k-1}, n_{k-1}) \{t_k - t_{k-1}\}} , \tag{5.15.30}$$

Thus, when the parameters to be estimated appear linearly in the attrition rates, very simple estimates result. Furthermore, all our previous results are just special cases of this one. We have presented these special cases first, however, in order to show the reader the basic idea of the maximum-likelihood method without his being overencumbered with notation the first time.

In all the above developments, we have had the same stopping rule for collecting our combat data: data was collected until the $K^{\underline{th}}$ casualty occurred. Let us now suppose, however, that we collect data (or run our "combat experiment") <u>for a specified length of time</u> $t_f$ <u>or until one side or the other has been annihilated</u>. Again, let us say that $K$ casualties have been observed at times $t_1, t_2, \ldots, t_K$. We have then that

$$t_K \leq t_f , \tag{5.15.31}$$

$$\sum_{k=1}^K c_k^X \leq m_0 , \qquad \sum_{k=1}^K c_k^Y \leq n_0 , \tag{5.15.32}$$

and (5.15.2) through (5.15.4) again hold. Here $m_0$ and $n_0$ denote the initial numbers of $X$ and $Y$ combatants. Furthermore, we will now consider the general continuous-time MARKOV-chain attrition-process model (5.1.2) (again, <u>see</u> Figure 5.11), with infinitesimal transition probabilities $P[X \text{ casualty in } \Delta t] = A(m,n)\Delta t$ and $P[Y \text{ casualty in } \Delta t] = B(m,n)\Delta t$.

In this case, there will be an additional contribution to the likelihood function of

$$\exp[-\{A(m_K,n_K) + B(m_K,n_K)\}\{t_f - t_K\}] \ , \qquad (5.15.33)$$

when $t_f > t_K$, i.e. when neither side is annihilated before $t_f$. Accordingly, the likelihood function for the observed sequence of casualties is given by

$$L = \left[\prod_{k=1}^{K} \{A(m_{k-1},n_{k-1})\}^{c_k^X} \{B(m_{k-1},n_{k-1})\}^{c_k^Y} \right.$$

$$\left. \times \exp[-\{A(m_{k-1},n_{k-1}) + B(m_{k-1},n_{k-1})\}\{t_k - t_{k-1}\}\right]$$

$$\times \exp[-\{A(m_K,n_K) + B(m_K,n_K)\}\{t_f - t_K\}] \quad , \qquad (5.15.34)$$

where (5.15.2) through (5.15.4), (5.15.31), and (5.15.32) hold. The natural logarithm of the likelihood function is then given by

$$\ln L = \sum_{k=1}^{K} c_k^X \ln A(m_{k-1},n_{k-1}) + \sum_{k=1}^{K} c_k^Y \ln B(m_{k-1},n_{k-1})$$

$$- \sum_{k=1}^{K} \{A(m_{k-1},n_{k-1}) + B(m_{k-1},n_{k-1})\}\{t_k - t_{k-1}\}$$

$$- \{A(m_K,n_K) + B(m_K,n_K)\}\{t_f - t_K\} \ , \qquad (5.15.35)$$

and hence when (5.15.28) holds we find that

$$\hat{a} = \frac{c_T^X}{\sum_{k=1}^{K+1} g_a(m_{k-1},n_{k-1})\{t_k - t_{k-1}\}} , \tag{5.15.36}$$

and

$$\hat{b} = \frac{c_T^Y}{\sum_{k=1}^{K+L} g_b(m_{k-1},n_{k-1})\{t_k - t_{k-1}\}} \tag{5.15.37}$$

where $t_{K+1} = t_f$. We also have that $t_K = t_f$ if and only if either $\sum_{k=1}^{K} c_k^X = m_0$ or $\sum_{k=1}^{K} c_k^Y = n_0$, i.e. if and only if either side is annihilated before $t_f$. Thus, we see that the maximum-likelihood estimate of a LANCHESTER attriton-rate coefficient depends (slightly) on the circumstances under which the combat data has been collected, although for the stochastic analogue of (5.15.1) we have that, for example, $\hat{a}$ = (total number of X casualties)/(total length of time that battle has been observed).

If we had $J$ replications of the "combat experiment," we would redefine our notation as follows:

$t_k^j$ = time of occurrence of $k^{\underline{th}}$ casualty in $j^{\underline{th}}$ replication,

$m_k^j$ = number of $X$ combatants "alive" during the interval $[t_k^j, t_{k+1}^j)$,

$n_k^j$ = number of $Y$ combatants "alive" during the interval $[t_k^j, t_{k+1}^j)$,

and $K_j$ = total number of casualties on both sides for the $j^{\underline{th}}$ replication of the battle.

It then follows [say for the second stopping rule and the model (5.1.2)

with (5.15.28)] that, for example,

$$\hat{a} = \frac{(c_T)_{\text{all replications}}}{\sum_{j=1}^{J}\sum_{k=1}^{K_j} g_a(m_{k-1}^j,\, n_{k-1}^j)\,\{t_k^j - t_{k-1}^j\}}\,, \tag{5.15.38}$$

where $(c_T^X)_{\text{all replications}}$ denotes the total number of X casualties for all replications of the "combat experiment."

We will wrap up this section by briefly sketching historical developments and possible future trends in the use of maximum-likelihood estimation of attrition-rate coefficients in combat analysis. This approach has been intimately related with the idea of hierarchy of models (see Section 7.20) in which the output data from, for example, a high-resolution combat model of small-unit operations is used as input data to a low-resolution combat model of large-unit operations (again, refer to Figure 5.1).

Although the concept of a hierarchy of combat models has apparently been on the minds of a number of military OR workers in the United States since at least about the mid-1950's, recent interest in the United States and an accompanying analytical framework apparently dates from the Ph.D. thesis of G. Clark [24] in 1969 (see also [25]). He developed a satellite model [called the COMAN (COMbat ANalysis) model] that must be used[40] in conjunction with a high-resolution combat-simulation model (usually Monte Carlo type) in order to interpolate/extrapolate the results of the higher-resolution model (in terms of numbers and types of casualties for a given force mix or mixes) to other force mixes not explicitly evaluated by the high-resolution model. The COMAN model was a stochastic LANCHESTER-type heterogeneous-force combat model (i.e. the continuous-time

MARKOV-chain analogue of certain deterministic heterogeneous-force LANCHESTER-type equations) and involved the following two modifications of the then existing LANCHESTER combat theory (see CLARK [24, pp. 139-164] for further details):

(M1) incorporation of weapon-system target-acquisition capability into the model through introduction of the probabilities that a target is unacquired by an enemy firer,

and (M2) introduction of target priorities.

The former modification (M1) was implemented through the introduction of target-acquisition probabilities, which then were used to modify (i.e. degrade) the inherent kill capabilities of weapon-system types, while the latter modification (M2) was implemented through the input of two target-priority lists (every weapon-system type on a particular side had the same target-priority list) and the modelling of the engagement of target types with priorities[41]. Let us now examine in greater detail how this former aspect [i.e. modification (M1)] was handled. For simplicity, we will consider a constant-parameter homogeneous-force version of CLARK's COMAN model.

CLARK's [24, pp. 157-158] basic idea for incorporating weapon-system target-acquisition capabilities into the LANCHESTER paradigm[42] may be seen by considering the MARKOV-chain model (5.1.2) (see Figure 5.11 again) with total-force kill rates given by

$$A(m,n) = a\{1 - (p_{XY})^m\}n \, , \qquad B(m,n) = b\{1 - (p_{YX})^n\}m \, , \tag{5.15.39}$$

where

$a$ denotes the kill rate for a single $Y$ weapon system having <u>acquired</u> targets at which to fire,

$$p_{XY} = P\begin{bmatrix}\text{a specific } X \text{ target is unacquired} \\ \text{by an individual } Y \text{ firer}\end{bmatrix},$$

and $b$ and $p_{YX}$ are similarly defined for the $X$ force.

Here, for example, $a$ denotes an acquisition-independent attrition-rate coefficient and represents the "inherent" kill capability of a single $Y$ firer in the sense that it is his kill rate when one or more enemy targets are available for him to fire at (i.e. there are acquired targets at which he can fire).

The total-force kill rates (5.15.39) may be developed in the following manner. One assumes that the total-force attrition rate for each side is equal to the sum of the individual firing-weapon kill rates for the opposing force. Interactions due to multiple firers attacking a single target are neglected by this assumption. Consider now, for example, a single $Y$ firer engaging $X$ targets of which there are a total of $n$ at time t. The probability that this firer has one or more $X$ targets at which to fire is given by $1 - (p_{XY})^m$, whence it follows by the above additivity assumption that the $Y$-force kill rate $A(m,n)$ is given by (5.15.39). Furthermore, it is readily shown that when targets are easy to acquire (e.g. $p_{XY}$ is near 0), then $A(m,n)$ is very nearly given by

an (i.e. the X-force attrition rate is proportional to only the number of enemy firers as in LANCHESTER's equations for modern warfare). Also, when targets are difficult to acquire (e.g. $p_{XY}$ is near 1), then $A(m,n)$ is very nearly given by $amn$ (i.e. the X-force attrition rate is proportional to the product of the numbers of firers and targets as in LANCHESTER's equations for area fire). Thus, we should think of (5.15.39) as a general attrition-rate model that incorporates weapon-system target-acquisition capabilities into the model and reduces to those corresponding to LANCHESTER's classic formulations in the above two important limiting cases. From an examination of DYNTACS[43] data CLARK [26] found that the probability of a target being unacquired is quite sensitive to the nature of the terrain profile between the opposing forces. This terrain profile can change abruptly and cause the target-acquisition probabilities to appear as almost discontinuous functions of battle time.

CLARK's idea of the COMAN model was adopted by the Research Analysis Corporation (RAC), which later became part of General Research Corporation (GRC), and evolved[44] into COMANEX (<u>COMAN</u> <u>EX</u>tended), which (like COMAN itself) was composed of two basic sub-programs: the pre-processor and the simulator (<u>see</u> CLARK [24] or [36] for further details). Figure 5.12 shows how these programs were used, with CARMONETTE serving as the high-resolution model.

Data for weapons characteristics, combat environment, mission, etc. for a particular mix of opposing forces were input into CARMONETTE. CARMONETTE was then run for a specified number of replications of the battle. The computer program then output (for each replication) a

COMANEX Extrapolation Process

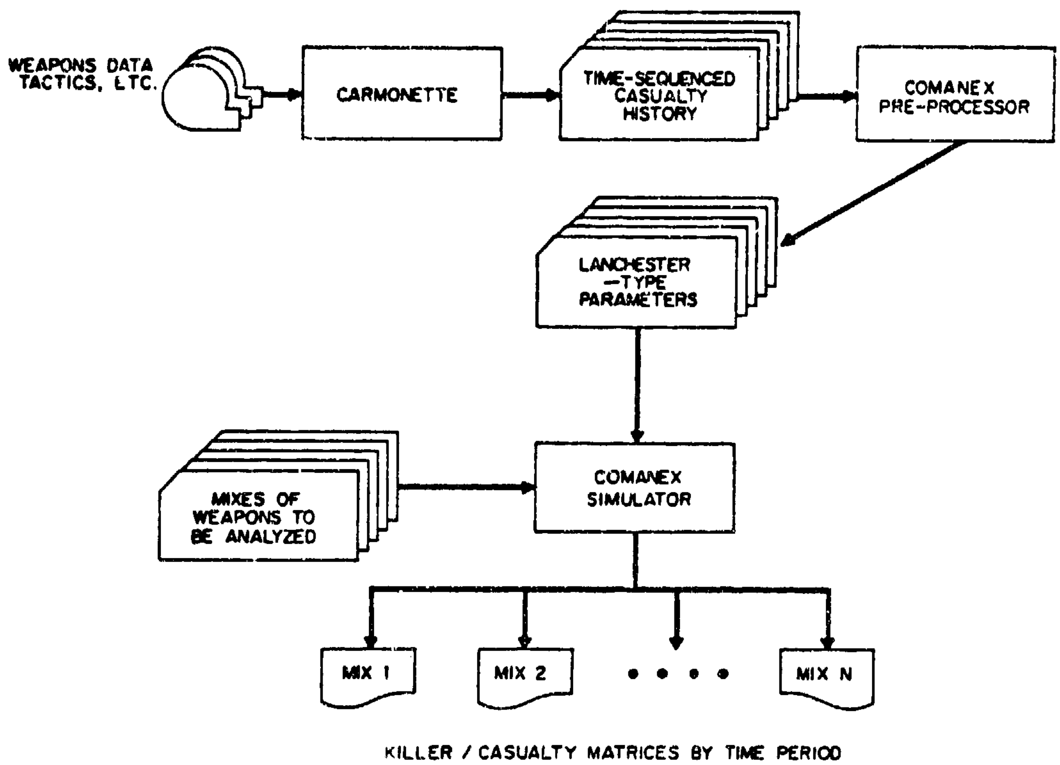


Figure 5.12. Implementation of the COMANEX model (from [36]), which is an example (apparently the first to be developed in the United States) of the fitted-parameter analytical model (see Figure 5.1). The COMANEX model was composed of two basic sub-programs: the pre-processor and the simulator, with CARMONETTE serving as the high-resolution model for generating input data to the pre-processor.

time-sequenced casualty history, with the time at which each casualty occurred (as well as the casualty type and the killer type) being given. This output was, in turn, input into the COMANEX pre-processor. This pre-processor massaged the data and output a set of values for LANCHESTER attrition-rate coefficients, which represent the kill rates for each firer/target combination on the battlefield. The values for these attrition-rate coefficients were then stored in the COMANEX simulator to be subsequently used in predicting the outcomes of battles involving force mixes "close" to the original mix (i.e. mixes involving the same types but different numbers of weapons).

The force mixes to be analyzed were then specified and input into the simulator. In practice, for test purposes, the first such mix was usually the original one from which the values for the LANCHESTER attrition-rate coefficients were determined. The simulator was exercised for up to 100 replications of the battle. It output the expected results of the battle in the form of killer/casualty matrices which listed the number of casualties (averaged over all replications) by time period, for each of the target types, and for each of the killer types. After it was verified that the simulator indeed reproduced the results of the original CARMONETTE run, the remaining force mixes were processed, and their expected outcomes were listed (again in the form of killer/casualty matrices). COMANEX was used in this fashion in a number of analyses for the U. S. Army (e.g. see [32]).

Later the same general idea was used by a U. S. Army systems-analysis agency called TRASANA (TRADOC Systems ANalysis Activity) with a few further modifications in the form of COMANEW [COMAN (N)EW].

Target priorities and target-acquisition probabilities were eliminated and replaced by heterogeneous-force allocation factors (see Section 7.7), and also ammunition expenditure was explicitly considered (see GRAHAM [39] for further details). Quite encouraging results have been reported.

Future trends would appear to be centered around the use of further additional functional forms for attrition-rates in this satellite-model approach. The theory of this approach has now been rather fully developed, and the author anticipates that future activities will be centered around computational work and that further computational results will be reported, especially as to which functional forms for LANCHESTER-type attrition rates give the "best" results. It is surprising, however, that there have been so few results reported so far about which forms for LANCHESTER-type equations are at least not inconsistent with simulated combat results generated by high-resolution Monte Carlo simulations.

## 5.16. Attrition-Rate Coefficients for Heterogeneous-Force Combat.

The modern battlefield contains many different weapon-system types that operate together with complementary capabilities as "combined-arms teams." For example, there might be both mounted and dismounted infantry, infantry with rifles, infantry with machine guns, tanks, different types of anti-tank weapon systems, artillery, mortars, other types of fire-support systems, etc. Since each of these various different weapon-system types would generally inflict and sustain casualties at different rates, when one wants to model the attrition process for combat between such combined-arms teams, one is obliged to keep track of the number of each type of casualty and consider combat between heterogeneous forces.

For such heterogeneous-force combat, the natural generalization of the homogeneous-force F|F deterministic LANCHESTER-type attrition process may be written as (see Section 7.7 for further details)

$$\begin{cases} \dfrac{dx_i}{dt} = -\sum_{j=1}^{n} A_{ij} y_j \quad \text{with} \quad x_i(0) = x_i^0 , \\ \\ \dfrac{dy_j}{dt} = -\sum_{i=1}^{m} B_{ji} x_i \quad \text{with} \quad y_j(0) = y_j^0 , \end{cases} \tag{5.16.1}$$

where $x_i(t)$ (for $i = 1,2,\ldots,m$) denotes the number of the $i^{\underline{th}}$ weapon-system type of the X force at time t, $B_{ji}$ denotes the rate at which one $X_i$ firer kills $Y_j$ targets[45], and the quantities $y_j(t)$ (for $j = 1,2,\ldots,n$) and $A_{ij}$ are similarly defined for the Y force (see Figure 7.11). Here (as in Section 7.7) we will always let the subscript i refer to the X force

(and take on the integer values 1 through $m$) and the subscript $j$ refer to the $Y$ force (and take on the integer values 1 through $n$). We will call a nonnegative quantity such as, for example, $A_{ij}$ a <u>heterogeneous-force LANCHESTER attrition-rate coefficient</u>. It represents the fire effectiveness of a $Y_j$ firer against $X_i$ targets and denotes the rate at which a typical $Y_j$ firer kills $X_i$ targets in the opposing heterogeneous enemy force (see Figure 5.13). BONDER and FARRELL [17] (see also Section 5.3 above) have argued that one should take such a heterogeneous-force LANCHESTER attrition-rate coefficient to be given, for example, by

$$A_{ij} = \frac{1}{E[T_{X_iY_j}]} , \qquad (5.16.2)$$

where $E[T_{X_iY_j}]$ denotes the expected time for a single $Y_j$ firer to kill an $X_i$ target. As we have stressed above, the development of credible methodology for computing numerical values for such LANCHESTER attrition-rate coefficients has greatly facilitated the use of LANCHESTER-type combat models as viable defense-planning tools.

Heterogeneous-force attrition-rate coefficients such as $A_{ij}$ and $B_{ji}$ in the model (5.16.1) reflect a much greater complexity in the attrition process than do homogeneous-force attrition-rate coefficients such as $a$ and $b$ in the model (5.2.1): besides being complex functions of weapon-system-type capabilities and target-type characteristics, the attrition-rate coefficients $A_{ij}$ and $B_{ji}$ also depend on additional operational factors such as the distribution of target types, relative rates of target-type acquisition for the various different types of firer-target pairs, procedures

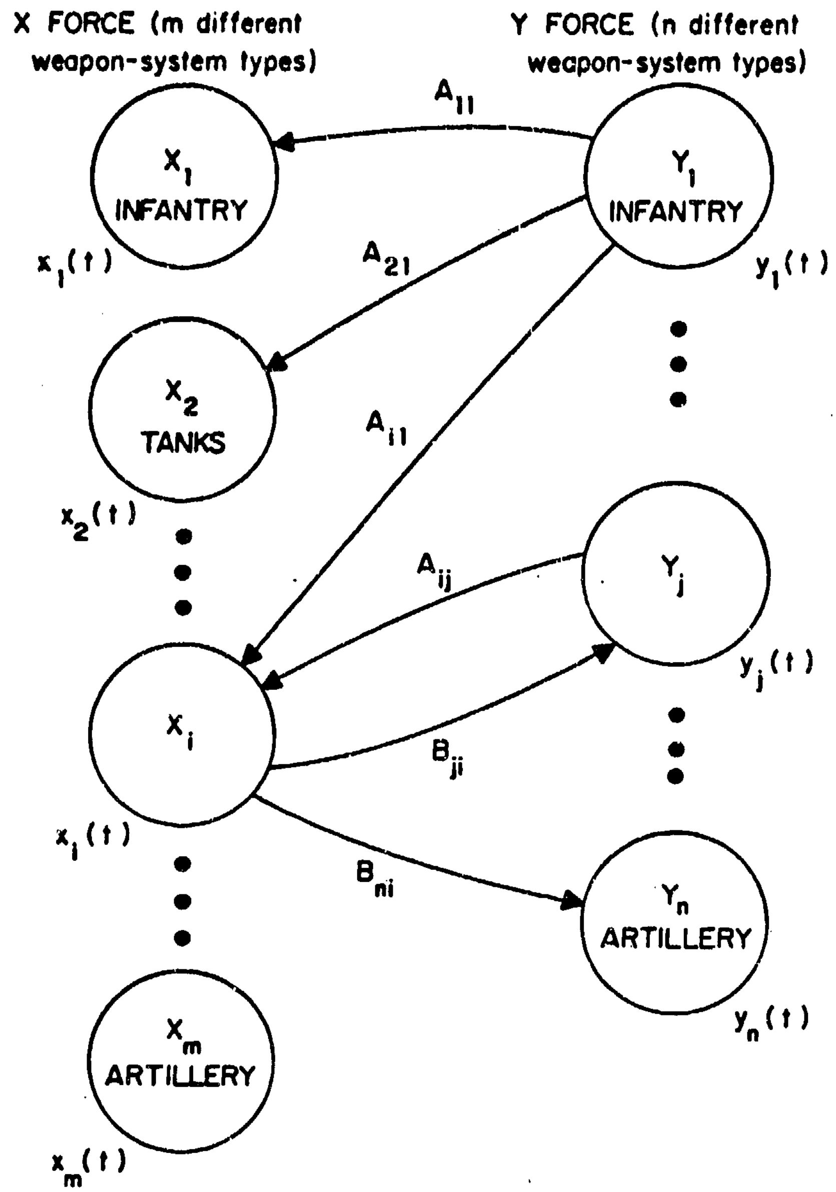


Figure 5.13. Schematic showing notation convention for subscripts on attrition-rate coefficients in heterogeneous-force combat. Our convention is that the first subscript denotes the target type and the second subscript denotes firer type, e.g. $A_{ij}$ denotes the rate at which a typical $Y_j$ firer kills $X_i$ targets in the opposing enemy force.

and priorities for assigning weapon-system types to target types, etc. In other words, not only must one consider how a given weapon-system type causes attrition to a particular engaged-enemy-weapon-system type (as one does in modelling homogeneous-force-on-force combat attrition), but also one must account for different such pairings occurring at different times and places on the battlefield and also possible changes in these pairings over time. Thus, attrition-rate coefficients for heterogeneous-force combat must reflect much greater complexities of the attrition process than those for homogeneous-force combat. It is of fundamental importance, though, that all approaches known to this author for modelling heterogeneous-force attrition-rate coefficients take homogeneous-force results [e.g. (5.4.1)] as key "building blocks" for constructing their heterogeneous-force results. Thus, although there will occasionally be some minor modifications, we will use (in the appropriate way) all the above homogeneous-force-attrition-rate-coefficient results for developing heterogeneous-force attrition-rate coefficients.

It is convenient for modelling attrition-rate coefficients (e.g. see BONDER and FARRELL [17, pp. 15-16] or CHERRY [21, pp. 6-7]) to reflect such complexities of heterogeneous-force combat as discussed above by partitioning the attrition process into four distinct subprocesses[46]:

(SP1) the fire effectiveness of weapon-system types firing at live targets,

(SP2) the allocation process of assigning weapon-system types to target types,

(SP3) the inefficiency of fire when weapon-system types engage other than live targers,

and (SP4) the effects of terrain on limiting firing activities of weapon-system types and on the mobility of the systems.

Two general ways in which these effects have been included in LANCHESTER attrition-rate coefficients are as follows: to model such a coefficient as, for example,[47]

$$\text{(W1)} \quad A_{ij} = \psi_{ij} f^{Y}_{ij} a_{ij} \;, \tag{5.16.3}$$

or

$$\text{(W2)} \quad A_{ij} = F^{Y}_{ij}\left(\alpha_{ij}, \begin{array}{c}\text{all other variables describing the}\\ \text{acquisition and engagement of targets}\end{array}\right), \tag{5.16.4}$$

where

$\psi_{ij}$ denotes the allocation factor (the fraction of $Y_j$ assigned to engage $X_i$) ,

$a_{ij}$ denotes the "inherent" single-firer weapon-system kill rate (the rate at which one $Y_j$ firer type kills $X_i$ target types when it is engaging only them),

$f^{Y}_{ij}$ denotes a factor aggregating the effects of all other variables that are not included in the "inherent" single-firer kill rate $a_{ij}$ and modifying the effectiveness of an individual $Y_j$ firer type against $X_i$ target types,

$F^{Y}_{ij}$ denotes a function that yields the attrition-rate coefficient for a $Y_j$ firer type engaging $X_i$ target types (with arguments as indicated),

and $\alpha_{ij}$ denotes the <u>conditional single-firer weapon-system kill rate</u> (the rate at which one $Y_j$ firer type kills acquired $X_i$ target types when it is engaging them).

The reader should note the distinction between the <u>"inherent" single-firer kill rate</u> $a_{ij}$ (the rate at which one $Y_j$ firer type kills $X_i$ target types when it is engaging only them) and the <u>single-firer kill rate against acquired targets</u> $\alpha_{ij}$ (the rate at which one $Y_j$ firer type kills acquired $X_i$ target types when it is engaging only them). In other words, $a_{ij} = \alpha_{ij}$ when the time to acquire a target is equal to zero. The "inherent" single-firer kill rate for a particular firer-type/target-type pair $a_{ij}$ may be calculated by using data for the pair together with the appropriate attrition-rate-coefficient formula given above. For the reader's convenience, we have summarized in Table 5.V the conditions under which such formulas have been developed and have also cited the equation number for each such formula given above. The conditional single-firer kill rate (i.e. the single-firer kill rate against acquired targets) for a particular firer-type/target type pair $\alpha_{ij}$ may then be calculated by setting the time to acquire a target equal to zero in the appropriate expression for $a_{ij}$. For example, the conditional single-firer kill rate for a weapon-system type using MARKOV-dependent fire and an impact-lethality mechanism is given by

$$\frac{1}{\alpha_{ij}} = E[T_{fr}] - E[T_h] + \frac{\{E[T_h] + E[T_f]\}}{P(K|H)} + \frac{\{E[T_m] + E[T_f]\}}{P(h|m)} \left\{\frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1\right\} , \tag{5.16.5}$$

TABLE 5.V. Summary of Conditions Under Which Expressions for LANCHESTER Attrition-Rate Coefficients Have Been Developed, With Equation Number of Each Expression Given.

(C1) MARKOV-dependent fire and impact-lethality mechanism (5.8.2)

(C2) MARKOV-dependent fire and lethality mechanism by which a target can be killed not only by a hit but also by a miss (5.14.1)

(C3) burst fire and impact-lethality mechanism (5.14.2) or (5.14.5)

(C4) multivolley fire and area-lethality mechanism[48] (5.13.3) or (5.13.19).

where all symbols are as defined in Section 5.8.

Before providing a few selective detailed results on the modelling of heterogeneous-force-attrition-rate coefficients $A_{ij}$ in the two general forms (5.16.3) and (5.16.4), we will present a brief overview of this entire field[49]. The model (5.16.3) and the corresponding form of (5.16.4) [namely, $A_{ij} = \psi_{ij} f_{ij}^{Y} \alpha_{ij}$] have received by far the widest use. Let us note here that the heterogeneous-force model presented in Section 7.7, i.e. (7.7.3), corresponds to (5.16.3) with $f_{ij}^{Y} = 1$. In other words, in Section 7.7 we have developed a heterogeneous-force model with[50]

$$A_{ij} = \psi_{ij} a_{ij} . \tag{5.16.6}$$

The modelling of attrition-rate coefficients $A_{ij}$ by the expression (5.16.3) has been used in operational models such as (at the battalion level of combat) BONDER/IUA [18] and its many derivatives (e.g. AIRCAV [85], BLDM[5], AMSWAG [41], FAST [19]) and (at the theater level of combat) IDAGAM [1; 67] (see also TAYLOR [74-78; 79, pp. 797-800]). The modelling of attrition-rate coefficients $A_{ij}$ by the expression (5.16.4) and its special form

$$A_{ij} = g_{ij}^{Y} \alpha_{ij} \tag{5.16.7}$$

has been used in operational models such as (at the battalion level of combat) COMAN [24; 25] and its derivatives COMANEX [36; 73] and COMANEW [39] (see also R. M. THRALL et al. [82]) and (at the theater level of combat) the VECTOR series of models [28; 54; 86; 87]. Here $g_{ij}^{Y}$ denotes a factor

[similar to $f^Y_{ij}$ in (5.16.4)] aggregating the effects of all other variables that are not included in the conditional single-firer kill rate $\alpha_{ij}$ and modifying the effectiveness of an individual $Y_j$ firer type against $X_i$ target types. COMAN and its derivatives have used (5.16.7), while VECTOR has used the nonlinear form (5.16.4).

We will now provide a few selective detailed results pertaining to the above brief overview. In the BONDER/IUA [18] and its many derivatives such as AIRCAV [85], BLDM [5], AMSWAG [41], and FAST [19], the first three subprocesses (SP1) through (SP3) given above are incorporated into an attrition-rate coefficient such as $A_{ij}$ as follows (<u>see</u> also Section 7.7):

$$A_{ij} = \psi_{ij} I^Y_{ij} a_{ij} , \qquad (5.16.8)$$

where $\psi_{ij}$ and $a_{ij}$ are as defined after equation (5.16.3) and (5.16.4), and $I^Y_{ij}$ denotes the <u>intelligence factor</u> (the fraction of those $\dot{Y}_j$ allocated against $X_i$ who are actually engaging live $X_i$ target types). This intelligence factor, however, has not been considered in any applications at least through 1975 (see CHERRY [21, p. 7]), i.e. $I^Y_{ij} = 1.0$ for all $i$ and $j$ and hence (5.16.8) reduces to (5.16.6). A submodel based on target-acquisition considerations is used to calculate the allocation factors $\psi_{ij}$. The procedure used in the original version of BONDER/IUA is similar to that used in AMSWAG and discussed below[51]. In the AIRCAV and BLDM models the factors were calculated based on parallel acquisition of targets[52] and a target-prioity list (inwhich more than one type of target was allowed to be tied at the same level of priority to a firing weapon-system type). In actual computation,

an algorithm based on a simplifying approximation was used to compute numerical values for such allocation factors (see [85, pp. 29-32] or [5, pp. III-6 through III-8]).

In the AMSWAG [41] model attrition-rate coefficients are modelled as

$$A_{ij} = \psi_{ij} U_j a_{ij} , \tag{5.16.9}$$

where $U_j$ denotes the fraction of the firer-type $Y_j$ that are unsuppressed. Submodels are used for

(a) the suppression factor $U_j$ [41, pp. 15-17],

and (b) the fire-allocation factor $\psi_{ij}$ [41, pp. 18-21].

We will now discuss in detail the fire-allocation submodel used in AMSWAG.

The following factors influence which target types will be engaged by a particular firer type in AMSWAG and what allocation of fire they will receive[53]

(F1) target-type priority,

(F2) range to target,

(F3) intervisibility,

(F4) round choice,

and (F5) target-type acquisition.

In AMSWAG each firing weapon-system type has its own target priority scheme which allows different target types to have the highest priority at various ranges. An example of one such firer-type target-priority scheme is shown in Figure 5.14. It is assumed that a firer type will attempt to allocate its firepower against the enemy target type currently having the higher priority, with the closest target not necessarily having the highest priority (see Figure 5.14). However, if two potential targets are of the same type, the one at the shortest range always has the higher priority. Besides being an important factor in target priority, the range (distance) between firer and target also determines firing feasibility, i.e. no firing event can take place beyond the specified maximum effective range of the firing weapon-system type. Moreover, no target (regardless of priority or proximity) can receive any fire allocation if line of sight from the firer to that particular target (i.e. intervisibility) does not exist. However, if line of sight does exist, the fact that a target is seen either partially exposed or fully exposed does not affect either the target's priority or its allocation.

The availability of ammunition of the appropriate type also influences the allocation of fire in AMSWAG: a proper round choice must exist before a firer type can allocate its fire against a particular target types. Round choice is modelled for each firer-type--target-type combination by a table of first and second choices of rounds at both short and long ranges, plus a threshold range used to determine whether the current firer-target range will be classified as either short or long (see Table 5.VI). If for some reason the first choice of round type cannot be fired, the model tries to carry out the firing event with the second-choice round type. If neither round type can be fired, the target type receives no allocation of fire

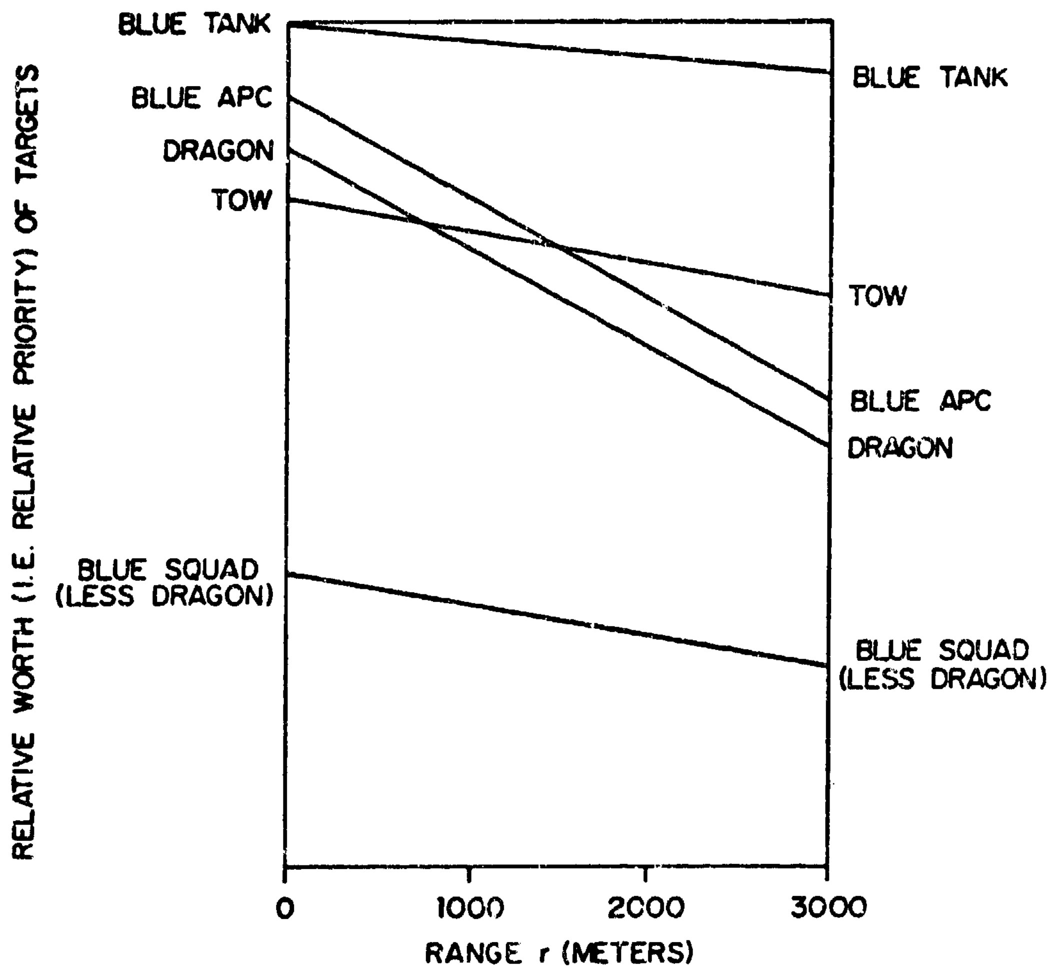


Figure 5.14. Typical target-type priorities used in AMSWAG for a BMP firer in Europe with Blue on the attack (from [41]).

during this time interval. [Here the term time interval refers to the fact that the battle has been segmented into a large number of small time steps (i.e. intervals) for computational reasons as per the numerical integration of the LANCHESTER-type attrition equations (see Appendix E, especially Figure E.1).] Currently in AMSWAG, there are two reasons why a particular round type might not be used: (1) the particular firer type does not have available that type of round, and (2) the firer is moving and that type of round cannot be fired from a moving platform. Thus, a target type will receive an allocation of fire only when all the following conditions have been met:

($\bar{C}1$) the firer type has not allocated more than ninety-eight percent of its firepower;

($\bar{C}2$) the target type is the highest priority target type that has not already received an allocation;

($\bar{C}3$) the target type is within the maximum effective range of the firer type;

($\bar{C}4$) line of sight exists;

and ($\bar{C}5$) a proper choice of round type exists.

Finally, target-acquisition probabilities determine in the following way exactly what the allocation by a firer type against a particular target type will be when all the above conditions have been met. The cumulative

TABLE 5.VI. Sample Round Choices Used in AMSWAG (from [41]).

| Firer Type | Target Type | First Choice at Short Range | First Choice at Long Range | Second Choice at Short Range | Second Choice at Long Range | Threshold Range (in meters) Used to Distinguish between Short and Long Range |
|---|---|---|---|---|---|---|
| M60A3 | BMP | HEAT | APDS | APDS | HEAT | 1500 |
| M60A3 | SQUAD | COAX | HEP | HEP | HEAT | 1000 |
| MICV | ATGM | COAX | HE | HE | COAX | 1000 |
| BMP | MICV | 73mm HEAT | SAGGER | 73mm HEAT | SAGGER | 800 |
| M113/TOW | T62 | TOW | TOW | TOW | TOW | 1250 |

detection probability for each firer type (say the $i^{\underline{th}}$) against each target type (say the $j^{\underline{th}}$) is computed at each time step since the existence of intervisibility. If we let $p_{ij}$ denote this cumulative detection probability, then in such an "expected-value" model as AMSWAG $p_{ij}$ is interpreted as representing the fraction of the $i^{\underline{th}}$ firer type that has detected the $j^{\underline{th}}$ target type. Then the fraction of fire allocated by the $i^{\underline{th}}$ firer type against the $j^{\underline{th}}$ target type cannot exceed $p_{ij}$ times the unallocated portion of the firer type's fire. A firer type continues to allocate its fire until it runs out of target types or has allocated more than ninety-eight percent of its firepower (see HAWKINS [41, p. 21] for further details).

In IDAGAM [1; 67] attrition-rate coefficients are also modelled by (5.16.6), but completely different submodels are used to compute the "inherent" single-firer kill rate $a_{ij}$ and the allocation factor $\psi_{ij}$ than are used in BONDER/IUA. The "inherent" single-firer kill rates are computed according to the heterogeneous-force version of (5.2.4) (but at a much lower level of resolution than that of a fire fight considered in BONDER/IUA), while a submodel based on the concept of a "standard force" (see SHUPACK [67, pp. 45-49] for further details) is used to determine the allocation factors. These are computed, for example, for the $Y$ force by

$$\psi_{ij} = \frac{\psi_{ij}^{SF}\left\{\dfrac{x_i(t)}{x_i^{SF}}\right\}}{\sum_{i=1}^{m} \psi_{ij}^{SF}\left\{\dfrac{x_i(t)}{x_i^{SF}}\right\}}, \qquad (5.16.10)$$

where $x_i^{SF}$ denotes the number of $X_i$ weapons in a standard force, $\psi_{ij}^{SF}$ denotes the fraction of $Y_j$ weapons that would fire at $X_i$ targets if $X$

were the standard force, $x_i(t)$ denotes the number of $X_i$ weapons in the sector (or geographical region of interest), and the summation extends over all weapon-system types in the sector. Thus, the allocation factors used in IDAGAM are internally computed based on what the allocation of fire by each weapon-system type in the given force would be against each opposing weapon-system type in a standard force and corrected by the relative force compositions. Both $x_i^{SF}$ and $\psi_{ij}^{SF}$ are externally determined and are inputs into IDAGAM. Thus, the fraction of fire allocated by a weapon-system type against an enemy weapon-system type in an opposing force is roughly proportional to what would be allocated against the standard force but modified by the relative force composition of the actual opposing force. The denominator of (5.16.10) insures that $\sum_{i=1}^{m} \psi_{ij} = 1.0$.

As noted above, both the COMAN model [24; 25] (and its derivatives COMANEX [36] and COMANEW [39]) and the VECTOR series of models [28; 54; 86; 87] (in particular, VECTOR-2) use the conditional single-firer kill rate $\alpha_{ij}$ to calculate the attrition-rate coefficient $A_{ij}$. COMANEX [73] considers target priorities and computes attrition-rate coefficients according to

$$A_{ij} = (p_X)^{x_i^H} \left\{ 1 - (p_X)^{x_i} \right\} \alpha_{ij} , \tag{5.16.11}$$

where

$$p_X = P \left[ \begin{array}{c} \text{a specific } X \text{ target is unacquired} \\ \text{by an individual } Y \text{ firer} \end{array} \right] ,$$

$x_i$ denotes the number of $X_i$ targets, and $x_i^H$ denotes the number of surviving $X$ weapon-system types of higher priority than $X_i$. Let us now introduce $S_i$ denoting the set of indices of all target types having a

higher priority than $X_i$. It follows that $x_i^H = \sum_{k \in S_i} x_k$. The parameters $p_X$ and $\alpha_{ij}$ are calculated as maximum-likelihood estimates from simulated combat data generated by a high-resolution Monte Carlo simulation such as CARMONETTE [36] (see Section 5.15 for further details). For a closely related alternative approach, see THRALL et al. [82, pp. 99-104]. COMANEW computes attrition-rate coefficients according to [cf. (5.16.6) above]

$$A_{ij} = \psi_{ij} \alpha_{ij} , \tag{5.16.12}$$

where both factors (i.e. $\psi_{ij}$ and $\alpha_{ij}$) are estimated from simulated combat data by the maximum-likelihood method (see [39] for further details).

VECTOR-2 [28; 54] also considers the conditional single-firer kill rates $\alpha_{ij}$ and uses different formulas to compute the attrition-rate coefficients $A_{ij}$ according to whether the target-acquisition process is done in series with or in parallel with the killing of acquired targets[54]. Thus, the two major factors determining the numerical value of an attrition-rate coefficient in VECTOR-2 are

(F1) the acquisition and selection of targets,

and (F2) the conditional single-firer kill rate against acquired target types, $\alpha_{ij}$.

The acquisition and selection of targets in VECTOR-2 is conceptualized as consisting of the following three processes:

(P1) the line-of-sight process, which determines whether a given target type is visible or not to a particular firer type,

(P2) the target-acquisition process, which determines the time for a firer type to acquire a particular target type,

and (P3) the target-section process, which represents how a particular target type is selected for engagement from among those acquired.

The interaction of these three processes depends on whether target acquisition is done in series or in parallel. In both cases each firer type orders all opposing enemy target types into a priority list, which the model uses to determine which target types are to be engaged first.

In serial acquisition in VECTOR-2 the acquired target type of highest priority is engaged by a particular firer type until it has been destroyed or until line of sight has been lost. At this time the serial acquirer must acquire a new target. Moreover, past acquisitions are not remembered by the serial acquirer. Also, in searching for a new target, the timeliness of acquisition is given consideration through a series of search-cutoff times. When there are $m$ target types, the selection of the next target type involves a sequence of $(m-1)$ search-cutoff times. Prior to the $k^{\underline{th}}$ cutoff time (where $k < m$), the observer looks for only target types of priorities 1 through k and ignores any lower priority targets. If the observer has not acquired a target by the $(m-1)^{\underline{st}}$ cutoff time, he will

then engage the first target acquired (regardless of its priority). Once a target is acquired in serial acquisition, it cannot be preempted by a higher priority target, and only its destruction or loss of line of sight can cause fire to be shifted away from it. In parallel acquisition search for new targets continues even during the engagement of acquired targets. When the target has been destroyed, a higher priority target type has been acquired, or line of sight has been lost; fire is instantaneously shifted to the highest priority acquired enemy target type. A parallel acquirer does remember all past target-type acquisitions. It should be noted here that these two different conceptual models of target acquisition lead to two completely different expressions for the LANCHESTER attrition-rate coefficient: the attrition-rate coefficient for serial acquisition may be developed using the mean-first-passage-time result given in Section 5.9 for a continuous-time-semi-MARKOV process, while that for parallel acquisition may be developed by straightforward probability arguments.

The following is a summary of the assumptions made in VECTOR-2 concerning target-type acquisition and selection in maneuver-unit combat [28; pp. 53-54]:

(A1) the time to acquire a target, given that it is continuously visible, is an exponentially distributed random variable with parameter $\lambda_{ij}$, where $i$ is an index denoting the weapon-system type of the target and $j$ is an index denoting the weapon-system type of the firer;

(A2) the line-of-sight process between a pair of opposing weapon-system types is an alternating MARKOV process with two states --visible and invisbile;

(A3) the line-of-sight process for an observer-target pair is independent of that for all other pairs;

(A4) there are two modes of acquiring targets; an observer using the parallel mode acquires targets continuously, even while engaging other targets; an observer using serial acquisition can acquire only between engagements of targets;

(A5) when an observer in the parallel mode acquires a target of higher priority than the one being engaged, he shifts his fire instantanteously to the target of higher priority;

and (A6) an observer in the serial mode selects a new target whenever he loses line of sight to the previous target or the previous target is killed (the model assumes that the firer can perfectly distinguish between active and killed weapon systems and never engages killed systems); there is a sequence of cutoff times to limit the time spent searching for certain target types, such that prior to the $n^{\underline{th}}$ cutoff time only weapon-system types of priorities 1 through $n$ are eligible as targets.

Thus, the target-acquisition-and-selection process transforms a $Y$ weapon-system type's (say the $j^{\underline{th}}$) kill rates against acquired $X$ target types ($\alpha_{ij}$ for $i = 1,2,\ldots,m$) into an achieved kill rate against a particular enemy target type (say the $i^{\underline{th}}$) $A_{ij}$ that accounts for target priorities and the various competing activities in which a single firer may be engaged over time. Moreover, the amount of attrition actually assessed against a force is limited by a tactically acceptable maximum attrition rate (<u>see</u> [28, pp. 54-55] for further details). We will now give attrition-rate-coefficient results for the two cases

(CA1) serial acquisition of targets,

and (CA2) parallel acquisition of targets.

For the former case (CA1), it is additionally assumed for the derivation of an expression for $A_{ij}$ that the time to kill an acquired target is exponentially distributed [with parameter $\alpha_{ij}$, where $i$ is an index denoting the weapon-system type of the target (here $X_i$) and $j$ is an index denoting the weapon-system type of the firer (here $Y_j$)]. Also, in VECTOR-2 the maximum number of weapon-system types in a maneuver element is currently 11, i.e. within a homogeneous portion of the battlefield $m = n = 11$ where $m$ and $n$ are X- and Y-force integer index limits appearing in (for example) summations below.

For <u>serial acquisition of targets</u> in VECTOR-2, the heterogeneous-force LANCHESTER attrition-rate coefficient $A_{ij}$ is taken to be given by

$$A_{ij} = \frac{h_{ij} P_{ij}}{\sum_{k=1}^{m} E[T_{kj}^{as}] + \left\{ \frac{1}{\alpha_{kj} + \mu_{ij} + \tilde{A}_{kj}} \right\}} \tag{5.16.13}$$

where

$$h_{ij} = P \begin{bmatrix} \text{a group-i target (here } X_i) \text{ being fired upon a acquired} \\ \text{by a group-j firer (here } Y_j) \text{ will be destroyed by that} \\ \text{firer before either line of sight is lost or the target} \\ \text{is destroyed by another firer.} \end{bmatrix},$$

$$P_{ij} = P \begin{bmatrix} \text{a group-j weapon which employs serial acquisition acquires} \\ \text{and selects a group-i target type when it selects a target} \end{bmatrix},$$

$E[T_{ij}^{as}]$ = expected time on a given acquisition that a group-j weapon spends acquiring and selecting a group-i target [here $T_{ij}^{as} = 0$ if the acquisition is of a non-group-i target; also if $T_{ij}^{as} > 0$ for some i, then $T_{ij}^{as} = 0$ for all other i],

$\frac{1}{\alpha_{ij}}$ = expected time that a group-j weapon firing at a group-i target requires to achieve a kill, i.e. the single-firer weapon-system kill rate against an acquire target [it should be recalled that the corresponding time to achieve a kill (a r.v.) has been assumed to be exponentially distributed with parameter $\alpha_{ij}$],

$\frac{1}{\mu_{ij}}$ = expected time that a weapon system in group i spends in the visible state (for a weapon in a group j) each time that it enters that that state [it is assumed that the corresponding time (a r.v.) is exponentially distributed with parameter $\mu_{ij}$],

$\frac{1}{\eta_{ij}}$ = corresponding value for the invisible state,

and $\frac{1}{\tilde{A}_{ij}}$ = expected time for any firer other than the single group-j firer in question to kill a particular target in group i.

In somewhat simpler words, $P_{ij}$ denotes the <u>selection probability</u> of an $X_i$-type target by a $Y_j$-type firer, and $h_{ij}$ denotes the corresponding <u>destruction probability</u>. The above expression (5.16.13) was developed by taking the LANCHESTER attrition-rate coefficient to be the reciprocal of the expected time to kill a target and then by involing BARLOW's [4] mean-first-passage-time result for a continuous-time semi-MARKOV process [e.g. <u>see</u> (5.9.13)], and consequently in VECTOR-2 the target-destruction process has been conceptualized in such a way that this latter result could be invoked (<u>see</u> [28, pp. 55-67] for further details). We will now give expressions for all the remaining computed quantities in (5.16.13) (again, <u>see</u> [28] for further details). Accordingly, we have

$$h_{ij} = \frac{\alpha_{ij}}{\alpha_{ij} + \mu_{ij} + \tilde{A}_{ij}}, \tag{5.16.14}$$

and

$$P_{IJ}$$

$$= D_{IJ}(t^{CO}_{I-1,J}) \{ \prod_{i=1}^{I-1} \bar{D}_{iJ}(t^{CO}_{i-1,J})\} \exp \{ - \sum_{i=1}^{I-1} R_{iJ}N_{iJ}[t^{CO}_{I-1,J} - t^{CO}_{i-1,J}]\}$$

$$+ R_{IJ}N_{IJ} \sum_{\ell=I-1}^{m-1} \{ \prod_{k=1}^{\ell+1} \bar{D}_{k,J}(t^{CO}_{k+1,J})\} \exp \{ \sum_{k=0}^{\ell} R_{k+1,J}N_{k+1,J}t^{CO}_{RJ}\}$$

$$\times \frac{1}{Z_{\ell J}} \{[\exp(-Z_{\ell J}t^{CO}_{\ell J})] - [\exp(-Z_{\ell J}t^{CO}_{\ell+1,J})]\} , \tag{5.16.15}$$

where

$$D_{IJ}(t) = P\left[\begin{array}{l}\text{observer in group J (here } Y_J\text{) has a target in group I}\\ \text{(here } X_I\text{) under surveillance at time t after initial of search}\end{array}\right],$$

$$\bar{D}_{IJ}(t) = 1 - D_{IJ}(t) ,$$

$t_{IJ}^{CO}$ = cut-off time for an observer in group J searching for targets to exclusively engage acquired targets of priority classes 1 through I (i.e. a target of priority class I + 1 will not be engaged in acquired before $t_{IJ}^{CO} < t_{I+1,J}^{CO}$) (see Table 5.VII; also KARR [47, pp. 32-33]),

$N_{IJ}$ = expected number of currently surviving group-I targets within range of a group-J firer,

$$R_{IJ} = \frac{\lambda_{IJ}\eta_{IJ}}{\eta_{IJ} + \mu_{IJ}} , \qquad (5.16.16)$$

$\frac{1}{\lambda_{IJ}}$ = expected time for a weapon in group J (here $Y_J$) to detect a visible target in group I (here $X_I$) [it should be recalled that the corresponding time to detect (a r.v.) has been taken by assumption (A1) to be exponentially distributed with parameter $\lambda_{IJ}$] ,

and

$$Z_{\ell J} = \sum_{k=1}^{\ell+1} R_{kJ}N_{kJ} . \qquad (5.16.17)$$

Here the two conventions have been followed that (1) a summation over an empty index set is always taken to be equal to zero, and (2) a product taken over an empty index set is always taken to be equal to one, e.g. $\sum_{k=1}^{0} T_k = 0$ and $\Pi_{k=1}^{0} T_k = 1$. Also, the complement of a cumulative distribution function

TABLE 5.VII. Rules for Target Selection by Serial Acquirer in VECTOR-2.

| Time | Priorities of Targets to be Engaged Immediately Upon Acquisition | Priorities of Targets to be Engaged if Previously Acquired and Still Visible |
|---|---|---|
| $[0, t_{1J}^{CO}]$ | 1 | |
| $t_{1J}^{CO}$ | | 2 |
| $(t_{1J}^{CO}, t_{2J}^{CO})$ | 1, 2 | |
| $t_{2J}^{CO}$ | | 3 |
| $(t_{2J}^{CO}, t_{3J}^{CO})$ | 1, 2, 3 | |
| ⋮ | | |
| $(t_{m-2,J}^{CO}, t_{m-1,J}^{CO})$ | 1, 2, ... , m-1 | |
| $t_{m-1,J}^{CO}$ | | m |
| $(t_{m-1,J}^{CO}, +\infty)$ | 1, 2, ..., m-1, m | |

like (for example) $D_{IJ}(t)$ has been denoted as $\bar{D}_{IJ}(t)$, and we then (of course) have $\bar{D}_{IJ}(t) = 1 - D_{IJ}(t)$. Let us observe that $0 \leq N_{IJ} \leq x_I$. The target types have been indexed in such a way that $X_1$ denotes the highest priority target, $X_2$ denotes the next highest, etc. It remains for us to give an expression for $D_{IJ}(t)$ in order that $P_{IJ}$ as given by (5.16.15) may be computed: the following expression has been developed for $D_{IJ}(t)$ (see [28, pp. 62-63] for further details)

$$D_{IJ}(t)$$

$$= 1 - \left[1 - \frac{R_{IJ}}{\mu_{IJ} + \tilde{A}_{IJ} - R_{IJ}} \{\exp(-R_{IJ}t) - \exp[-(\mu_{IJ} + \tilde{A}_{IJ})t]\}\right]^{N_{IJ}}. \quad (5.16.18)$$

Returning now to the computation of the LANCHESTER attrition-rate coefficient $A_{ij}$ by (5.16.13), we see that it remains for us to give expressions for the expected time to acquire and select a target $E[T_{IJ}^{as}]$ and the single-firer kill rate of $X_i$-type targets by other than $Y_j$-type firers $\tilde{A}_{ij}$. The following expression has been developed for $E[T_{IJ}^{as}]$ (see [28, pp. 65-66] for further details)

$E[T_{IJ}^{aa}]$

$$= t_{I-1,J}^{CO}\, D_{IJ}(t_{I-1,J}^{CO}) \left\{\prod_{i=1}^{I-1} \bar{D}_{iJ}(t_{i-1,J}^{CO})\right\} \exp\left\{\sum_{i=1}^{I-1} R_{iJ}N_{iJ}[t_{I-1,J}^{CO} - t_{i-1,J}^{CO}]\right\}$$

$$+ R_{IJ}N_{IJ} \sum_{\ell=I-1}^{m-1} \left\{\prod_{k=0}^{\ell} \bar{D}_{k+1,J}(t_{kJ}^{CO})\right\} \exp\left\{\sum_{k=0}^{\ell} R_{k+1,J}N_{k+1,J}\, t_{kJ}^{CO}\right\}$$

$$\times \frac{1}{Z_{\ell J}^2} \left\{(Z_{\ell J}t_{\ell J}^{CO} + 1)\exp(-Z_{\ell J}t_{\ell J}^{CO}) - (Z_{\ell J}t_{\ell+1,J}^{CO} + 1)\exp(-Z_{\ell J}t_{\ell+1,J}^{CO})\right\}. \quad (5.16.19)$$

Finally, the following approximation has been developed for $\tilde{A}_{ij}$ and is used in VECTOR-2

$$\tilde{A}_{ij}(t + \Delta t) = \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} A_{i\ell}(t)\, f_{\ell}^{j}(t) \ , \quad (5.16.20)$$

where

$f_{\ell}^{j}(t) = y_{\ell}(t)/\{\sum_{k=1, k\neq j}^{n} y_k(t)\}$ = fraction of total $Y$ weapons exclusive of group $j$ that $Y$ weapons of group $\ell$ comprise.

Here, the fact that the differential-equation force-on-force attrition model is numerically integrated by discretizing time into time steps (<u>see</u> Appendix E) has been used to develop this approximation, with the right-hand side of (5.16.20) being evaluated at the old time step and the left-hand side being taken at the new one. In way of summary, <u>the computation of $A_{ij}$ for weapons that employ serial acquisition requires the following inputs</u>: $\alpha_{ij}$, $\mu_{ij}$, $\eta_{ij}$, $\lambda_{ij}$, $N_{ij}$, $y_j$, and $t_{ij}^{CO}$.

The interested reader can find the derivation of the above serial-acquisition attrition-rate-coefficient results sketched in [28, pp. 55–68]

(see also KARR [47, pp. 38-44]). It will be instructive, however, for us to briefly consider the development of the expression (5.16.15) for $P_{IJ}$, the probability of selecting a target from target-type group I. This probability is given by

$$P_{IJ} = D_{IJ}(t^{CO}_{I-1,J}) \prod_{i=1}^{I-1} \{\bar{F}_{T^a_{iJ}}(t^{CO}_{I-1,J} - t^{CO}_{i-1,J})\, \bar{D}_{iJ}(t^{CO}_{i-1,J})\}$$

$$+ \sum_{\ell=I-1}^{m-1} \int_{t^{CO}_{\ell J}}^{t^{CO}_{\ell+1,J}} \{\prod_{k=I}^{\ell} \bar{D}_{k+1,J}(t^{CO}_{kJ})\, \bar{F}_{T^a_{k+1,J}}(t - t^{CO}_{kJ})\}$$

$$\times \{\prod_{k=0}^{I-2} \bar{F}_{T^a_{k+1,J}}(t - t^{CO}_{kJ})\, \bar{D}_{k+1,J}(t^{CO}_{kJ})\}\, \bar{D}_{IJ}(t^{CO}_{I-1,J})\, dF_{T^a_{IJ}}(t - t^{CO}_{I-1,J}), \quad (5.16.21)$$

where

$T^a_{ij}$ = the time (a r.v.) for an observer in group i to acquire a target in group j, with cumulative distribution function $F_{T^a_{ij}}(t) = P[T^a_{ij} \le t]$.

The first term on the right-hand side of (5.16.21) represents the probability that a target in group I (here $X_I$) is under surveillance at time $t^{CO}_{I-1,J}$ and that no higher priority target was ever under surveillance at a time before $t^{CO}_{I-1,J}$ at which time it would have been engaged, while the second term represents the probability that a target in group I was acquired at some time t after $t^{CO}_{I-1,J}$ and that neither a higher priority target nor a lower priority one was ever under surveillance at a time before t at which time it would have been engaged. It follows from assumptions (A1) through (A3) above that

$$F_{T^a_{ij}}(t) = 1 - \exp(-R_{ij}N_{ij}t) \; , \tag{5.16.22}$$

whence substitution of (5.16.22) into (5.16.21) yields (5.16.15). The expression (5.16.19) for $E[T^{as}_{IJ}]$ may be developed in a similar fashion. Finally, it is worthwhile to observe here that $\eta_{ij}/(\eta_{ij} + \mu_{ij})$ gives the probability that a target of type i is visible. Recalling that $\lambda_{ij}$ denotes the rate of acquisition of a group-i target by a group-j observer, we then immediately see the justification of (5.16.22).

For <u>parallel acquisition of targets</u> in VECTOR-2, the heterogeneous-force LANCHESTER attrition-rate coefficient $A_{ij}$ is taken to be given by

$$A_{ij} = Q^{XY}_{ij} \, \alpha_{ij} \; , \tag{5.16.23}$$

where

$$Q^{XY}_{ij} = P\begin{bmatrix}\text{at a random point in time a given group-j (here } Y_j\text{) weapon} \\ \text{employing parallel acquisition is firing at a group-i (here} \\ X_i\text{) target.}\end{bmatrix} .$$

We further have that $Q^{XY}_{1j} = S^{XY}_{1j}$ and

$$Q^{XY}_{ij} = S^{XY}_{ij} \prod_{k=1}^{i-1} (1 - S^{XY}_{kj}) \qquad \text{for} \quad 2 \le i \le m \; , \tag{5.16.24}$$

with

$$S^{XY}_{ij} = 1 - \left[1 - \frac{\eta_{ij}\,\lambda_{ij}}{(\mu_{ij} + \eta_{ij})\,(\mu_{ij} + \lambda_{ij})}\right]^{N_{ij}} \; , \tag{5.16.25}$$

where

$$S_{ij}^{XY} = P\left[\begin{array}{l}\text{at a random point in time a group-i (here } X_i\text{) target is} \\ \text{available to a group-j (here } Y_j\text{) firer, i.e. a target is} \\ \text{available and has been detected.}\end{array}\right].$$

The above expression for $S_{ij}^{XY}$ has been by considering the alternating-renewal visibility process for a $X_i$-type target (see [28, pp. 68-70] for further details).

Finally, let us give a brief overview of the data-base requirements for computation of attrition-rate coefficients in VECTOR-2. Current values of the following parameters are required for the calculation of attrition-rate coefficients at each time step:

(P1) number of survivors in each weapon-system-type group;

(P2) conditional single-firer kill rate, $\alpha_{ij}$ or $\beta_{ji}$;

(P3) acquisition rate for each weapon-system type in each observing and observed group[55], $\lambda_{ij}^{XY}$ or $\lambda_{ij}^{YX}$;

(P4) rates for the alternating-MARKOV-renewal line-of-sight process, $\mu_{ij}$ and $\eta_{ij}$;

(P5) fraction of targets within range for every pair of firer type and target type;

and (P6) rate of fire for each weapon-system type.

The parameters (P1) are obtained from other parts of VECTOR-2, while (P6) is an external-user input. Parameters (P2) through (P5) are internally computed in the model. These computations involve more detailed input data from the following four classes (see [28, pp. 70-71] for further details):

(DC1) scenario data expressing differences in force employment (e.g. between armored, mechanized, and dismounted infantry units); such data reflect the initial geometry and maneuver patterns of forces and the making of such tactical decisions as, for example, when to mount and dismount infantry into APCs,

(DC2) movement data consisting of the speed of each weapon-system type (indexed on terrain trafficability),

(DC3) line-of-sight data consisting of the rates of entering and leaving the visible state in each of the terrain visibility classes,

(DC4) weapon-system-performance data (including the firing rate for each weapon-system type) used to compute the conditional single-firer kill rate, acquisition rate, and the fraction of the target group within range for each firer-type/target-type pair.

From the above brief sketch, the reader undoubtedly senses that the data-base requirements for VECTOR-2 are rather demanding. In fact, upwards of 350,000 pieces of input data are required for its running (see BONDER [16, p. 36]),

and many man-months of effort are involved in the use of this much data in such a complex operational model, e.g. the time required to acquire the input data, the time required to structure this data into the model's input format, the time required to run the model, and the time required to analyze and evaluate the model's results (see [6] for further details).

FOOTNOTES for Chapter 5

1. Methodology for the prediction of LANCHESTER attrition-rate coefficients from weapon-system performance characteristics has been developed by S. BONDER [11; 14] and others (namely, BARFOOT [3], BONDER and FARRELL [17], and KIMBLETON [49]). In particular, BONDER [11; 14] has given an analytical expression involving various weapon-system performance parameters for the so-called LANCHESTER attrition-rate coefficient, i.e. coefficient of the attrition rate for a $F|F$ process. This approach (in contrast to that of G. CLARK discussed in Footnote 2) does not involve the complimentary use of a high-resolution Monte Carlo combat simulation. Thus, we may say that we have a "freestanding" analytical model in the sense that it is complete in itself and does not require the complimentary use of a Monte Carlo simulation.

   Furthermore, RUSTAGI and SRIVASTAVA [62] have given results concerning the maximum-likelihood estimation of the MARKOV-dependent-fire parameters in BONDER's [11; 14] attrition-rate expression. Thus, experimental firing data can be used to generate maximum-likelihood estimates of the parameters in LANCHESTER attrition-rate coefficients and consequently of the coefficients themselves. However, these maximum-likelihood estimates require information about the outcome of each and every round in a sequence of firing trials. Consequently, RUSTAGI and LAITINEN [61] have given results for the moment estimation of the parameters, which is applicable when only partial information is available on the observed firing sequences.

2. Methodology for the maximum-likelihood estimation of LANCHESTER attrition-rate coefficients from Monte Carlo simulation output data

has been developed by G. CLARK [24]. His basic idea is to use a combat analysis model (COMAN Model) in conjunction with a high-resolution Monte Carlo combat simulation.

3. Unfortunately, the historical combat data base does not contain information about the times between casualties, which is needed for the basic estimation procedure (cf. Figure 5.1). Furthermore, it is unlikely that it ever will although such experimental data is recorded under simulated combat conditions by the U.S. Army Combat Developments Experimentation Command (CDEC) at Fort Ord, California. We must bear in mind, however, that CDEC data is not real combat data.

4. As usual, random variables are denoted by capital letters, while their realizations are denoted by the corresponding lower-case letters.

5. The reader should recall that these equations were also called in Section 2.12 Lanchester-type equations for a F|F attrition process.

6. If $N$, a r.v., denotes the number of rounds required to kill a target [i.e. $N$ denotes the trial on which the target is (first) killed], then

$$E[N] = \frac{1}{P_{SSK}} ,$$

when there is statistical independence between the outcomes of any two rounds fired, since the $N$ obeys a geometric probability law, i.e.

$$\text{Prob } [N = n] = P_{SSK} (1 - P_{SSK})^{n-1} ,$$

which is well known to yield an expected value of $1/P_{SSK}$ for $N$.

7. Although our basic idea for this justification is taken from BONDER and FARRELL [17], our development here differs from theirs in several essential aspects. For example, BONDER and FARRELL did not point out that (5.3.1) holds exactly for exponentially-distributed times between kills.

8. As pointed out previously by the author (<u>see</u> TAYLOR [81, p. 47]), this justification is not universally accepted and is apparently somewhat controversial. Moreover, there apparently has been some computational evidence against the appropriateness of (5.3.1).

9. Since we assume that $\lim_{\Delta t \to 0} P[X$ casualty in $(t, t + \Delta t)] = 0$, it follows that the expected number of $X$ casualties in $(t, t + \Delta t]$ is the same as in $(t, t + \Delta t)$.

10. A more appropriate taxonomy than that shown in Table 5.I would appear to this author to be

<u>Aiming Doctrine</u>

(1) "Aimed Fire"

(2) "Area Fire"

<u>Firing Doctrine</u>

<u>Lethality Mechanism</u>,

where <u>aiming doctrine</u> would refer to how aim points are selected, <u>firing doctrine</u> would refer to how consecutive rounds are related to one another (i.e. how they are correlated), and <u>lethality mechanism</u>

would be as defined by BONDER and FARRELL [17, pp. 86-87] (see main text above). Under aiming doctrine, "aimed fire" would refer to the situation in which fire is aimed at particular targets, while "area fire" would refer to the situation in which fire is directed into only the general area thought to contain targets (see Section 5.13 for further discussion). BONDER and FARRELL's classification of firing doctrine would be retained, except that the term firing doctrine would now refer to how consecutive rounds are related to one another. As a colleague (LTC Richard S. Miller, USA, of the Naval Postgraduate School) has pointed out, weapon systems with an area-lethality mechanism (see main text) almost always engage their targets with "area fire." Thus, for weapon systems with an impact-lethality mechanism (again, see main text), one might be tempted to omit the aiming-doctrine portion of the above proposed alternate taxonomy. However, weapon systems with an impact-lethality mechanism frequently are fired in the "area-fire" mode, for example, when engaging very poorly located targets (cf. VON NEUMANN [88] or WEISS [90]).

11. Strictly speaking, the lethality mechanism of a weapon system's projective also depends on the target's vulnerability, and consequently we should speak of the weapon-target damage mechanism (see SNOW and RYAN [71, p. 5] or WEISS [89, p. 7] for a further discussion). It is convenient, however, to simply refer to this as the weapon's damage (or lethality) mechanism. Furthermore, terminology is far from uniform in this field, and different authors frequently use the same word with quite different meanings. For example, the U. S. Army's Engineering Design Handbook [84, p. 15-9] says that

"vulnerability is ordinarily a term used for the case where actual hits are obtained on targets such as tanks and aircraft. Lethality, on the other hand, refers primarily to the case where lethal or incapacitating fragments, for example, are projected over an area on the battlefield to incapacitate personnel." This terminology should be compared with that used by BONDER and FARRELL [17, pp. 86-87] and also with that used by us above. Moreover, SNOW and RYAN [71, p. 2] classify targets as being either (1) impact sensitive, or (2) fragment sensitive; and projectiles are usually classified as being either (1) nonfragmenting, or (2) fragmenting. The weapon/target-damage-characteristics taxonomy outlined in the preceding sentence would yield a four-fold classification scheme for weapon/target-damage mechanisms (e.g. a fragmenting projectile fired against an impact-sensitive target [an example of which would be an artillery shell fired against tanks]).

12. Other categories of weapon-system types have been analyzed and other expressions for the LANCHESTER attrition-rate coefficient developed by BONDER and FARRELL [17] (<u>see</u> also Table 5.I).

13. However, we give below in Section 5.8 an expression that applies under even more general conditions: namely, (C1) identical probability distributions for the number of rounds to achieve the $i^{\underline{th}}$ hit for $i \geq 2$, and (C2) any arbitrary distribution for the number of hits to kill.

14. As noted above in Section 5.3, BONDER [11] originally took the LANCHESTER attrition-rate coefficient to be given by $E[1/T]$, e.g. $a = E[1/T_{XY}]$. Subsequently, BARFOOT [3] has suggested that a more appropriate expression for the LANCHESTER attrition-rate coefficient under constant battle conditions (e.g. at a constant range) is the so-called harmonic mean of the rate at which a single firer kills enemy targets, i.e. $1/E[T]$. This latter definition is in better consonance with our introductory comments made in Section 5.1 (cf. (5.1.3)). BARFOOT based his arguments for the use of $1/E[T]$ on the fact that the harmonic mean of a set of rates yields a more appropriate estimate of the average rate than does the arithmetic mean (see Section 5.3 above and [3, p. 894]). It should also be pointed out that the definition of, for example, $a$ as $E[1/T_{XY}]$ is not analytically tractable (i.e. explicit analytical results apparently are not obtainable and numerical methods must be employed) whereas the definition of $a$ as $1/E[T_{XY}]$ does yield explicit analytical results. Thus, the harmonious mean of the rate of target attrition is superior on both theoretical and also computational grounds to the arithmetic-mean attrition rate in LANCHESTER combat theory.

15. In (5.4.1) all the subscripted event times, e.g. $t_a$, are taken to be fixed deterministic quantities. We show below in Section 5.8 that (5.4.1) also holds for the average values of these times taken to be random variables. Although this result is intuitively obvious, its proof has not apparently heretofore appeared anywhere, and we have used a simple new approach to prove this important result.

16. We will show in Section 5.8 below that (5.4.1) also holds for the average values of these times taken to be random variables (see also Footnote 15 above).

17. By saying logical analysis, we are emphasizing here that there has not been any empirical verification of BONDER's model for the LANCHESTER attrition-rate coefficient. Furthermore, considering the nature and quality of available historical combat data (see McQUIE [51], McQUIE et al. [52], or HUBER, LOW, and TAYLOR [45]; also see Section 7.21 below), such empirical verification against real combat data does not seem to be possible.

18. Originally BONDER [11] tried to compute $E[1/T]$ (see Footnote 14 above). Here we have taken the liberty of integrating together the ideas of BONDER [11] and BARFOOT [3] (e.g. see BONDER [14] or BONDER and FARRELL [17]).

19. The reader should recognize this decomposition as an application of the general modelling principle of factoring a complex system problem into simpler problems (see MORRIS [55, p. B-711] for a further discussion).

20. Here we are again following BONDER's [11] (see also BONDER [12, pp. III-4 through III-11]) development based on determination of the distribution of the number of rounds required to obtain $z$ hits $p_{N|Z}(n|z)$. A more general result that reduces to the distribution of the number of rounds to obtain $z$ hits (5.5.27) was developed

earlier by GNEDENKO [38, pp. 138-139] (*see* also RUSTAGI and SRIVASTAVA [62, p. 1223] and RUSTAGI and LAITINEN [61, pp. 918-919]). In Section 5.6 below we present a simpler, more general approach that does not involve determination of this complicated distribution.

21. Although justification of this important result, which is a special case of our more general result (5.8.1), apparently appears here for the first time, the statement of an equivalent result does appear elsewhere (e.g. *see* [28, p. 171] or [54, p. 165]). However, no proof of this result is given in [28] or [54], but in such places the reader is referred to BONDER and FARRELL [17] for its development. The author, however, could not find any such development in [17], only a development for deterministic event times (*cf*. Section 5.5 above) and an accompanying statement that when the event times are random variables, "expressions for the LAPLACE-STIELTJES transform of the time to kill may be obtained" [17, p. 132] (*see* also KIMBLETON [49, p. 704]).

22. For a critique of the determination of attrition-rate coefficients in VECTOR-2 (which is essentially the same as that in VECTOR-0 and VECTOR-1), see KARR [47, pp. 31-47], who has discussed their development in terms of "an important limit theorem for semi-MARKOV processes (*cf*. ÇINLAR [23, Theorems (10.4.3) and (10.5.22])." KARR [47, p. 39] has pointed out that except for this limit theorem, none of the results given in ÇINLAR [22; 50] are required for such developments.

23. So far our discussion has more or less paralleled that given by FARRELL [17, pp. 136-137]. We now will depart from FARRELL's development by expressing results in terms of the ratios of stationary probabilities $r_j = \pi_j/\pi_1$.

24. Here we mean that target-type attrition occurs at a rate proportional to the product of the numbers of firers and targets (cf. the convention adopted in Section 2.12 for two-sided LANCHESTER-type attrition processes).

25. See WEISS [89, pp. 708]. See also HAYWARD [42] for a very closely related discussion in a slightly different context. HAYWARD has proposed the organization of variables upon which combat effectiveness depends into three categories: those that relate to (C1) capabilities, (C2) environment, and (C3) mission.

26. An early discussion of such a model with range-dependent attrition-rate coefficients appears in WEISS [91, p. 88]. WEISS's model was apparently later elaborated upon by BONDER [9].

27. See Footnote 11 above.

28. Strictly speaking, the vulnerability of a target also depends on the nature of the attacking weapon system's projectile (for further details, see [84, p. 15-2] and also Footnote 11 above).

29. The explicit statement of this approach apparently first appeared in BONDER [11, p. 231], although it had appeared implicitly in earlier work by WEISS [89; 91].

30. In actuality (as discussed in Footnote 11 above), the lethal area also depends on the target's vulnerability and this may change over time. Consider, for example, artillery being fired at dismounted-infantry troops. For modelling purposes, the lethal area of an artillery round is usually taken to depend on the posture (e.g. standing, kneeling, prone, or in foxholes) of the infantry soliders, and this may change over time (see [84, pp. 15-9 through 15-13] for further details; also BONDER and FARRELL [17, pp. 154-155]).

31. The formula (5.13.6) given in the main text is readily modified if the region occupied by the $X$ targets does not coincide with the region perceived by the $Y$ firers to contain them and into which their fire is directed. Furthermore, one must then consider the probability that a round fired at the perceived region lands in the region actually containing the $X$ targets.

32. This concept goes back at least to WEISS [90, p. 6]. It underlies essentially all analytical computations of the expected number of kills for indirect-fire weapons (e.g. artillery), although it is usually not explicitly stated (e.g. see GRUBBS [40, p. 1022]). For a lucid explicit statement of this precept, see McNOLTY [50, p. 1028].

33. By considering TAYLOR's formula with LAGRANGE's form of the remainder (e.g. see COURANT and JOHN [29, pp. 446-449]), one can readily show that for $x \in [0, a]$ with $0 < a < 1$

$$\ln(1 - x) = -x - R ,$$

where $0 \leq R \leq (1/2)\,\{a/(1-a)\}^2$. It follows that $-x$ is a good approximation to $\ln(1-x)$ when $x \in [0, 0.2]$, with a maximum error not greater than $1/32$ at $x = 0.2$. However, by considering the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, one can show that $R(x) = \int_0^x \frac{u\,du}{1-u}$, which yields the better error bound $0 \leq R \leq (1/2)a^2/(1-a)$. Hence, for $x \in [0, 0.2]$ the maximum error occurs at $x = 0.2$ and is actually not greater than $1/40$.

34. However, BONDER and FARRELL's [17, pp. 141-162] development is quite different than that given here. The use of the fundamental precept of target coverage and how it is related to "area" fire is never explicitly mentioned by them.

35. These assumptions are taken from BONDER and FARRELL [17, pp. 143-144 and p. 149] (see also [54, pp. 169-170] or [28, pp. 175-176]).

36. The corresponding expression used in models built by VECTOR RESEARCH, INC. such as, for example, VECTOR-2 apparently contains an algebraic error, since it does not simplify to the known result (5.4.1) for MARKOV-dependent fire when there is zero probability of a kill by a miss. Moreover, slightly different assumptions were taken to hold for this expression's development: namely, the time to fire being

the same on all subsequent rounds, and the probability of a kill given a miss on the first round taken to be not necessarily the same as the probability of a kill given a miss on any subsequent round (e.g. see [28, pp. 172-173] for further details). The latter assumption may be readily incorporated into our expression for the expected time to kill a target and (5.14.1) accordingly modified, but we have not presented these results here because they are so complex.

37. The first four modes (M1) through (M4) were explicitly given by BONDER and FARRELL [17, pp. 107-108], while the last is implicit in, for example, VECTOR-2 [28, pp. 174-175].

38. For simplicity, we have chosen to invoke the result for the expected time to kill a target for the case in which all the subscripted event times are deterministic (cf. Footnote 15 above). We could have, of course, chosen to particularize more general results, e.g., (5.8.2) which has random event times.

39. In reality, actual historical combat does not (and probably cannot) supply the required data inputs. Therefore, in practice one must use data generated either by combat field experimentation or by a high-resolution Monte Carlo combat simulation. Moreover, one must always bear in mind that such data is not real combat data and of unsubstantiated validity. However, in the combat-modelling business there unfortunately is no better data available than such simulated data (see McQUIE et al. [52] and HUBER, LOW, and TAYLOR [45] for further discussions).

40. Usually the cost of such use is only a very small fraction of that for the detailed (i.e. high-resolution) model. For example, COMANEX has been reported [73] to take only about 0.003 of the computer time required by CARMONETTE.

41. A simple model of target-type engagement based on the assumption that there is a constant probability that a specific enemy target is unacquired by an individual firer in a specific time interval was used by CLARK [24, pp. 156-158]. This assumption simplified considerably the expression obtained for the probability of engaging a particular enemy target type. Otherwise, concepts used in the analysis and modelling of priority queues (see, for example, MORSE [56, pp. 121-137] or SAATY [63, pp. 348-352; 64, pp. 231-242]), e.g. whether service for high-priority units is preemptive or nonpreemptive, must be used (as they are in, for example, VECTOR-2 [28]).

42. Here we mean a lucid simple example of the approach of using differential equations to model the force-on-force combat-attrition process.

43. Here we mean output data from DYNTACS (e.g. see [8] or [27]), which is a high-resolution Monte Carlo simulation of armored combat at battalion level.

44. The main changes were that the COMANEX model was deterministic and a matrix of target priorities (i.e. each weapon-system type had its own target-priority list) were used (see STOCKTON [73] for further details).

45. It is not assumed here that $B_{ji}$ is constant. In fact, for present purposes one need not make any assumption about the variables upon which $B_{ji}$ depends, i.e. no particular functional dependence is assume here.

46. Our list here follows the discussion of BONDER and FARRELL [17, pp. 15-16].

47. Throughout the rest of this section we will always focus on $A_{ij}$, with $B_{ji}$ being symmetrically determined.

48. Since equation (5.13.18) does not contain a time for target acquisition (i.e. it is implicitly assumed that the time to acquire a target is equal to zero), it applies to both $a_{ij}$ and also $\alpha_{ij}$.

49. At least to the extent that available literature and model documentation permit. As we have discussed in Chapter 1, documentation of any combat model (particularly its underlying methodology) is generally quite bad, and much work that is done is never documented for "external consumption" [44; 66] (see Footnotes 17 and 23 of Chapter 1 for further details). Furthermore, essentially all of the major developments reported in this section have never been published in the open literature.

50. The first place where such a fire-distribution model appears [although not explicitly in the form of (5.16.6)] is the remarkable RAND research memorandum by GIAMBONI, MENGEL, and DISHINGTON [37, pp. 3-4]

(see also MENGEL [53]). The first place where allocation factors explicitly in the form given by (5.16.6) have appeared is (to the best of this author's knowledge) in SISKA, GIAMBONI, and LIND [68, p. 12] and in the open literature in ISBELL and MARLOW [46, p. 76] (see also WEISS [91, pp. 94-95]).

51. SMOLER [69, pp. 10-11] has pointed out that both the detection and fire-allocation submodels in AMSWAG contain several features that are at variance with military experience and judgment. He has consequently proposed an alternative fire-allocation procedure [69, pp. 31-36].

52. For a detailed discussion of parallel acquisition, see the below discussion on VECTOR-2.

53. Our discussion here is drawn from HAWKINS [41]. Also, see Footnote 51 above.

54. See KARR [47, pp. 31-47] for a critique of the determination of attrition-rate coefficients in VECTOR-2, which in this respect is essentially the same as VECTOR-0 and VECTOR-1. See also Footnote 22 above.

55. Here $\lambda_{ij}^{XY}$ denotes the acquisition rate of a $Y_j$-type observer against $X_i$-type targets, while $\lambda_{ij}^{YX}$ denotes that of an $X_j$-type observer against $Y_i$-type targets. In our previous discussion of heterogeneous-force LANCHESTER attrition-rate coefficients above, e.g. <u>see</u> (5.16.16), it was not considered necessary to be absolutely precise, and for simplicity's sake we used the symbols $\lambda_{ij}$, $R_{ij}$, $t_{ij}^{CO}$, etc. without superscripts.

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## Chapter 6. HOMOGENEOUS-FORCE MODELS

### 6.1. Introduction

The classic LANCHESTER theory of combat assumed constant attrition-rate coefficients for its combat models[1]. A so-called attrition-rate coefficient (see Chapter 5) in such a model represents the fire effectiveness of a weapon-system type against a particular target type, i.e. its effective firepower. All the models that we have considered previously in this book have had constant attrition-rate coefficients. Time-dependent attrition-rate coefficients are used to model temporal variations in firepower on the battlefield. This chapter considers LANCHESTER-type combat between two homogeneous forces with temporal variations in each combatant's fire effectiveness.

In general, we may model such combat with the following LANCHESTER-type equations for $x, y > 0$ [the first equation, for example, becomes $dx/dt = 0$ for $x = 0$]

$$\begin{cases} \dfrac{dx}{dt} = -G(t,x,y) & \text{with } x(0) = x_0, \\ \\ \dfrac{dy}{dt} = -H(t,x,y) & \text{with } y(0) = y_0, \end{cases} \tag{6.1.1}$$

where $x(t)$ and $y(t)$ denote, resepctively, the $X$ and $Y$ force levels at time $t$. For cases of no replacements and withdrawals such as we will consider here, $G$ and $H$ are the attrition rates of the $X$ and $Y$ forces, respectively. As we have seen in Section 2.12 for constant attrition-rate coefficients, various different military situations have been hypothesized to yield different functional forms for the attrition rates $G = A(x,y)$ and $H = B(x,y)$. We will consider time-dependent versions of such attrition rates $A(x,y)$ and $B(x,y)$ in this chapter.

We emphasize analytical results[2] for obtaining insights into the dynamics of combat for the following three types of time-dependent attrition processes (<u>see</u> Section 2.12 for explanation of notation):

(P1) $F|F$,

(P2) $FT|FT$,

(P3) $(F+T)|(F+T)$.

Let us recall that, for example, an attrition-rate coefficient in a $F|F$ LANCHESTER-type model is different from and related to different physical quantities than one in an $FT|FT$ model. Moreover, the analytical results that we present here allow one to study these particular variable-coefficient models almost as easily and thoroughly as LANCHESTER's classic constant-coefficient ones.

S. BONDER's [4; 5; 7; 10] pioneering work on methodology for the evaluation of military systems (particularly mobile systems such as tanks, mechanized infantry combat vehicles, etc.) provides a motivation for interest in variable-coefficient, deterministic, LANCHESTER-type combat models such as we consider in this chapter. BONDER [6] has pointed out that in many cases (for example, in the case of mobile weapon systems) the validity of the assumption of constant attrition-rate coefficients is seriously open to question (<u>see</u> also BONDER [4; 5; 7]). Two significant LANCHESTER-theory developments of the 1960's that have generated interest in time-dependent attrition-rate coefficients have been the development of methodology for

(D1) the prediction of LANCHESTER attrition-rate coefficients from weapon-system-performance data by S. BONDER [6; 8] and others[3],

(D2) the (maximum-likelihood) estimation of such coefficients from Monte Carlo simulation output by G. CLARK [13].

Both these developments and others[4] have generated interest in variable-coefficient homogeneous-force models of the general form (6.1.1) and have facilitated their use (and that of corresponding heterogeneous-force models) in defense-planning studies.

How do temporal variations in each combatant's fire effectiveness affect the outcome of battle? When is the outcome significantly influenced (or even changed) by such temporal variations? These are important questions for the military operations research worker to answer. We will try to answer them (at least in a few specific cases) by considering several specific instances of a LANCHESTER-type combat model with time-dependent attrition-rate coefficients. Thus, we begin with a specific example of such a model, S. BONDER's model of a constant-speed attack on a static defensive position in which the fire effectiveness of each side's weapons is range dependent (i.e. it depends on the range between firer and target).

In this model, we will assume that both sides use "aimed" fire and target acquisition times are negligible. Consequently, we will model attrition as a variable-coefficient F|F LANCHESTER-type process (i.e. use variable-coefficient LANCHESTER-type equations of modern warfare). Consideration of this model will (1) suggest several classes of time-dependent attrition-rate coefficients that are of tactical interest, and (2) show that temporal variations in such coefficients may have a really big impact on battle outcome.

## 6.2. BONDER's Constant-Speed-Attack Model.

In this section we will consider S. BONDER's [4; 5; 7] model of a constant-speed attack on a static defensive position in which the fire effectiveness of each side's weapon system is range dependent (i.e. it depends on the range between firers and targets). This model will motivate our interest in certain functional forms for time-dependent attrition-rate coefficients that we will consider subsequently in this chapter.

Let us accordingly consider "aimed-fire" combat between two homogeneous forces and assume that target-acquisition times do not depend on the number of targets. We further assume that one force attacks at constant speed the other force's static defensive position. Assuming that the fire effectiveness of each side's weapon system is range dependent, BONDER hypothesized (see Section 2.12 for a further discussion on physical assumptions) that such an engagement could be modelled by the following LANCHESTER-type equations for $x$ and $y > 0$ [the first equation, for example, becomes $dx/dt = 0$ for $x = 0$]

$$\begin{cases} \dfrac{dx}{dt} = -\,\alpha(r)y & \text{with } x(t=0) = x_0, \\[2ex] \dfrac{dy}{dt} = -\,\beta(r)x & \text{with } y(t=0) = y_0, \end{cases} \tag{6.2.1}$$

where $x(t)$ and $y(t)$ denote, respectively, the X and Y force levels at time $t$, $r$ denotes the range between the opposing forces, and $\alpha(r)$ and $\beta(r)$ denote range-dependent attrition-rate coefficients (see Section 5.12).

Range is related to time by

$$r(t) = r_0 - vt, \tag{6.2.2}$$

where $r_0$ denotes the opening range of battle and $v > 0$ denotes the constant attack speed. For example, let us consider the constant-speed attack of a mobile homogeneous $Y$ force against the static defensive position of a homogeneous $X$ force (see Figure 6.1). The basic idea emphasized in BONDER's model (6.2.1) is that force separation, i.e. range between the opposing forces, changes over time, and the fire effectiveness of, for example, a single $Y$ firer, denoted as $\alpha(r)$, depends on this force separation (see also WEISS [61, pp. 87-88]).

For the combat situation modelled by (6.2.1) we can take either time $t$ or range $r$ as the independent variable in our differential combat model. In our work we have found it to be more convenient to take time as the independent variable. In other words, observing that $r = r(t)$, we see that we may eliminate range $r$ from the attrition-rate coefficients $\alpha$ and $\beta$, i.e.

$$\alpha(r(t)) = a(t) \quad \text{and} \quad \beta(r(t)) = b(t), \tag{6.2.3}$$

to obtain time-dependent attrition-rate coefficients, and thus the model (6.2.1) may be converted into

$$\begin{cases} \dfrac{dx}{dt} = -\,a(t)y & \text{with } x(0) = x_0 \\[2ex] \dfrac{dy}{dt} = -\,b(t)x & \text{with } y(0) = y_0. \end{cases} \tag{6.2.4}$$

Thus, any model such as (6.2.1) with range-dependent attrition-rate coefficients can always be converted into one with time-dependent ones.

As we have seen in Section 5.6 above, in many cases of tactical interest we may model the fire effectiveness of $Y$'s weapon system as a

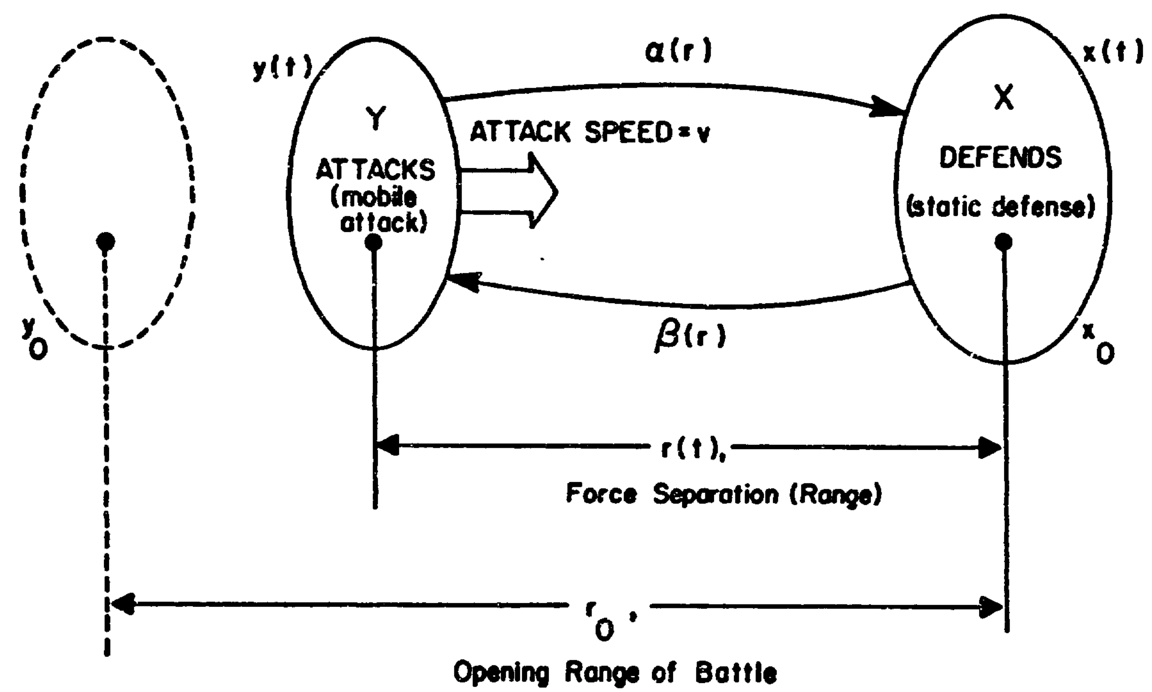


Figure 6.1. Diagram of BONDER's constant-speed-attack model. Force separation, $r(t)$, is given by $r(t) = r_0 - vt$.

function of range with the power attrition-rate coefficient.

$$\alpha(r) = \begin{cases} \alpha_0 \left(1 - \frac{r}{r_\alpha}\right)^\mu & \text{for } 0 \le r \le r_\alpha, \\ 0 & \text{for } r_\alpha \le r, \end{cases} \tag{6.2.5}$$

where $r_\alpha$ denotes the maximum effective range of Y's weapon system and $\mu \geq 0$. Here $\mu$ is used to model the range dependence of Y's power attrition-rate coefficient and is called the "shape" parameter (see Figure 6.2). We may similarly model the fire effectiveness of X's weapon system as a function of range with the power attrition-rate coefficient

$$\beta(r) = \begin{cases} \beta_0 \left(1 - \frac{r}{r_\beta}\right)^\nu & \text{for } 0 \le r \le r_\beta, \\ 0 & \text{for } r_\beta \le r, \end{cases} \tag{6.2.6}$$

where $r_\beta$ denotes the maximum effective range of X's weapon system and $\nu \geq 0$. As we have discussed in Chapter 5, the parameter values chosen for the models (6.2.5) and (6.2.6) depend on both the kill capabilities of the weapon system (as functions of range) and also the vulnerabilities of the two target types.

Let us also consider another range-capability model that will turn out to be in some sense equivalent to the above model, although this equivalence will certainly not be obvious at this moment. Thus, another relevant model for the fire effectiveness of Y's weapon system as a function of range is given by the exponential attrition-rate coefficient

$$\alpha(r) = \alpha_0 e^{-\alpha_1 r}, \tag{6.2.7}$$

where $\alpha_0$ denotes the kill rate of a single Y system at zero force separation and $\alpha_1$ is a positive constant that is used to model the decline in

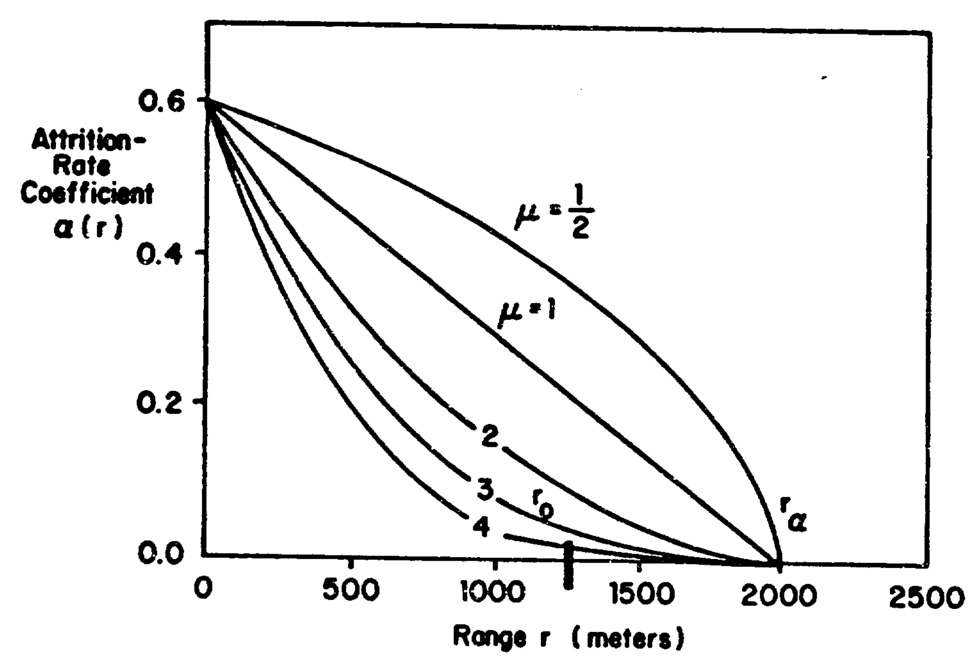


Figure 6.2. Dependence of Y's power attrition-rate coefficient $\alpha(r)$ on the exponent $\mu$ with the maximum effective range of the weapon system and kill rate at zero range held constant. [NOTES: 1. The maximum effective range of the system is denoted as $r_\alpha$ = 2000 meters. 2. $\alpha(0) = \alpha_0 = 0.6$ X casualties/(unit time × number of Y firers) denotes the weapon-system kill rate for Y at zero force separation (range). 3. The opening range of battle is denoted as $r_0$ = 1250 meters and (as shown) $r_0 < r_\alpha$.]

weapon systems (as functions of range) and also the vulnerabilities of the two target types.

Let us also consider another range-capability model that will turn out to be in some sense equivalent to the above model, although this equivalence will certainly not be obvious at this moment. Thus, another relevant model for the fire effectiveness of Y's weapon system as a function of range is given by the exponential attrition-rate coefficient

$$\alpha(r) = \alpha_0 e^{-\alpha_1 r}, \tag{6.2.7}$$

where $\alpha_0$ denotes the kill rate of a single Y system at zero force separation and $\alpha_1$ is a positive constant that is used to model the decline in kill rate with increasing range and is called the "shape" parameter (<u>see</u> Figure 6.3). Although the Y weapon-system type theoretically has an infinite maximum effective range according to (6.2.7), its fire effectiveness is essentially equal to zero for large values of force separation. Similarly, we have for the X weapon-system type

$$\beta(r) = \beta_0 e^{-\beta_1 r}. \tag{6.2.8}$$

In any case, irrespective of such a theoretical property for the maximum effective ranges of the weapon systems, the range-dependent attrition-rate coefficients (6.2.7) and (6.2.8) will in many instances give a good fit to each weapon system's kill rate between the opening range of battle and the final one.

As we have discussed in general terms above, we may use (6.2.2) to eliminate range r from the range-dependent attrition-rate coefficients in the model (6.2.1). Doing this for the range-dependent attrition-rate

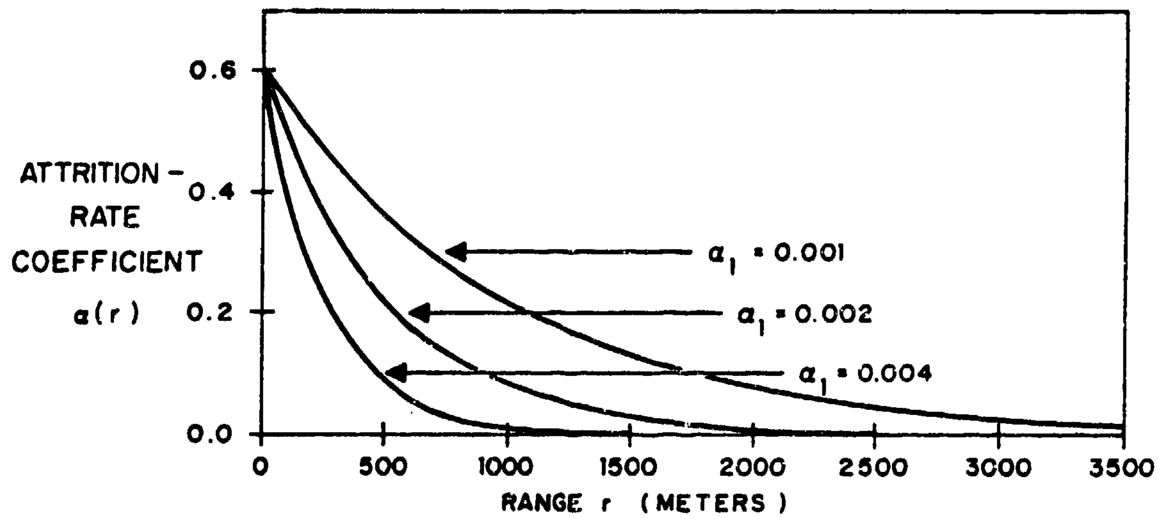


Figure 6.3. Dependence of Y's exponential attrition-rate coefficient $\alpha(r) = \alpha_0 \exp\{-\alpha_1 r\}$ on range and the "shape" parameter $\alpha_1$ with the kill rate at zero force separation (range) $\alpha(0) = \alpha_0$ held constant. Although the Y weapon-system type theoretically has an infinite maximum effective range according to this model, its fire effectiveness is readily seen to be essentially equal to zero for large enough values of force separation.

coefficients (6.2.5) and (6.2.6), we obtain the time-dependent-coefficient model (6.2.4) with general power attrition-rate coefficients.

$$a(t) = k_a(t+C)^{\mu}, \quad \text{and} \quad b(t) = k_b(t+C+D)^{\nu}, \tag{6.2.9}$$

where

$$C = \left(\frac{r_\alpha - r_0}{v}\right), \qquad D = \left(\frac{r_\beta - r_\alpha}{v}\right), \tag{6.2.10}$$

$$k_a = \alpha_0\left(\frac{v}{r_\alpha}\right)^{\mu}, \quad \text{and} \quad k_b = \beta_0\left(\frac{v}{r_\beta}\right)^{\nu} \ . \tag{6.2.11}$$

We will call $C$ the starting parameter, since it allows us to model (with $\mu$ and $\nu \geq 0$) battles that begin within the maximum effective range of the $Y$ weapon system (see Figure 6.2). We will call $D$ the offset parameter, since it allows us to model (again, with $\mu$ and $\nu \geq 0$) battles between opposing weapon systems with different maximum effective ranges, i.e. opposing weapon systems whose maximum effective ranges are "offset" (see Figure 6.4). We observe that $C$ and $D \geq 0$ if and only if $r_\beta \geq r_\alpha \geq r_0$. $C > 0$ means that the battle begins within the maximum effective range of the $Y$ weapon system, while $D > 0$ means that the maximum effective range of the $X$ weapon system is greater than that of the $Y$ system.

In a similar fashion, we may use (6.2.2) to eliminate range $r$ from the range-dependent attrition-rate coefficients (6.2.7) and (6.2.8) in the model (6.2.1) and obtain the time-dependent-coefficient model (6.2.5) with exponential attrition-rate coefficients

$$a(t) = k_a e^{\lambda_a t}, \quad \text{and} \quad b(t) = k_b e^{\lambda_b t}, \tag{6.2.12}$$

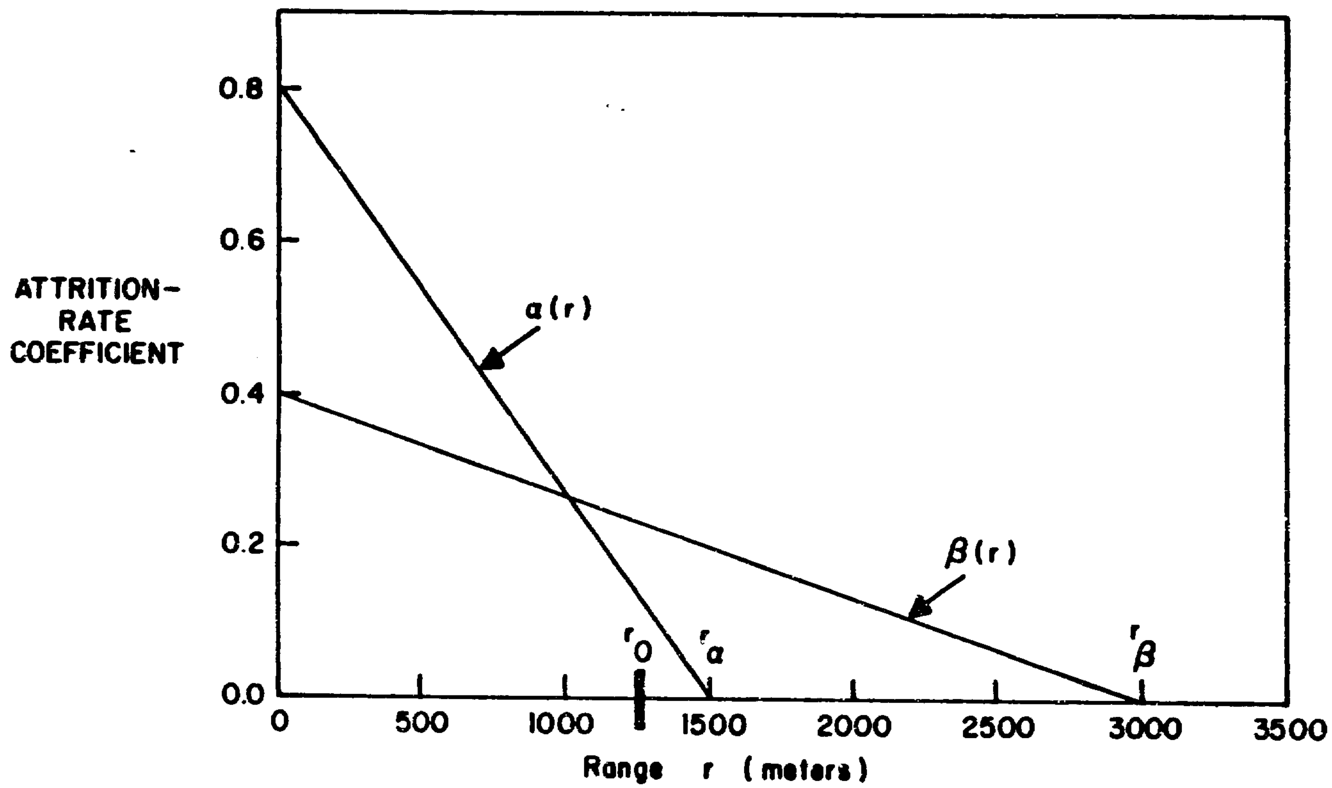


Figure 6.4. Explanation of the starting parameter C and the offset parameter D for the general power attrition-rate coefficients modelling a constant-speed attack. In this example both attrition-rate coefficients vary linearly with range, and the X weapon-system type corresponding to $\beta(r)$ has a greater maximum effective range $r_\beta$ than does the Y-weapon-system type with maximum effective range $r_\alpha$, i.e. $r_\beta > r_\alpha$. In other words, the opposing weapon-system types have fire effectivenesses that are "offset" in the sense that one weapon-system type can reach out further on the battlefield than the other. Also, the opening range of battle (i.e. the initial separation between forces) is denoted as $r_0$ and (as shown) $r_0 <$ minimum $(r_\alpha, r_\beta)$. Finally, the <u>starting parameter</u> is given by $C = (r_\alpha - r_0)/v$, and the <u>offset parameter</u> is given by $D = (r_\beta - r_\alpha)/v$.

where

$$k_a = \alpha_0 e^{-\alpha_1 r_0}, \qquad k_b = \beta_0 e^{-\beta_1 r_0}, \tag{6.2.13}$$

$$\lambda_a = \alpha_1 v, \qquad \text{and} \qquad \lambda_b = \beta_1 v \ . \tag{6.2.14}$$

We close this section with some illustrative numerical results from BONDER's constant-speed-attack model. Let us therefore examine the contant-speed-attack model. We will consider the constant-speed attack of a mobile homogeneous $Y$ force against the static defensive position of a homogeneous $X$ force (see Figure 6.1). We assume that combat attrition can be modelled by (6.2.1) with range-dependent attrition-rate coefficients (6.2.5) and (6.2.6). The dependence upon range of the attrition-rate coefficient $\alpha(r)$ (which represents the fire effectiveness of the $Y$ weapon system) is shown in Figure 6.2. Let us assume that the attacking $Y$ force initially numbers 30 and attacks at a constant speed of 5 miles per hour. We assume that the defending $X$ force initially numbers 10. We will see that exactly what will happen in such a battle is quite sensitive to the variations in the kill rates of the opposing weapon systems with range.

In Figure 6.5 we have plotted force-level trajectories for three different battles, denoted as battles (A), (B), and (C). These force-level curves have been developed from analytical results to be discussed subsequently in this chapter, but at this point in time we are not quite ready to discuss how we have developed them. In these battles both types of opposing weapon systems have the same maximum effective range, i.e. $r_\alpha = r_\beta = r_e$, and the battle begins at this range, i.e. $r_0 = r_e$. For these battles we have held constant the kill rates at zero force separation, i.e. $\alpha_0 = \alpha(0)$ and $\beta_0$, and have varied in these three battles the manner in which $\alpha(r)$ and $\beta(r)$ depend upon range, i.e. for $0 \leq r \leq r_e$ the

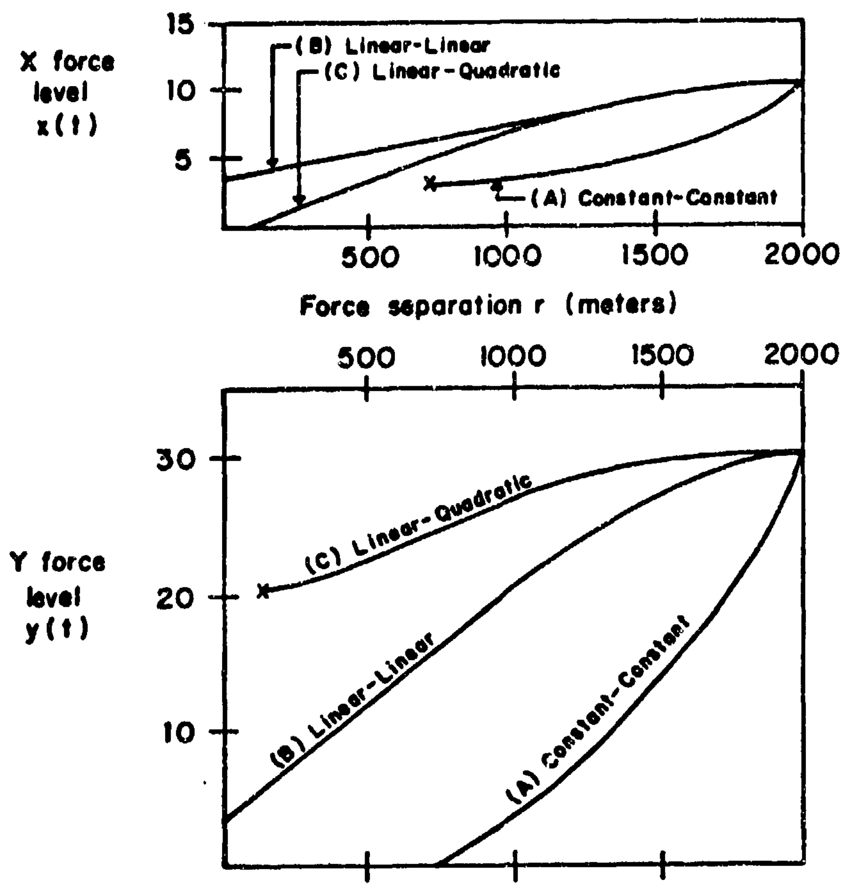


Figure 6.5. Force-level trajectories of X and Y forces for three different battles [denoted in the figure as (A), (B), and (C) and explained in the main text] with each side's fire effectiveness modelled by the power attrition-rate coefficients for $r_0 = r_\alpha = r_\beta = r_e = 2000$ meters, $\alpha_0 = 0.06$ X (casualties/minute) per Y firer, $\beta_0 = 0.6$ Y (casualties/minute) per X firer, $v = 5$ mph, $x_0 = 10$, and $y_0 = 30$. The symbol × denotes the end of a force-level trajectory due to annihilation of the enemy force.

attrition-rate coefficients are given by the following expressions in each of the three battles:

(A) constant-constant: $\alpha(r) = \alpha_0$ and $\beta(r) = \beta_0$,

(B) linear-linear $\alpha(r) = \alpha_0(1-r/r_e)$ and $\beta(r) = \beta_0(1-r/r_e)$,

(C) linear-quadratic: $\alpha(r) = \alpha_0(1-r/r_e)$ and $\beta(r) = \beta_0(1-r/r_e)^2$,

In other words, in battle (C) (the linear-quadratic case) the term "linear" denotes that $\alpha(r)$ (the fire effectiveness of the $Y$ weapon system type) varies linearly with range, while the term "quadratic" denotes that $\beta(r)$ varies quadratically with range. In battle (A) both attrition-rate coefficients are constant, and thus in this case we have assumed no variation in fire effectiveness with range for either weapon-system type.

We see from Figure 6.5 that battle outcome is quite sensitive to the variation in weapon-system kill rate with range: in battle (A) the attacking $Y$ force is annihilated at a range of about 750 meters, while in battle (C) the defending $X$ force is annihilated before the attackers have approached within 100 meters of the defender's position. Figure 6.5 shows us the inadequacy of using constant attrition-rate coefficients in battles with appreciable variations in force separation to model the kill rates of weapon-system types whose true capabilities actually vary appreciably with range. The constant-coefficient results can be quite misleading for such battles. We also see from Figure 6.5 that we can use the initial trend of battle to forecast battle outcome only when we know how the fire effectiveness of each weapon-system type depends on range. If the reader will compare results for the three battles, the truth of this statement should be clear. We finally note the "compounding" effect of casualties over time: a small advantage in range capability rapidly "grows in its effect on force-level trajectories," and such a small difference can have a large effect on battle outcome.

Figure 6.6 shows similar force-level curves for the same battle-parameter values except that the battle begins at an opening range of 1250 meters, i.e. $r_0 = 1250$ meters, instead of 2000 meters as it did for Figure 6.5. The force-level curves corresponding to the constant-coefficient case, i.e. battle (A) in Figure 6.6 with $r_0 = 1250$ meters are exactly the same for the same time intervals (but not range intervals) as those shown in Figure 6.5 with $r_0 = 2000$ meters. Other force-level trajectories decay faster in Figure 6.6 than they do in Figure 6.5 because the "intensity" of combat is greater, i.e. as a function of time the attrition-rate coefficients are larger here than for Figure 6.5. Again we see that battle outcome is sensitive to the range dependence of the attrition-rate coefficients. From comparing the force-level curves shown in Figure 6.5 with those in Figure 6.6, we see that the differences between battles (A), (B), and (C) are smaller when the opening range of battle $r_0$ is much less than the maximum effective range of the two opposing weapon-system types. In fact, when $r_e \to +\infty$, the force-level trajectories converge to the classic constant-coefficient ones (see BONDER [7, p. IV-33] for a further discussion).

Thus, we see that the range dependence of weapon-system kill rates has a very significant impact on battle outcome for BONDER's constant-speed-attack model. We have reached this conclusion after examining three specific battles, denoted as (A), (B), and (C) in Figures 6.5 and 6.6 and classified according to the combination of two attrition-rate-coefficient range dependencies (e.g. linear-quadratic). In these figures each different battle is represented by a separate force-level curve. Moreover, it will be instructive for us to examine further parametric variations in attrition-rate-coefficient range dependencies. It will be convenient, however, to identify battles in a slightly different manner: we will denote exponent combinations for the attrition-rate coefficients (6.25) and (6.26) as $\mu:\nu$, where $\mu$

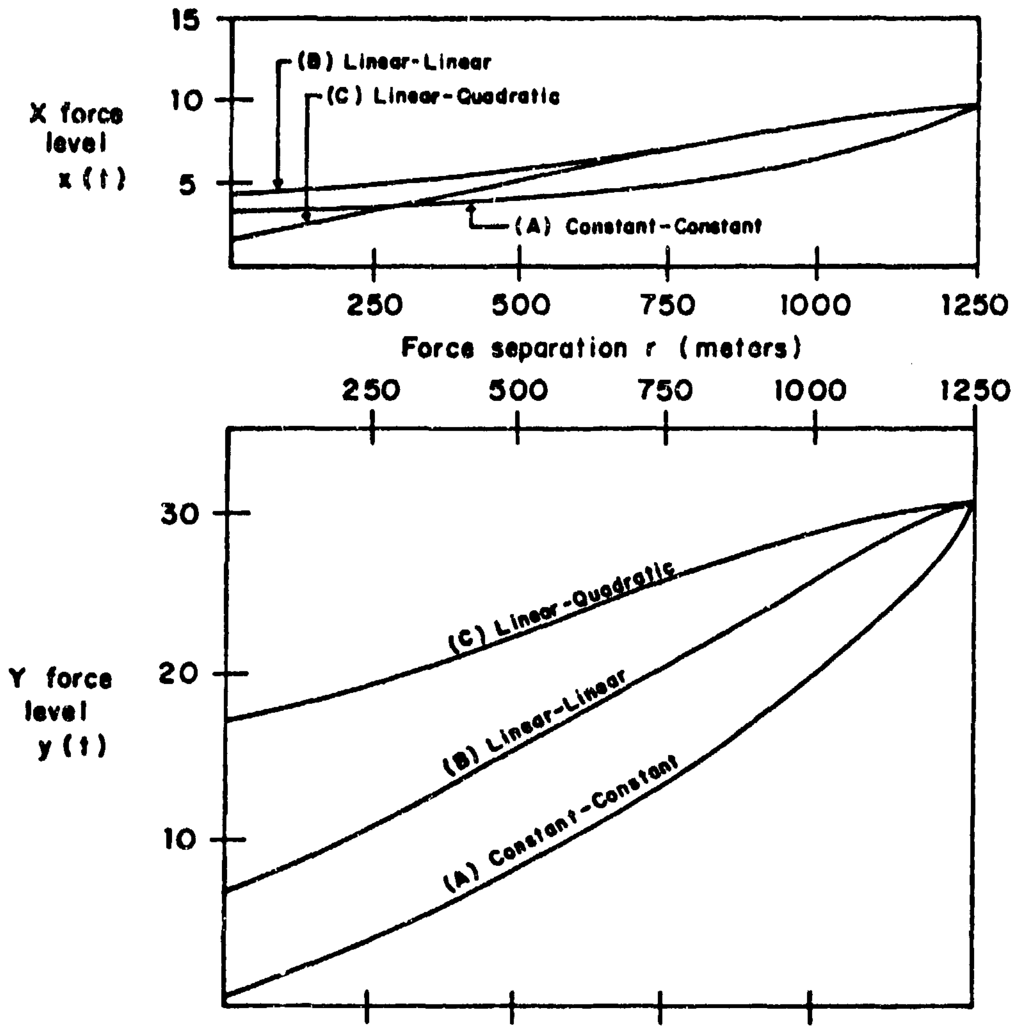


Figure 6.6. Force-level trajectories of X and Y forces for an additional three different battles modelled with the power attrition-rate coefficients for the same parameter values chosen for Figure 6.5 except that the opening range of battle $r_0$ is given by $r_0 = 1250$ meters (still with $r_\alpha = r_\beta = r_e = 2000$ meters). The symbol conventions are also the same as in Figure 6.5.

denotes the exponent for the $Y$ weapon-system-type kill rate $\alpha(r)$ and $\nu$ denotes the exponent for the $X$ weapon-system-type kill rate $\beta(r)$.

Accordingly, further battle results for a wider variety of exponent combinations in BONDER's constant-speed attack modelled with (6.2.1) and the attrition-rate coefficients (6.2.5) and (6.2.6) are shown in Figures 6.7, 6.8, and 6.9. In Figure 6.7 we have expanded the range of exponent combinations from those for the battles shown in Figure 6.5. Furthermore, battles are identified differently in these figures (i.e. Figures 6.7, 6.8, and 6.9) than they were in Figures 6.5 and 6.6. For example, battle (C) with the linear-quadratic range-dependent attrition-rate coefficients is now denoted simply as 1:2, i.e. $\mu = 1$ and $\nu = 2$ for the coefficients (6.2.5) and (6.2.6). As in Figures 6.5 and 6.6, we have held $\alpha_0 = \alpha(0)$ and $\beta_0$ constant for these computations, i.e. the kill rates at zero force separation are the same for all these battles.

Figure 6.7 further shows us that the nature of a force-level trajectory is quite sensitive to the particular combination of exponent values $\mu$ and $\nu$ and that these exponents are additional parameters that help determine who wins and who loses. Returning to the constant-speed attack of a mobile $Y$ force against the static defensive position of a defending $X$ force, we see that, for example, for $\mu = 1$ (i.e. the kill rate $\alpha(r)$ of the attacker's weapon system varying linearly with range) a battle may have quite different outcomes depending on the value of $\nu$: the reader should contrast the force-level trajectories denoted as 1:0, 1:1, 1:2, and 1:3 in Figure 6.7. We also see that we can use the initial trend of battle to predict battle outcome only when we know the nature of the dependency of each weapon-system type's kill capability on range; the results shown in Figure 6.7 should make this clear. For example, compare the outcomes for the curves denoted as 1:2, 2:2, and 3:2. We also note the "compounding"

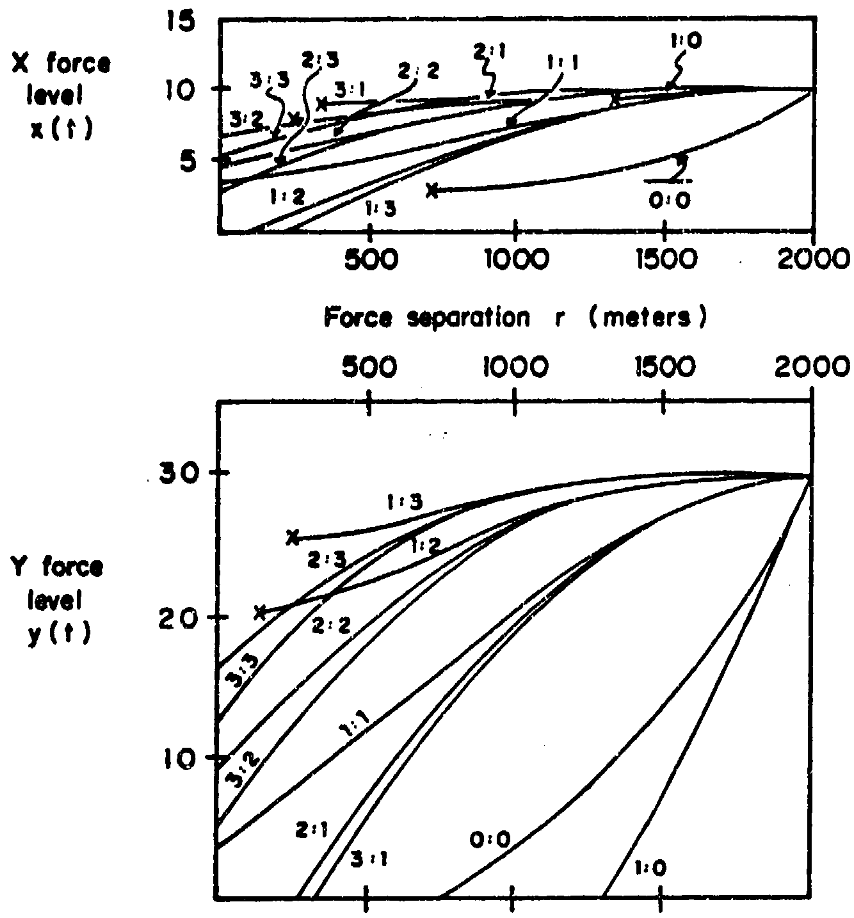


Figure 6.7. Results for BONDER's constant-speed-attack model when both sides' weapon-system types have the same maximum effective range: force-level trajectories of X and Y forces for different battles corresponding to different combinations of the exponents $\mu$ and $\nu$ in the power attrition-rate coefficients for $r_0 = r_\alpha = r_\beta = r_e = 2000$ meters, $\alpha_0 = 0.06$ X (casualties/minute) per Y firer, $\beta_0 = 0.6$ Y (casualties/minute) per X firer, $v = 5$ mph, $x_0 = 10$, and $y_0 = 30$. Each exponent combination is expressed as $\mu{:}\nu$ in the figure, and the symbol $\times$ denotes the end of a force-level trajectory due to the annihilation of the enemy force.

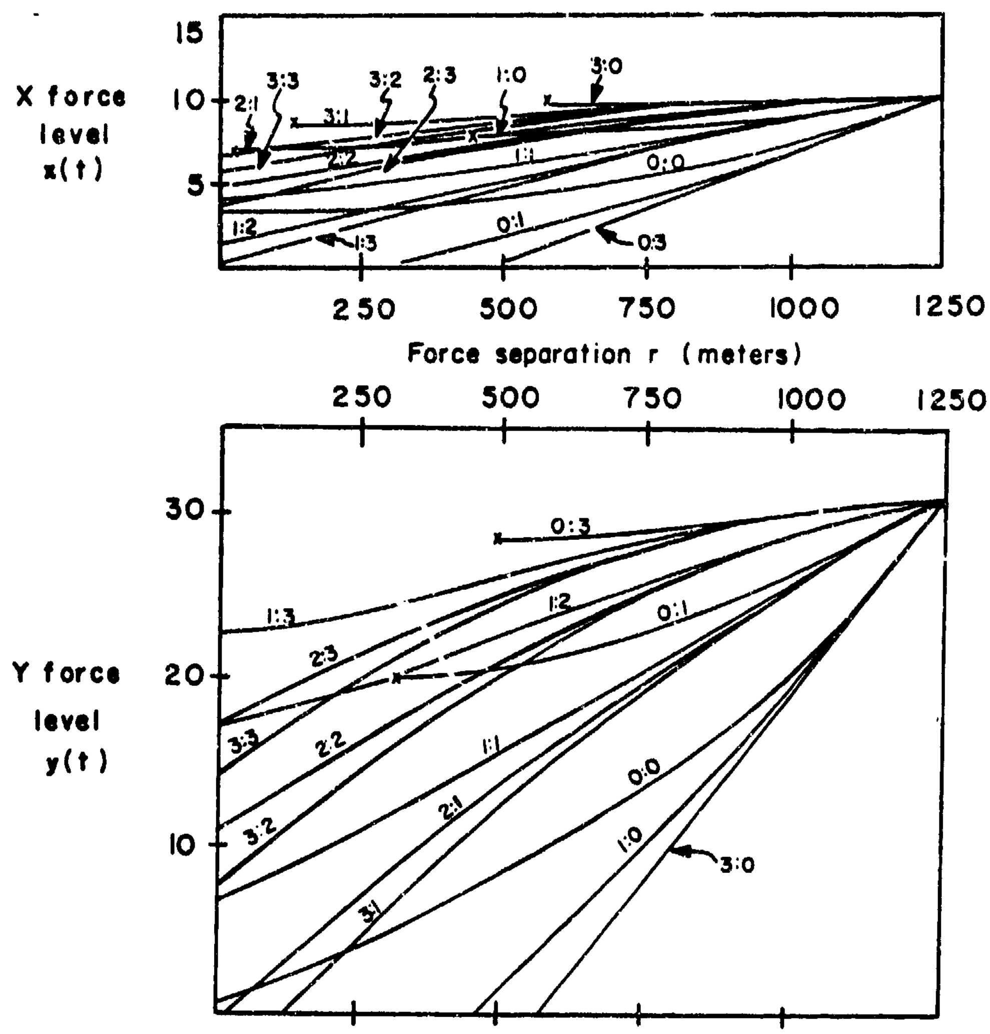


Figure 6.8. Further results for BONDER's constant-speed-attack model when both sides' weapon-system types have the same maximum effective range: force-level trajectories of X and Y forces for different battles corresponding to different combinations of the exponents $\mu$ and $\nu$ in the power attrition-rate coefficients for the same parameter values chosen for Figure 6.7 except that $r_0 = 1250$ meters. The symbol conventions are also the same as in Figure 6.7.

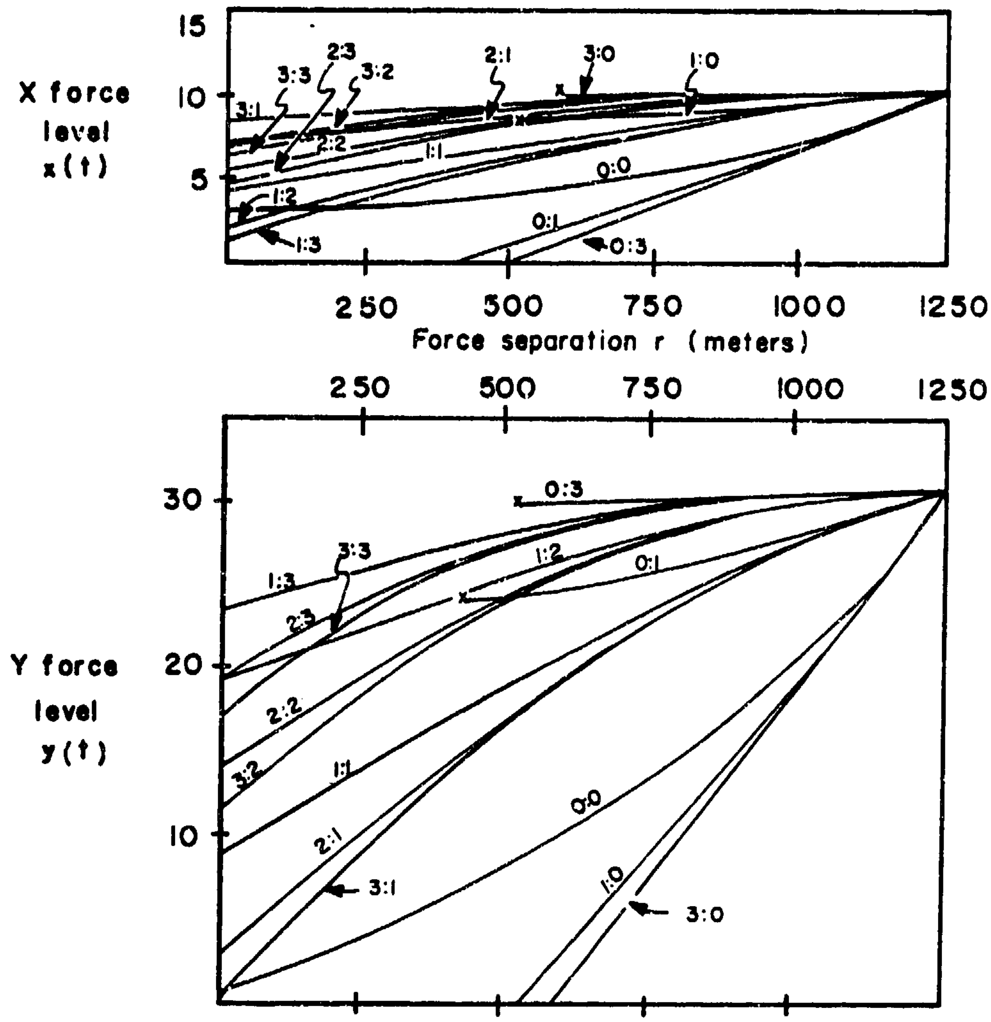


Figure 6.9. Further results for BONDER's constant-speed-attack model when both sides' weapon-system types have the same maximum effective range: force-level trajectories of X and Y forces for different battles corresponding to different combinations of the exponents $\mu$ and $\nu$ in the power attrition-rate coefficients for the same parameter values chosen for Figure 6.7 except that $r_0 = 1250$ meters and $r_\alpha = r_\beta = r_e = 1500$ meters. Again, the symbol conventions are also the same as in Figure 6.7.

effect over time: a small advantage in range capability can eventually materially affect battle outcome.

In Figure 6.8 we have similarly expanded the range of exponent combinations for the battles shown in Figure 6.6, i.e. all battle parameters are the same as for Figure 6.7 except that $r_0 = 1250$ meters instead of 2000 meters. Similar to the case shown in Figure 6.6, the force-level curves shown in Figure 6.8 with $r_0 = 1250$ meters are similar to those shown in Figure 6.7 with $r_0 = 2000$ meters except that as a function of time they decrease faster in Figure 6.8 for $\mu$ and $\nu > 0$ because the "intensity" of combat is greater, i.e. as a function of time both attrition-rate coefficients are larger here than in Figure 6.7. Figure 6.9 shows similar force-level curves for the same parameter values except that $r_e = r_\alpha = r_\beta = 1500$ meters. Observing that for $\mu \geq 1$ we have $\alpha(r;r_\alpha) < \alpha(r;\bar{r}_\alpha)$ if and only if $r_\alpha < \bar{r}_\alpha$, we may consider that the "intensity" of combat is less intense for the engagements depicted in Figure 6.9 than for those shown in Figure 6.8.

Figure 6.10 shows the effect of increasing maximum effective range of the defender's weapons, i.e. that of the $X$ force (*cf*. Figure 6.1), when each weapon-system type's kill rate is assumed to vary linearly with range (*see* Figure 6.4). For the family of battles depicted in Figure 6.10, we have held the opening range of battle constant at $r_0 = 1250$ meters and the maximum effective range of the attacking $Y$ weapon system constant at $r_\alpha = 1500$ meters. Both attrition-rate coefficients vary linearly with range [i.e. $\mu = \nu = 1$ in (6.2.5) and (6.2.6)], $\alpha_0$ and $\beta_0$ have been held constant, and $r_\beta$ has been varied. The force-level curves in Figure 6.10 quantitatively show the benefit from increasing the long-range kill capability of the defender's weapon system: more attacker casualties occur earlier

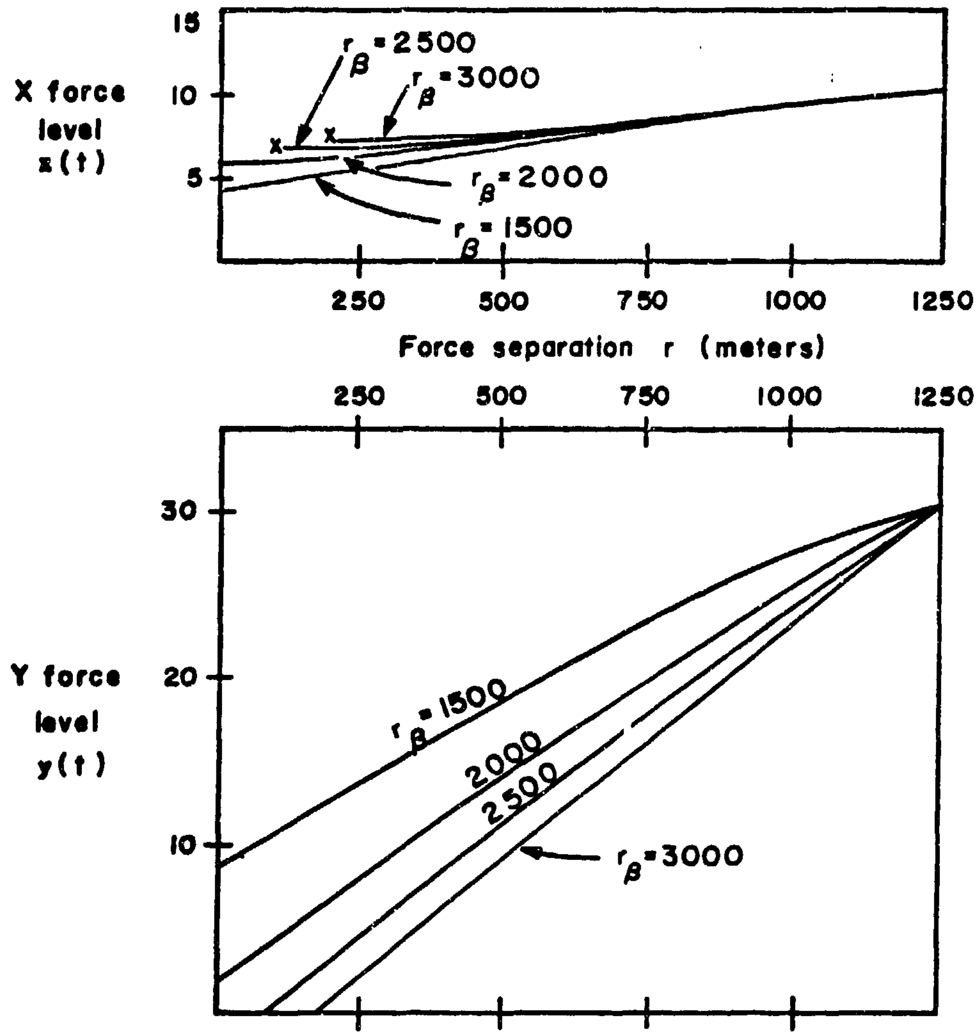


Figure 6.10. Results for BONDER's constant-speed-attack model when opposing weapon-system types have different maximum effective ranges: force-level trajectories of X and Y forces for various different maximum effective ranges of the X-force-weapon-system type for linear attrition-rate coefficients with $r_0$ = 1250 meters, $r_\alpha$ = 1500 meters, and the same values of the other parameters (i.e. $\alpha_0$, $\beta_0$, and v) listed in the legend of Figure 6.7. The symbol × has the same meaning as in that figure.

in the battle, and these are then magnified over time by the "compounding nature" of the LANCHESTER-type equations (6.2.1). Again, these numerical results have been generated from analytical results that are given later in this chapter. However, using an analogue computer, BONDER and FARRELL [10, pp. 296-367] have developed extentive parametric results for this model.

The important thing to glean from all these battle examples is that <u>variations in weapon-system kill rates with range in mobile operations</u> (equivalently, temporal variations in fire effectiveness over the course of a battle) <u>have a significant impact on the battle's outcome</u>. Consequently, we should use time-dependent attrition-rate coefficients to model temporal variations in fire effectiveness when, for example, the range between firers and targets changes appreciably during battle.

As noted above, we have generated all the force-level curves shown in Figures 6.5 through 6.10 from analytical results, i.e. infinite-series solutions, to be subsequently developed in this chapter. However, we could have equally well generated them by a step-by-step numerical integration of a finite-difference approximation to our differential-equation combat model. We can, of course, always numerically do this for a specific set of battle-parameter values. However, the structure of combat results is not at all evident from such specific numerical evaluations, but it may be deduced from further analysis of the analytical results. Of course, before we embark on an analytical examination of force-level trajectories for the model (6.2.4), we should consider what information one wants to extract from the model.

## 6.3. Information to be Obtained from the Model.

As we have discussed many times above, our goal in this book is to help the reader to obtain insights into the dynamics of combat from relatively simple combat models rather than enriching such models in details (see W. T. MORRIS [29] for a lucid discussion of the process of such enrichment). Consequently, both our research and also the developments of this chapter have been guided by this goal of obtaining insights into the dynamics of combat.

We will emphasize extracting as much operational information as possible from the model with a minimum of effort. What information should we seek to obtain? Although the specific information to extract from any combat model depends, of course, on the purpose of the OR study, we have used the questions shown in Table 6.I to guide our efforts. We have tried to make the extraction of such information from variable-coefficient homogeneous-force models almost as easy as obtaining it from LANCHESTER's classic constant-coefficient models. As we have just seen in the previous section, such variable-coefficient combat models are required when there are appreciable temporal variations in fire effectiveness during a battle.

In the rest of this chapter we will present analytical results for time-dependent $F|F$, $FT|FT$, and $(F+T)|(F+T)$ attrition processes. S. BONDER [10, pp. 30-31] has stressed the importance of analytical solutions to such models for developing insights into the dynamics of combat by explicitly portraying the relation between various factors in the combat attrition process and the surviving numbers of forces and also for facilitating sensitivity and other parametric analyses (see BONDER [9]). Consequently, we will consider developing and analyzing solutions to variable-coefficient differential models for $F|F$, $FT|FT$, and $(F+T)|(F+T)$ combat.

Table 6.I. Information to Extract from Combat Model.

(Q1) Who will "win" the engagement? Be annihilated?

(Q2) How do the force levels change over time in the battle?

(Q3) How many survivors will the winner have?

(Q4) What force ratio is required to guarantee victory?

(Q5) How long will the battle last?

(Q6) How do changes in the initial force levels and weapon-system parameters affect the battle's outcome?

(Q7) What will be the casualty-exchange ratio?

(Q8) Is concentration of forces a good tactic?

Most of these developments for analytically investigating variable-coefficient LANCHESTER-type combat have only recently appeared in the literature. In particular, the theory of variable-coefficient F|F combat is now essentially almost as complete as that for LANCHESTER's classic constant-coefficient equations for modern warfare. In other words, it is now almost as easy to extract information (recall Table 6.I) from these variable-coefficient LANCHESTER-type combat models as it is from the corresponding constant-coefficient ones.

## 6.4. The Special Case of Quasi-Autonomous Equations.

Before elaborating upon general results concerning analytical solutions of LANCHESTER-type equations with time-dependent attrition-rate coefficients, let us consider a very important special case that bridges the gap between constant-coefficient and variable-coefficient models. As stressed by S. BONDER [10, pp. 30-31], analytical solutions to LANCHESTER-type equations are important for developing insights into the dynamics of combat by explicityly portraying the relation between the parameters of the attrition process and the numbers of survivors. Unfortunately, it is generally impossible to express the solution to such a system of equations with time-dependent attrition-rate coefficients in terms of any of the classic "elementary" functions of mathematics[5], e.g. exponential functions, hyperbolic functions, etc. Thus, we are grateful when constant-coefficient results may be used in some sense for analyzing combat modelled with time-dependent coefficients.

Let us therefore note that any homogeneous-force model of the form

$$\begin{cases} \dfrac{dx}{dt} = -h(t)\, A(x,y) & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -h(t)\, B(x,y) & \text{with } y(0) = y_0, \end{cases} \tag{6.4.1}$$

may be transformed into the autonomous system of differential equations (i.e. the right-hand sides of the differential equations do not contain the time parameter)

$$\begin{cases} \dfrac{dx}{d\tau} = -A(x,y) & \text{with } x(\tau=0) = x_0, \\ \dfrac{dy}{d\tau} = -B(x,y) & \text{with } y(\tau=0) = y_0, \end{cases} \tag{6.4.2}$$

by the substitution

$$\tau = \int_0^t h(s)ds, \tag{6.4.3}$$

where we assume that the integral exists. Thus, the model (6.4.1) with time-dependent attrition-rate coefficients may be transformed into a constant-coefficient one by a transformation of the battle's time scale. We will say that such LANCHESTER-type equations are <u>quasi-autonomous</u>.

We have already encountered in Section 3.6 an important example of such quasi-autonomous equations for an $F|F$ attrition process, namely

$$\frac{dx}{dt} = -a(t)y \qquad \text{and} \qquad \frac{dy}{dt} = -b(t)x, \tag{6.4.4}$$

where

$$a(t) = k_a h(t), \qquad b(t) = k_b h(t), \tag{6.4.5}$$

$h(t) > 0$ for all $t \geq 0$, and $k_a$ and $k_b$ are positive constants. The substitution

$$\tau = \sqrt{k_a k_b} \int_0^t h(s)ds, \tag{6.4.6}$$

then transforms (6.4.4) with (6.4.5) into

$$\frac{dx}{d\tau} = -\sqrt{\frac{k_a}{k_b}}\, y, \qquad \text{and} \qquad \frac{dy}{d\tau} = -\sqrt{\frac{k_b}{k_a}}\, x, \tag{6.4.7}$$

whence[6], for example,

$$x(t) = x_0 \cosh \tau - y_0 \sqrt{\frac{k_a}{k_b}} \sinh \tau, \tag{6.4.8}$$

which may be written as

$$x(t) = x_0 \cosh (\overline{\sqrt{a(t)b(t)}}\, t) - y_0 \sqrt{\frac{a(t)}{b(t)}} \sinh (\overline{\sqrt{a(t)b(t)}}\, t), \qquad (6.4.9)$$

where $\overline{\sqrt{a(t)b(t)}}$ denotes the average intensity of combat, i.e.

$$\overline{\sqrt{a(t)b(t)}} = \frac{1}{t} \int_0^t \sqrt{a(s)b(s)}\, ds. \qquad (6.4.10)$$

Finally, we note that for combat modelled with the quasi-autonomous equations (6.4.4) and (6.4.5) a "square law" still holds[7]

$$k_b(x_0^2 - x^2) = k_a\,(y_0^2 - y^2). \qquad (6.4.11)$$

## 6.5. General Force-Level Results for Variable-Coefficient LANCHESTER-Type Equations of Modern Warfare.

Let us consider the following LANCHESTER-type equations for a F|F attrition process with time-dependent attrition-rate coefficients

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y & \text{with } x(0) = x_0, \\ \\ \dfrac{dy}{dt} = -b(x)x & \text{with } y(0) = y_0 . \end{cases} \tag{6.5.1}$$

These equations may be hypothesized to model combat under either of the following two sets of circumstances (cf. Sections 2.2 and 2.11 above):

either (S1) both sides use "aimed" fire and target-acquisition times do not depend on the number of targets [61],

or (S2) both sides use "area" fire and a constant-density defense [12].

Mathematically, we assume that the attrition-rate coefficients $a(t)$ and $b(t)$ are defined, positive, and continuous for $t_0 < t < +\infty$ with $t_0 \leq 0$. For convenience, we introduce the notation that $a(t) \in L(t_0,T)$ means $\int_{t_0}^{T} a(t)dt$ exists (and is given by a finite quantity). From our assumptions about $a(t)$ it follows that $a(t) \notin L(t_0,T)$ implies that $\int_{t_0}^{T} a(t)dt = +\infty$, and similarly for $b(t)$. We also assume that both $a(t)$ and $b(t) \in L(t_0,T)$ for any finite $T$. It follows that, for example, $a(t) \notin L(t_0,+\infty)$ implies that $\lim_{T\to+\infty} \int_{t_0}^{T} a(t)dt = +\infty$. We will further take $a(t)$ and $b(t)$ to be given in the form

$$a(t) = k_a\, g(t)\ , \quad \text{and} \quad b(t) = k_b\, h(t)\ , \tag{6.5.2}$$

where $k_a$ and $k_b$ are positive constants chosen so that $a(t)/b(t) = k_a/k_b$ when $g(t) \equiv h(t)$. In other words, $k_a$ and $k_b$ are basically "scale factors," which are useful for the parametric study of battle outcomes as related to various system parameters.

We will now introduce some useful notation for two important parameters of such "aimed-fire" battles with time-dependent attrition-rate coefficients (6.5.1). In Chapter 2, we considered the $F|F$ attriton process with constant attrition-rate coefficients and found out that the force-level trajectories depended on the following three quantities: (1) the initial force ratio $u_0 = x_0/y_0$, (2) the intensity of combat $I = \sqrt{ab}$ , and (3) the relative fire effectiveness $R = a/b$, where $a$ and $b$ denote constant attrition-rate coefficients. With these constant-coefficient results in mind, we introduce for the model (6.5.1) the <u>intensity of combat</u> $I(t)$ and the <u>relative fire effectiveness</u> $R(t)$ defined by

$$I(t) = \sqrt{a(t)\, b(t)}\ , \quad \text{and} \quad R(t) = a(t)/b(t)\ . \tag{6.5.3}$$

We similarly introduce the <u>combat-intensity parameter</u> $\lambda_I$ and the <u>relative-fire-effectiveness parameter</u> $\lambda_R$ defined by

$$\lambda_I = \sqrt{k_a k_b}\ , \qquad \text{and} \qquad \lambda_R = k_a/k_b. \tag{6.5.4}$$

Before considering the representation of solutions to (6.5.1), let us establish an important mathematical property of such solutions:

all solutions to (6.5.1) with both $a(t)$ and $b(t) \geq 0$ for all $t \geq 0$ and also with both $x_0$ and $y_0 > 0$ are nonoscillatory in the sense that $x(t)$ and $y(t)$ can have at most one zero for $t \geq 0$. To see this, we multiply the first equation of (6.5.1) by $y$, the second by $x$, add, and integrate to obtain

$$x(t)\, y(t) = x_0 y_0 - \int_0^t \{a(s)\, y^2(s) + b(s)\, x^2(s)\} ds\ , \qquad (6.5.5)$$

whence follows the assertion by recalling that on physical grounds we must have (and therefore we will assume that) both $a(t)$ and $b(t) \geq 0$ for all $t \geq 0$ and also that both $x_0$ and $y_0 > 0$.

> THEOREM 6.5.1: All solutions to (6.5.1) are non-oscillatory in the sense that at most one of the force levels $x(t)$ and $y(t)$ can ever vanish in finite time.

As we have discussed in Section 2.2 above, we should "turn off" the combat model (6.5.1) when either side is annihilated [cf. (2.2.2)]. For many purposes, however, it is convenient to "let the equations run for all $t \geq 0$." Theorem 6.5.1 then tells us that if, for example, the X force is ever annihilated [i.e. there is a finite $t_a^X$ such that $x(t_a^X) = 0$], then $y(t) > 0$ for all $t \geq 0$. This property is useful for developing force-annihilation-prediction conditions for the model (6.5.1). Furthermore, it does not hold for all differential combat models.

We will now show how the well-known constant-coefficient results (2.2.11) for the force levels as functions of time, i.e. $x(t)$ and

$y(t)$, may be generalized to the model (6.5.1) for battles with time-dependent attrition-rate coefficients. The basic idea is to construct the solution out of certain generalizations of the classic hyperbolic functions. Thus, the X force level as a function of time, $x(t)$, may be represented as (see TAYLOR and BROWN [53])

$$x(t) = x_0\{C_Y(0)\ C_X(t) - S_Y(0)\ S_X(t)\} - y_0\sqrt{\lambda_R}\ \{C_X(0)\ S_X(t) - S_X(0)\ C_X(t)\} , \qquad (6.5.6)$$

where the hyperbolic-like general LANCHESTER functions (GLF) $C_X(t)$ and $S_X(t)$ are linearly independent solutions to the X force-level equation

$$\frac{d^2x}{dt^2} - \left\{\frac{1}{a(t)}\frac{da}{dt}\right\}\frac{dx}{dt} - a(t)\ b(t)x = 0 , \qquad (6.5.7)$$

with initial conditions

$$\begin{aligned} &C_X(t_0) = 1 , && S_X(t_0) = 0 , \\ &\{1/a(t_0)\}dC_X/dt(t_0) = 0 , && \{1/a(t_0)\}dS_X/dt(t_0) = 1/\sqrt{\lambda_R} . \end{aligned} \qquad (6.5.8)$$

Here $t_0 \le 0$ denotes the largest finite time at which $a(t)$ or $b(t)$ ceases to be defined, positive, or continuous. We will set $t_0 = 0$ if no such finite time exists.

In a similar fashion, the Y force level as a function of time, $y(t)$, may be represented as

$$y(t) = y_0\{C_X(0)\, C_Y(t) - S_X(0)\, S_Y(t)\}$$

$$- \frac{x_0}{\sqrt{\lambda_R}} \{C_Y(0)\, S_Y(t) - S_Y(0)\, C_Y(t)\} , \qquad (6.5.9)$$

where the hyperbolic-like GLF $C_Y(t)$ and $S_Y(t)$ are linearly independent solutions to the <u>Y force-level equation</u>

$$\frac{d^2y}{dt^2} - \left\{\frac{1}{b(t)} \frac{db}{dt}\right\} \frac{dy}{dt} - a(t)\, b(t) y = 0 , \qquad (6.5.10)$$

with initial conditions

$$C_Y(t_0) = 1 , \qquad S_Y(t_0) = 0 , \qquad (6.5.11)$$

$$\{1/b(t_0)\} dC_Y/dt(t_0) = 0 , \qquad \{1/b(t_0)\} dS_Y/dt(t_0) = \sqrt{\lambda_R} .$$

It may be shown (and we will do so below) that

$$C_X(t)\, C_Y(t) - S_X(t)\, S_Y(t) = 1 , \qquad (6.5.12)$$

whence (6.5.6) and (6.5.9) are readily seen to satisfy the initial conditions to (6.5.1).

It is often convenient to view the above GLF as solutions to the following two systems

$$\left\{ \begin{array}{ll} \dfrac{dC_X}{dt} = \dfrac{a(t)}{\sqrt{\lambda_R}} S_Y & \text{with } C_X(t_0) = 1 , \\[2ex] \dfrac{dS_Y}{dt} = \sqrt{\lambda_R}\; b(t)\, C_X & \text{with } S_Y(t_0) = 0 , \end{array} \right. \qquad (6.5.13)$$

and the dual system obtained by making the substitutions $X \to Y$, $Y \to X$, $a(t) \to b(t)$, $b(t) \to a(t)$, and $\lambda_R \to 1/\lambda_R$ in (6.5.13)

$$\begin{cases} \dfrac{dC_Y}{dt} = \sqrt{\lambda_R}\ b(t)\ S_X & \text{with } C_Y(t_0) = 1 , \\ \\ \dfrac{dS_X}{dt} = \dfrac{a(t)}{\sqrt{\lambda_R}}\ C_Y & \text{with } S_X(t_0) = 0 . \end{cases} \qquad (6.5.14)$$

Equation (6.5.12) is now a trivial consequence of (6.5.13) and (6.5.14).

Thus, the X and Y force levels may be constructed from the GLF, which we may consider to be the basic "building blocks" of all analytical results for the differential combat model (6.5.1). In other words, once we have determined the GLF defined by (6.5.7), (6.5.8), (6.5.10), and (6.5.11) (or, equivalently, by the two systems (6.5.13) and (6.5.14)), we can, for example, construct the X force level $x(t)$ by means of (6.5.7).

Thus, it remains to discuss the calculation of the hyperbolic-like GLF. Two approaches that may be used to calculate the hyperbolic-like GLF from their definitions are as follows:

(A1) method of succesive approximations,

and (A2) infinite-series methods.

The infinite-series methods essentially consist of assuming an infinite series of a given form with undetermined coefficients and then determining these coefficients (see, for example, INCE [23], KAMKE [24], MURPHY [32], or RAINVILLE [35]). We have primarily used, however,

successive approximations in our work (see, for example, TAYLOR [43]), and we will now further discuss this approach.

We will now illustrate the method of successive approximations by developing an expression for the hyperbolic-like GLF $C_X(t)$. From (6.5.13) we find that $C_X(t)$ satisfies the following VOLTERRA integral equation

$$C_X(t) = 1 + \int_{t_0}^{t} a(s_1)ds_1 \int_{t_0}^{s_1} b(s_2)\, C_X(s_2)ds_2 \ . \tag{6.5.15}$$

We may also write that

$$C_X(s_2) = 1 + \int_{t_0}^{s_2} a(s_3)ds_3 \int_{t_0}^{s_3} b(s_4)\, C_X(s_4)ds_4 \ ,$$

which we may then substitute into the right-hand side of (6.5.15) and recursively continue. Doing this, we find that we may write

$$C_X(t) = \sum_{n=0}^{\infty} F_n(t) \ , \tag{6.5.16}$$

where $F_0(t) = 1$ and for $n > 0$

$$F_n(t) = \int_{t_0}^{t} a(s)ds \int_{t_0}^{s} b(r)\, F_{n-1}(r)\, dr \ . \tag{6.5.17}$$

It may be shown that $F_n(t) \leq (1/n!)\{\int_{t_0}^{t} a(s)\ B(s)ds\}^n$, whence the infinite series (6.5.16) converges uniformly and absolutely on $S$ for $S = [0,T]$ with $T$ finite. In a similar fashion we may show that

$$S_X(t) = \frac{1}{\sqrt{\lambda_R}} \int_{t_0}^{t} a(s)\ \{\sum_{n=0}^{\infty} G_n(s)\}ds \ , \tag{6.5.18}$$

where $G_0(t) = 1$ and for $n > 0$

$$G_n(t) = \int_{t_0}^{t} b(s)ds \int_{t_0}^{s} a(r)\, G_{n-1}(r)dr \; . \tag{6.5.19}$$

Example 6.5.1. If $a(t) = k_a h(t)$ and $b(t) = k_b h(t)$ with $h(t) > 0$ for all $t > -\infty$, then $C_X(t) = \cosh\tau$ and $S_X(t) = \sinh\tau$, where $\tau(t) = \sqrt{\lambda_I} \int_0^t h(s)ds$.

Example 6.5.2. If $a(t) = k_a(t + C)^{\mu}$ and $b(t) = k_b(t + C)^{\nu}$ with $C \geq 0$ and both $\mu$ and $\nu > -1$, then

$$C_X(t) = \Gamma(q) \sum_{k=0}^{\infty} \left(\frac{\lambda_I}{\mu + \nu + 2}\right)^{2k} \frac{(t + C)^{k(\mu+\nu+2)}}{\{k!\ \Gamma(k + q)\}} \; ,$$

and

$$S_X(t) = \Gamma(p) \sum_{k=0}^{\infty} \left(\frac{\lambda_I}{\mu + \nu + 2}\right)^{2k+1} \frac{(t + C)^{k(\mu+\nu+2)+\mu+1}}{\{k!\ \Gamma(k + 1 + p)\}} \; ,$$

where $p = (\mu + 1)/(\mu + \nu + 2)$ and $q = 1-p$.

Before leaving the topic of time solutions to (6.5.1), let us record here some further important properties of such solutions. First of all, if the reader compares, for example, the X force level (6.5.6) with the corresponding constant-coefficient result (2.2.9), he will see that it is more complex. TAYLOR and BROWN [53] have shown that (6.5.6) only simplifies for $t_0 < 0$ when

$$a(t)/b(t) = k_a/k_b = \text{CONSTANT} \; , \tag{6.5.20}$$

since only then does a so-called algebraic addition theorem (see below) hold between the hyperbolic-like GLF.

THEOREM 6.5.2 (TAYLOR and BROWN [53]): For $t_0 < 0$, one can further simplify (6.5.6) if and only if $a(t)/b(t) = k_a/k_b = \text{CONSTANT}$ (constant ratio of attrition-rate coefficients).

Let us now give an example of how such an algebraic addition theorem helps us to simplify (6.5.6). Consider a constant-coefficient battle that begins at $t = t_1$. Equation (6.5.6) then yields

$$x(t) = x_0\left\{\cosh\sqrt{ab}\,t_1 \cosh\sqrt{ab}\,t - \sin\sqrt{ab}\,t_1 \sinh\sqrt{ab}\,t\right\}$$
$$- y_0\sqrt{\frac{a}{b}}\left\{\cosh\sqrt{ab}\,t_1 \sin\sqrt{ab}\,t - \sinh\sqrt{ab}\,t_1 \cosh\sqrt{ab}\,t\right\}, \tag{6.5.21}$$

which simplifies to

$$x(t) = x_0 \cosh\sqrt{ab}\,(t-t_1) - y_0\sqrt{\frac{a}{b}}\sinh\sqrt{ab}\,(t-t_1)\ , \tag{6.5.22}$$

due to the well-known algebraic addition theorems for the ordinary hyperbolic functions, e.g. $\cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v$.

As we have seen above in Section 6.4, when the ratio of attrition-rate coefficients is constant, i.e. (6.5.20) holds, we can transform the X force-level equation into one with constant coefficients by a transformation of the independent variable t. As we have seen, this situation leads to particularly convenient results.

In this respect, TAYLOR and BROWN have proved the following result.

THEOREM 6.5.3 (TAYLOR and BROWN [53]): A necessary and sufficient condition to be able to transform the $X$ force-level equation (6.5.7) by a transformation of the independent variable $t$ into a linear second-order ordinary differential equation with constant coefficients is that

$$\frac{1}{I(t)} \frac{d}{dt} \ln R(t) = \text{CONSTANT} . \tag{6.5.23}$$

In this case the desired substitution is given by

$$\tau = K \int^{t} \sqrt{a(s)\, b(s)}\, ds , \tag{6.5.24}$$

where $\int^{t} \ldots ds$ denotes an indefinite integral and $K$ is an arbitrary constant conveniently chosen.

Finally, the reader may be interested in the author's assessment as to just how difficult it is to develop analytical solutions to such LANCHESTER-type equations for modern warfare when there are temporal variations in fire effectiveness. Figure 6.11 shows the author's subjective estimate of such difficulties.

| | No Replacements | | Replacements | |
|---|---|---|---|---|
| | Constant Coefficients | Variable Coefficients | Constant Coefficients | Variable Coefficients |
| Two Homogeneous Forces | *Very Easy* | *Difficult* | *Easy* | *Very Difficult* |
| Two Homogeneous Forces With Supporting Fires Not Subject to Attrition | *Easy* | *Very Difficult* | *Not Too Easy* | *Very Difficult* |
| Heterogeneous Forces (Several Combatant Types) | *Difficult* | *Essentially Impossible* | *Very Difficult* | *Impossible* |
| Heterogeneous Forces (Many Combatant Types) | *Essentially Impossible* | *Impossible* | *Impossible* | *Impossible* |

Figure 6.11. Classification of LANCHESTER-type equations for "modern warfare" and their ease of solution by analytical methods (after L. von BERTALANFFY [3]).

## 6.6. Force-Annihilation-Prediction Conditions

It is important for the military operations analyst to have a clear understanding of how the initial force ratio and weapon-system-performance parameters interact to determine a battle's outcome. For any particular battle, we can always, of course, determine its outcome by explicitly computing the force-level trajectories and plotting them over time: the loser is simply the side that first reaches its battle-termination condition (see Section 3.3). The force-level trajectories may be generated either from the analytical results discussed in the previous section or more simply by numerical integration of the differential equations. This approach, however, is time consuming and by itself provides no understanding about the parametric dependence of battle outcome on the initial force levels and weapon-system-performance parameters. Moreover, as work by BONDER and FARRELL [10] and TAYLOR [43; 53] unfortunately shows, even the analytical (i.e. infinite-series) solution to variable-coefficient equations generally provides by itself (i.e. without explicitly computing force-level trajectories) little information about battle outcome because of its complexity.

Moreover, frequently the military operations analyst may only want to determine who is going to "win" a battle without having to spend the time and effort of explicitly computing the force-level trajectories. It is therefore of interest to develop battle-outcome-prediction (or victory-prediction) conditions that help one obtain insights into the dynamics of combat by explicitly portraying the relation between the various factors in the combat-attrition process and battle outcome. Specifically, one would like to have a (hopefully) simple expression that

relates battle outcome to the model's parameters. Thus, the military OR analyst is interested in developing battle-outcome-prediction conditions. Battle outcome, however, depends on the battle-termination model chosen, and modelling battle termination is a somewhat controversial topic as we saw in Chapter 3.

Although we are well aware that engagement termination is a complex random process for which it is by no means certain that force levels are the significant variables (*see* Chapter 3), we will consider two types of battle-termination conditions in this section:

(T1) battle terminated by one side's force level reaching its "breakpoint" value while the other side's force level has always been above its breakpoint value (force-level-breakpoint battle),

and (T2) battle terminated by the force ratio first reaching either of two given "breakpoint" force ratio values (force-ratio-breakpoint battle).

Moreover, in both cases we will only consider *deterministic breakpoints* here (*see* Section 3.4 for a further discussion), and we will accordingly refer to these engagements with deterministic battle-termination conditions as

(E1) fixed-force-level-breakpoint battle,

and

(E2) fixed-force-ratio-breakpoint battle.

The first type of battle-termination condition (T1) and the corresponding engagement with deterministic breakpoints (E1) have been discussed in Section 2.8 and Chapter 3 above, and thus it remains to discuss battle-termination-condition type (T2) and the corresponding engagement model with deterministic breakpoints (E2). Let us as usual denote the force ratio $x/y$ as $u$. Then for a fixed-force-ratio-breakpoint battle, we denote the "breakpoint" force ratio as $u_{BP}^{X}$ when $X$ terminates the battle (i.e. tries to "break off" the engagement), and as $u_{BP}^{Y}$ when $Y$ terminates the battle. The idea here is that, for example, X will decide to "break off" the engagement when he perceives a certain very unfavorable force ratio against him. These "breakpoint" force ratios then satisfy $0 \leq u_{BP}^{X} < u_0 < u_{BP}^{Y} \leq +\infty$.

Corresponding to a fight until the annihilation of one side or the other is the case in which $u_{BP}^{X} = 0$ and $u_{BP}^{Y} = +\infty$. Such a "fight-to-the-finish" may consequently be examined under either of the above two battle-termination conditions (T1) and (T2). BONDER and HONIG [11] have pointed out, however, that force annihilation may not always be the best criterion for evaluating the outcomes of simulated military operations. See BONDER and FARRELL [10, pp. 192-242] for a detailed LANCHESTER-type analysis of an attack scenario for with other "end of battle conditions" play the principal role. Nevertheless, it is of considerable interest (especially for developing insights into the dynamics of combat) to be able to easily predict the occurrence of force annihilation.

Thus, as we have discussed in Section 2.8 above, battle outcome depends on not only the dynamics of combat but also the battle-termination model considered. Consequently, we will generally obtain different victory-prediction conditions for the above two types of engagements:

(E1) fixed-force-level-breakpoint battle, and (E2) fixed-force-ratio-breakpoint battle. Moreover, it turns out that there are two different kinds of battle-outcome-prediction conditions that have been developed for the model (6.5.1):

(A) exact force-annihilation-prediction conditions (necessary and sufficient for the occurrence of force annihilation),

and

(B) simple approximate battle-outcome-prediction conditions (sufficient, but not necessary, for the occurrence of a particular type of outcome).

The first type of condition is essentially developed from results on the representation of solutions to (6.5.1), see equations (6.5.6) and (6.5.9) above. In retrospect, the author feels that the main value of (6.5.6) is that it may be used to develop these force-annihilation-prediction conditions. The second type of battle-outcome-prediction condition may be developed from considering the equation satisfied by the force ratio.

We will see that so-called higher transcendental functions, unfortunately, are usually involved (i.e. for $t_0 < 0$ and $a(t)/b(t) \neq$ CONSTANT) in the "exact" force-annihilation-prediction conditions. On the other hand, no higher transcendental functions are usually involved in the "simple approximate" battle-outcome-prediction conditions for a fixed-force-ratio-breakpoint battle, but many times one is unable to predict the outcome, i.e. there is a "gap" in this type of condition.

Concerning <u>exact force-annihilation-prediction conditions</u>, the author [52] (extending earlier results by TAYLOR and COMSTOCK [58]) has developed the following general result.

THEOREM 6.6.1 (TAYLOR [52]): The X force will be annihilated in finite time in LANCHESTER-type combat modelled with (6.5.1) if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R}\, F(Q^*_{max}) , \tag{6.6.1}$$

where $F(Q)$ is given by

$$F(Q) = \frac{C_X(0) - QS_X(0)}{QC_Y(0) - S_Y(0)} . \tag{6.6.2}$$

Neither side will be annihilated in finite time if and only if

$$\sqrt{\lambda_R}\, F(Q^*_{max}) \le \frac{x_0}{y_0} \le \sqrt{\lambda_R}\, F(Q^*_{min}) , \tag{6.6.3}$$

where

$$\lim_{t \to +\infty} \frac{S_X(t)}{C_X(t)} = \frac{1}{Q^*_{max}} = \frac{1}{\sqrt{\lambda_R}} \int_{t_0}^{+\infty} \frac{a(s)ds}{\{C_X(s)\}^2} \tag{6.6.4}$$

and

$$\lim_{t \to +\infty} \frac{S_Y(t)}{C_Y(t)} = Q^*_{mix} = \sqrt{\lambda_R} \int_{t_0}^{t} \frac{b(s)}{\{C_Y(s)^2\}} \tag{6.6.5}$$

We always have $Q^*_{min} \leq Q^*_{max}$ with $Q^*_{min} < Q^*_{max}$, with $Q^*_{min} < Q^*_{max}$ if and only if both $a(t) + b(t) \in L(t_0,+\infty)$.

The deterministic inequality (6.6.1) is the generalization of the well-known constant-coefficient force-annihilation-prediction condition given in Section 2.2 above (recall Proposition 2.2.1). We will call the parameters $Q^*_{max}$ and $Q^*_{min}$ defined by (6.6.4) and (6.6.5) in Theorem 6.6.1 the parity-condition parameters, since parity between the two forces (i.e. neither force annihilated in finite time) may be associated with them [see (6.6.3) above]. As (6.6.1) shows us, force-annihilation prediction may be expressed in terms of the following three parameters:

(P1) the initial force ratio, $u_0 = x_0/y_0$,

(P2) the relative-fire-effectiveness parameter, $\lambda_R = k_a/k_b$,

and (P3) the partity-condition parameter, $Q^* = Q^*_{max}$ or $Q^*_{min}$.

As Theorem 6.6.1 tells us, different parity-condition parameters are involved in the prediction of annihilation of the X force and in that of the Y force. These two parity-condition parameters are functionals depending on only the attrition-rate-coefficient functions $a(t)$ and $b(t)$ [see (6.6.4) and (6.6.5) above]. Depending on the boundedness of the total cumulative fire effectiveness of both sides (i.e. the integrability of the attrition-rate coefficients over the interval $[t_0,+\infty)$), however,

the values of these two parameters $Q^*_{min}$ and $Q^*_{max}$ may not be the same [i.e. $Q^*_{min} \leq Q^*_{max}$ with $Q^*_{min} < Q^*_{max}$ if and only if both $a(t)$ and $b(t) \in L(t_0,+\infty)$]. Thus, unless both $a(t)$ and $b(t) \in L(t_0,+\infty)$, only a single parameter, denoted simply as $Q^*$, is actually involved in force-annihilation prediction.

Let us now give a physical interpretation for the parity-condition parameter. TAYLOR and COMSTOCK [58, p. 355] have pointed out that we may consider $Q^*$ to be the initial $Y$ force level that leads to a draw[8] in the following fight-to-the finish (i.e. parity exists between the two forces) against an $X$ force of "unit strength"

$$\begin{cases} \dfrac{dE^-_X}{dt} = -\dfrac{a(t)}{\sqrt{\lambda_R}}\, E^-_Y & \text{with } E^-_X(t_0;Q) = 1 , \\[2ex] \dfrac{dE^-_Y}{dt} = -\sqrt{\lambda_R}\; b(t)\, E^-_X & \text{with } E^-_Y(t_0;Q) = Q , \end{cases} \tag{6.6.6}$$

where $E^-_X(t;Q)$ and $E^-_Y(t;Q)$ are so-called <u>subdominant solutions</u> which play the role of decreasing exponentials for the $X$ and $Y$ force-level equations. Let us denote any $Q \in [Q^*_{min}, Q^*_{max}]$ as $Q^*$. It follows from (6.6.3) and (6.6.6) that

$$E^-_X(t;Q^*) \text{ and } E^-_Y(t;Q^*) > 0 \text{ for all finite } t \geq t_0 . \tag{6.6.7}$$

Considering (6.6.6) and (6.6.7), we may think of $Q^*$ as "the Y-force equivalent of an X force of unit strength," since neither force is annihilated in finite time.

Let us now consider two examples of LANCHESTER-type battles for which the parity-condition parameter may be explicitly analytically determined. The first example shows the possibility of the existence of a finite range of values for the initial force ratio $x_0/y_0$ such that neither side is ever annihilated in battle, while the second analytically determines the parity-condition parameter for a very important specific case of attrition-rate coefficients (namely, power attrition-rate coefficients with "no offset" modelling, for example, combat between two opposing weapon-system types with the same maximum effective range). Further examples and use of such results in tactical analysis is given in Section 6.9 below.

Example 6.6.1. Consider combat modelled by (6.5.1) with the following attrition-rate coefficients

$$a(t) = k_a h(t) , \qquad \text{and} \qquad b(t) = k_b h(t) . \tag{6.6.8}$$

We assume that $h(t) > 0$ for all $t > -\infty$, and then $t_0 = 0$. It follows (see Sections 6.4 and 6.5) that $C_X(t) = C_Y(t) = \cosh \tau$ and $S_X(t) = S_Y(t) = \sinh \tau$, where $\tau(t) = \tau_I \int_0^t h(s)ds$. Denote $\lim_{t\to+\infty} \tau(t)$ as $M$. It follows that

$$Q^*_{min} = \frac{1 - e^{-2n}}{1 + e^{-2n}} = \frac{1}{Q^*_{max}} \leq 1 . \tag{6.6.9}$$

Thus, $Q^*_{min} < Q^*_{max}$ if and only if $M < +\infty$ if and only if $h(t) \in L(0,+\infty)$.[9] Theorem 6.6.1 tells us that $X$ will be annihilated if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R}\left(\frac{1 - e^{-2n}}{1 + e^{-2n}}\right) .$$

Furthermore, neither X nor Y will be annihilated in finite time for

$$\sqrt{\lambda_R}\left(\frac{1 - e^{-2n}}{1 + e^{-2n}}\right) \le \frac{x_0}{y_0} \le \sqrt{\lambda_R}\left(\frac{1 + e^{-2n}}{1 - e^{-2n}}\right) .$$

Example 6.6.2. Consider combat modelled by (6.5.1) with the following power attrition-rate coefficients with no offset

$$a(t) = k_a(t + C)^{\mu}, \qquad \text{and} \qquad b(t) = k_a(t + C)^{\nu}, \tag{6.6.10}$$

where $C \geq 0$. It follows that $t_0 = -C$. As we saw in Section 6.2 above, these coefficients may be taken to model, for example, the constant-speed attack of a mobile force against the static defensive position of an enemy force in which each side's fire effectiveness varies as a power of the range between the two opposing forces. These particular coefficients (6.6.10) model combat between two opposing forces armed with weapon systems with the same maximum effective range, i.e. set $D = 0$ in (6.2.9). The assumption that both $a(t)$ and $b(t) \in L(t_0,T)$ for any finite $T \geq t_0$ yields that we must have $\mu$ and $\nu > -1$, and consequently both $a(t)$

and $b(t) \notin L(t_0,+\infty)$ so that $Q^*_{min} = Q^*_{max} = Q^*$. Considering (6.5.7), (6.5.8), (6.5.10), and (6.5.11), one may show that (see [53, p. 52])

$$C_X(t) = \Gamma(q)\left(\frac{\lambda_I}{\mu + \nu + 2}\right)^p (t + C)^{(\mu+1)/2} I_p(T) \;, \tag{6.6.11}$$

$$S_X(t) = \Gamma(p)\left(\frac{\lambda_I}{\mu + \nu + 2}\right)^q (t + C)^{(\mu+1)/2} I_{-p}(T) \;, \tag{6.6.12}$$

$$C_Y(t) = \Gamma(p)\left(\frac{\lambda_I}{\mu + \nu + 2}\right)^q (t + C)^{(\nu\;1)/2} I_q(T) \;, \tag{6.6.13}$$

and

$$S_Y(t) = \Gamma(q)\left(\frac{\lambda_I}{\mu + \nu + 2}\right)^p (t + C)^{(\nu+1)/2} I_{-q}(T) \;, \tag{6.6.14}$$

where $\lambda_I = \sqrt{k_a k_b}$, $I_p(T)$ denotes the modified BESSEL function of the first kind of order $p$ (e.g. see LEBEDEV [27, p. 108], OLVER [34, p. 60], or WATSON [60, p. 77]), $p = (\mu+1)/(\mu+\nu+2)$, $q = 1 - p$, and

$$T(t) = \lambda_I \frac{(t + C)^{(\mu+\nu+2)/2}}{\{(\mu + \nu + 2)/2\}} \;. \tag{6.6.15}$$

Hence,

$$\frac{1}{Q^*} = \lim_{t \to +\infty} \frac{S_X(t)}{C_X(t)} = \frac{\Gamma(p)}{\Gamma(q)} \left(\frac{\lambda_I}{\mu + \nu + 2}\right)^{q-p} \lim_{t \to +\infty} \frac{I_p(T)}{I_{-p}(T)} \;. \tag{6.6.16}$$

We observe that $\mu$ and $\nu > -1$ implies that $0 < p, q < 1$ and also that $T \to +\infty$ as $t \to +\infty$. Using the so-called asymptotic representation for modified BESSEL functions of the first kind (e.g. see OLVER [34, p. 269]), one may show that on the real line $\lim_{\xi \to +\infty} \{I_\alpha(\xi)/I_\beta(\xi)\} = 1$ for all real values of $\alpha$ and $\beta$. It follows from (6.6.16) that

$$Q^* = \frac{\Gamma(q)}{\Gamma(p)} \left( \frac{\lambda_I}{\mu + \nu + 2} \right)^{p-q} , \tag{6.6.17}$$

and hence [from (6.6.11) through (6.6.14) above]

$$F(Q^*) = C^{(\mu-\nu)/2} \frac{\{I_{-p}(T_0) - I_p(T_0)\}}{\{I_{-q}(T_0) - I_q(T_0)\}} , \tag{6.6.18}$$

where $T_0$ denotes $T(0)$. At the expense of some mathematical obscurity, the expression (6.6.18) may be written in the somewhat simpler form

$$F(Q^*) = \frac{q^p}{p^q} \left( \frac{\lambda_I}{\mu + \nu + 2} \right)^{q-p} \frac{A_\alpha(\xi_X)}{A_\beta(\xi_Y)} , \tag{6.6.19}$$

where $A_\alpha(\xi)$ denotes the generalized AIRY function of the first kind of (nonintegral) order $\alpha$ (see SWANSON and HEADLEY [42, pp. 1401-1402]), $\alpha = (\nu-\mu)/(\mu+1)$, $\beta = (\mu-\nu)/(\nu+1)$, $\xi_X = [\lambda_I/(\mu+1)]^{2p} C^{\mu+1}$, and $\xi_Y = [\lambda_I/(\nu+1)]^{2q} C^{\nu+1}$. Theorem 6.6.1 then tells us that the X force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R} \left( \frac{\lambda_I}{\mu + \nu + 2} \right)^{q-p} \frac{q^p A_\alpha(\xi_X)}{p^q A_\beta(\xi_Y)} , \tag{6.6.20}$$

which for $t_0 = 0$ simplifies to

$$\frac{x_0}{y_0} < \sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu + \nu + 2}\right)^{q-p} \frac{\Gamma(p)}{\Gamma(q)} \quad . \tag{6.6.21}$$

Concerning simple approximate battle-outcome-prediction conditions, the author [45] (see also TAYLOR and PARRY [59]) has shown that under the appropriate conditions $x_0/y_0 < \sqrt{a_0/b_0}$ implies that the X force will lose a fixed-force-ratio-breakpoint battle in finite time. Here $a_0$ denotes $a(0)$ and similarly for $b_0$. A fight-to-the-finish is, of course, just a special case of such a battle. More precisely, we have

THEOREM 6.6.2 (TAYLOR [45]): Assume that $b(t) \notin L(0,+\infty)$ and that $R(t) = a(t)/b(t)$ is nondecreasing. Then for LANCHESTER-type combat modelled with (6.5.1),

$$\frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}} \tag{6.6.22}$$

implies that the X force will lose a fixed-force-ratio-breakpoint battle in finite time.

PROOF. Introducing the force ratio $u = x/y$, we find that it satisfies the Riccaci equation (see Appendix A.3)

$$\frac{du}{dt} = b(t)u^2 - a(t) \qquad \text{with } u(0) = u_0 = x_0/y_0 \ . \tag{6.6.23}$$

Let $u_+(t) = \sqrt{R(t)} = \sqrt{a(t)/b(t)}$ denote the positive root of the quadratic equation $b(t)u^2 - a(t) = 0$, and observe that $du/dt < 0$ for any positive $u < u_+(t)$ (see Figure 2.7). The assumption that $R(t)$ is nondecreasing then yields that $u_+(t)$ is nondecreasing. It is readily shown that $du/dt(0) < 0$ and $u_+(t)$ nondecreasing imply that $du/dt(t) < 0$ for all $t \geq 0$ (e.g. see Section 2.2 above or TAYLOR and PARRY [59, pp. 526-527]). Consequently, when (6.6.22) holds and $R(t)$ is nondecreasing, it follows that $du/dt(t) < 0$ for all $t \geq 0$. It then remains to be shown that X's breakpoint force ratio is reached in finite time.[10] Observing that $a_0 < +\infty$ and $b_0 > 0$, we find that under the stated conditions

$$\frac{du}{dt} = b(t)\,\{u^2 - R(t)\} \leq \frac{b(t)}{b_0}\,\{b_0 u_0^2 - a_0\} = \left\{\frac{b(t)}{b_0}\right\}\frac{du}{dt}\,(0).$$

Thus,

$$u(t) = u_0 + \int_0^t \left(\frac{du}{dt}\right) dt \leq u_0 + \left\{\frac{1}{b_0}\,\frac{du}{dt}\,(0)\right\}\int_0^t b(s)ds\,, \qquad (6.6.24)$$

whence $b(t) \notin L(0,+\infty)$ implies that $u(t)$ goes to $u_{BP}^X$ in finite time.

Q.E.D

The above proof of Theorem 6.6.2 is particularly important, since it may be extended to more general models, e.g. (6.13.1) (see Theorem 6.13.3 below). Moreover, the role of the assumption that $b(t) \notin L(0,+\infty)$ in guaranteeing that the battle is driven to termination is clearly shown in the above proof.[11]

By considering LIOUVILLE's so-called normal form (see INCE [23, p. 271]) for the Y force-level equation, the author [45, p. 197] has also developed the following complementary result

THEOREM 6.6.3 (TAYLOR [45]): Assume that

$0 < R(0) < +\infty$ and that $\lim_{T \to +\infty} \int_{t_0}^{T} \sqrt{a(t)\, b(t)}\, dt \to \infty$.

Let $\tau(t) = \int_{t_0}^{T} \sqrt{a(s)\, b(s)}\, ds$,

$$G(\tau) = \frac{Q''(\tau)}{Q(\tau)}, \quad \text{and} \quad Q(\tau) = [R(t)]^{1/4}, \tag{6.6.25}$$

where $Q'(\tau)$ denotes $dQ/d\tau$. If $G(\tau) \leq 0$ for all $\tau \geq 0$, then

$$\frac{x_0}{y_0} > \sqrt{\frac{a_0}{y_0}}\,(1 + \varepsilon_0) \tag{6.6.26}$$

implies that the $Y$ force will be annihilated in finite time. Here $\varepsilon_0$ denotes $(1/\sqrt{a_0 b_0})[d/dt\ \ell n\{a(t)/b(t)\}^{1/4}]$. Furthermore, if $dR/dt \geq 0$ for all $t \geq 0$, then $Y$ will lose a fixed-force-ratio-breakpoint battle in finite time.

The deterministic inequalities (6.6.22) and (6.6.26) show us the complementary nature of Theorems 6.6.2 and 6.6.3: if the initial force ratio $u_0 = x_0/y_0$ is below a certain critical value, Theorem 6.6.2 predicts that $Y$ will win a fixed-force-ratio-breakpoint battle; while if $u_0$ exceeds a second critical value, Theorem 6.6.3 predicts the $X$ will win.

Example 6.6.3. Again we consider combat modelled by (6.5.1) with the power attrition-rate coefficients with no offset (6.6.10) and $C > 0$. Without loss of generality, we may assume that $\mu \geq \nu$ and then

$dR/dt \geq 0$ (i.e. $R(t)$ is nondecreasing). Theorem 6.6.2 then yields that $Y$ will win a fixed-force-ratio-breakpoint battle in finite time if

$$\frac{x_0}{y_0} < \sqrt{\frac{k_a}{k_b}} \; C^{(\mu-\nu)/2} \; . \tag{6.6.27}$$

In preparation for invoking Theorem 6.6.3, we compute

$$\tau(t) = \left(\frac{2\lambda_I}{\mu + \nu + 2}\right)(t + C)^{(\mu+\nu+2)/2} \tag{6.6.28}$$

and

$$G(\tau) = \frac{(\nu-\mu)(\mu + 3\nu + 4)}{4(\mu + \nu + 2)^2 \tau^2} \; . \tag{6.6.29}$$

We observe that $G(\tau) \leq 0$ for all $\tau(t) \geq \tau(0)$ and also $\varepsilon_0 \geq 0$ if and only if $\mu \geq \nu$. Hence, Theorem 6.6.3 yields that $X$ will win a fixed-force-ratio-breakpoint battle in finite time if

$$\frac{x_0}{y_0} > \sqrt{\frac{k_a}{k_b}} \; C^{(\mu-\nu)/2} + \frac{(\mu-\nu)}{4k_b} \; C^{-(\nu+1)} \; . \tag{6.6.30}$$

The complementary nature of Theorems 6.6.2 and 6.6.3 is clearly shown by the victory-prediction conditions (6.6.27) and (6.6.30). However, these deterministic inequalities also show us that these simple approximate victory-prediction conditions fail to predict the outcome of battle when

$$\sqrt{\frac{k_a}{k_b}} \, C^{(\mu-\nu)/2} \leq \frac{x_0}{y_0} \leq \sqrt{\frac{k_a}{k_b}} \; C^{(\mu-\nu)/2} + \frac{(\mu-\nu)}{4k_b} \, C^{-(\nu+1)} \; . \tag{6.6.31}$$

Further results and examples are given in TAYLOR [39].

Let us now elaborate further upon the general nature of the victory-prediction conditions given in Theorems 6.6.2 and 6.6.3. Our examination will also yield that there is a "gap" in these victory-prediction conditions: for a certain given range of values for the initial force ratio, we cannot forecast the outcome of battle. To see the complementary nature of these conditions, we observe that under the appropriate conditions, Theorem 6.6.2 yields (for $dR/dt \geq 0$ always)

$$Y \text{ will win if } \frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}} , \tag{6.6.32}$$

while Theorem 6.6.3 yields (for $G(\tau) \leq 0$ always and $\varepsilon_0 \geq 0$)

$$X \text{ will win if } \frac{x_0}{y_0} > (1 + \varepsilon_0) \sqrt{\frac{a_0}{b_0}} . \tag{6.6.33}$$

Moreover, for many attrition-rate coefficients of tactical interest (e.g. the power attrition-rate coefficients with no offset), we have that $dR/dt \geq 0$ if and only if $G(\tau) \leq 0$ if and only if $\varepsilon_0 \geq 0$, although these if-and-only-if statements do not generally hold. In such cases, though, we observe that for

$$\sqrt{\frac{a_0}{b_0}} \leq \frac{x_0}{y_0} \leq (1 + \varepsilon_0) \sqrt{\frac{a_0}{b_0}} \tag{6.6.34}$$

we cannot predict by this approach who will be the loser of a fixed-force-ratio-breakpoint battle. Thus, there is a "gap" in these simple approximate battle-outcome-prediction conditions (see Figure 6.12).

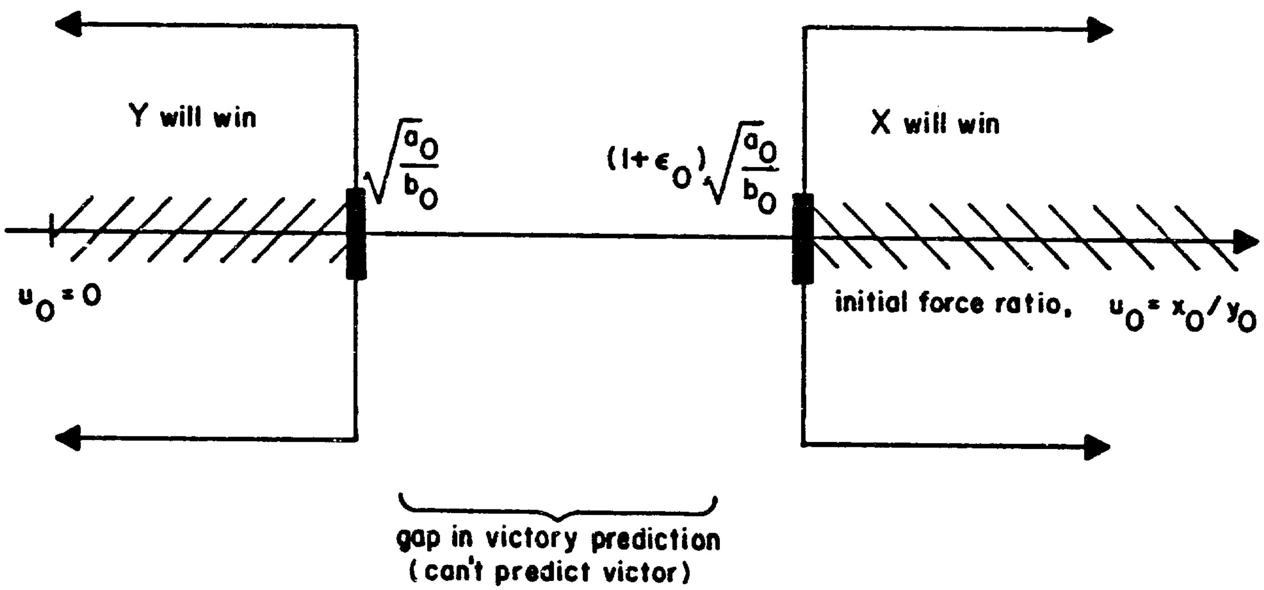


Figure 6.12. The "gap" in the simple approximate victory-prediction conditions (6.6.32) and (6.6.33) for $dR/dt \geq 0$, $G(\tau) \leq 0$, and $\epsilon_0 \geq 0$.

The significant thing to note about the simple approximate victory-prediction conditions (6.6.32) and (6.6.33) is that although they are rather strong sufficient conditions, they are very simple: they involve only simple functions of the initial conditions and initial values of the attrition-rate coefficients plus assumptions about the behavior over time of the attrition-rate coefficients. No "special" mathematical functions are involved, although this is not true for the exact force-annihilation-prediction conditions given in Theorem 6.6.1 except for the special case in which $a(t)/b(t) \equiv$ CONSTANT. However, as shown by both (6.6.34) and Figure 6.12, there is a "gap" in these simple approximate victory-prediction conditions. The price of removing this "gap" is the introduction of higher trandscendental functions (see, for example, TAYLOR and COMSTOCK [58, p. 350]). Furthermore, "exact" results with no such gap in victory prediction are apparently only possible for a fight-to-the-finish in which one side or the other is to be annihilated (see also Sections 3.5 and 3.6 above).

## 6.7. Parametric Dependence of the Parity-Condition Parameter.

We have seen in Section 2.2 that for a LANCHESTER-type $F|F$ attrition process with constant attrition-rate coefficients, $Y$ will win a fight-to-the-finish in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}} \quad . \tag{6.7.1}$$

Thus, when there are no temporal variations in fire effectiveness, annihilation of a force depends on only two relative factors, namely: (I) the initial force ratio $u_0 = x_0/y_0$, and (II) the relative fire effectiveness $R = a/b$. Theorem 6.6.1 generalizes (6.7.1) to homogeneous-force combat modelled by (6.5.1) with the temporal variations in fire effectiveness. It tells us that, for example, the annihilation of the $X$ force depends on the following three factors

(F1) the initial force ratio, $u_0 = x_0/y_0$,

(F2) the relative-fire-effectiveness parameter, $\lambda_k = k_a/k_b$,

and (F3) the parity-condition parameter, $Q^* = Q^*_{max}$ ,

when there are temporal variations in fire effectiveness. The first two factors, (F1) and (F2), are clearly relative ones, and explicitly depend on certain given parameters in our combat model.

How does the parity-condition parameter $Q^*$ depend on the input parameters to our simple combat model (6.5.1)? This is an important

question for the military OR worker to answer, since its answer will help him to better understand how force-level and weapon-system-performance factors interact to determine the outcome of battle. In our examination here we will show that for time-dependent attrition-rate coefficients the outcome of battle no longer depends on just relative factors but that the intensity of combat generally also influences the battle's outcome. Specifically, we will determine on which input parameters of the model (6.5.1) the parity-condition parameter depends for the special case of unlimited firepower for one or both sides, i.e. either $a(t) \notin L(0,+\infty)$ or $b(t) \notin L(0,+\infty)$. In this case $Q^*_{min} = Q^*_{max}$, and we will denote this common value simply as $Q^*$. Theorem 6.6.1 then takes the following form.

THEOREM 6.7.1: Assume that either $a(t) \notin L(0,+\infty)$ or $b(t) \notin L(0,+\infty)$. Then the $X$ force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R} \left\{ \frac{C_X(0) - Q^* S_X(0)}{Q^* C_Y(0) - S_Y(0)} \right\}, \tag{6.7.2}$$

where the parity-condition parameter $Q^*$ is unique and given by

$$\lim_{t \to +\infty} \frac{S_X(t)}{C_X(t)} = \frac{1}{Q^*} = \frac{1}{\sqrt{\lambda_R}} \int_{t_0}^{\infty} \frac{a(s)ds}{\{C_X(s)\}^2} . \tag{6.7.3}$$

We also have that

$$\lim_{t \to +\infty} \frac{S_Y(t)}{C_Y(t)} = Q^* = \sqrt{\lambda_R} \int_{t_0}^{\infty} \frac{b(s)ds}{\{C_Y(s)\}^2} \quad . \tag{6.7.4}$$

Also, neither side will be annihilated in finite time if and only if the inequality sign in (6.7.2) is replaced by an equality sign.

We will henceforth in this section assume that either $a(t) \notin L(0,+\infty)$ and/or that $b(t) \notin L(0,+\infty)$. For determining the parametric dependence of the parity-condition parameter $Q^*$, it is convenient to introduce a new independent variable $s$ defined by

$$s(t) = K \lambda_I \int_{t_0}^{t} g(\sigma)\, d\sigma, \tag{6.7.5}$$

where the parameter $K$ is to be chosen to simplify the form of $J(s)$ given by (6.7.7) below. We denote $s(0)$ as $s_0$, and then $s_0 \geq 0$ if and only if $t_0 \leq 0$. The substitution (6.7.5) transforms the $X$ force-level equation (6.5.7) into the normal form (e.g. see KAMKE [24]).

$$\frac{d^2x}{ds^2} - J(s)x = 0 \ , \tag{6.7.6}$$

where the so-called invariant $J(s)$ of the normal form is given by

$$J(s) = \frac{1}{K^2} \left\{ \frac{h(t)}{g(t)} \right\}, \tag{6.7.7}$$

and $t = t(s)$ via (6.7.5). We also define the normal-form hyperbolic-like GLF $c_X(s)$ and $s_X(s)$, which satisfy (6.7.6) with the initial conditions

$$c_X(0) = 1, \quad c_X'(0) = 0, \quad \text{and} \quad s_X(0) = 0, \quad s_X'(0) = 1, \tag{6.7.8}$$

where (for example) $c_X'(s)$ denotes $dc_X/ds$. It follows that

$$c_X(s) = C_X(t(s)) \ , \quad \text{and} \quad s_X(s) = KS_X(t(s)) \ , \tag{6.7.9}$$

where $t = t(s)$ by the inversion of (6.7.5). The corresponding Y functions (see TAYLOR [51] for further details) are analogously defined to satisfy $c_Y(s) = C_Y(t(s))$ and $s_Y(s) = (1/K)\ S_Y(t(s))$.

It then turns out that the parity-condition parameter $Q^*$ may only depend on the combat-intensity parameter $\lambda_I$ as the following theorem shows.

THEOREM 6.7.2 (TAYLOR [51]): The parity-condition parameter $Q^*$ does not depend on the relative-fire-effectiveness parameter $\lambda_R$ but may depend on the combat-intensity parameter $\lambda_I$. It is independent of $\lambda_I$ if and only if the ratio of attrition-rate coefficients is constant, i.e. $a(t)/b(t) = \text{CONSTANT}$.

The above theorem may be proved by considering the differential equation satisfied by the quotient $s_X/c_X$ (see TAYLOR [51] for further details). It is also worth noting that the force-annihilation-prediction condition (6.7.2) may be written in terms of the normal-form hyperbolic-like GLF as

$$\frac{x_0}{y_0} < \frac{\sqrt{\lambda_R}}{K} \quad \frac{c_X(s_0) - Z^* s_X(s_0)}{Z^* c_Y(s_0) - s_Y(s_0)} \quad , \tag{6.7.10}$$

where the modified parity-condition parameter $Z^*$ is given by

$$Z^* = Q^*/K \; . \tag{6.7.11}$$

We also have that

$$\lim_{s \to +\infty} \frac{s_X(s)}{c_X(s)} = \frac{1}{Z^*} \tag{6.7.12}$$

By choosing $K$ in (6.7.5) in the right way, we can sometimes factor $Q^*$ into two terms, one of which (i.e. $K$) depends on $\lambda_I$ and one (i.e. $Z^*$) that does not. Theorem 6.7.3 shows us when this factorization is probable.

THEOREM 6.7.3 (TAYLOR [51]): The modified parity-condition parameter $Z^*$ of (6.7.10) is independent of the combat-intensity parameter $\lambda_I$ if and only if the invariant $J(s)$ of the normal form is of the form $J(s) = s^\alpha$. In this case, the parameter $K$ depends on the combat-intensity parameter $\lambda_I$ and is free from $\lambda_I$ if and only if $a(t)/b(t)$ is constant.

TAYLOR [51] has also shown that when the invariant $J(s) = s^\alpha$,

$$Z^* = p^{2p-1} \frac{\Gamma(1-p)}{\Gamma(p)} \tag{6.7.13}$$

with $p = 1/(2 + \alpha)$. In this case

$$c_X(s) = F_w(S), \qquad s_X(s) = p^{(1-2p)} H_p(S),$$
$$c_Y(s) = F_p(S), \quad \text{and} \quad s_Y(s) = p^{(2p-1)} H_q(S), \tag{6.7.14}$$

where $q = 1-p$, $S(s) = 2ps^{1/(2p)}$, and $F_\nu$ and $H_\nu$ denote LANCHESTER-CLIFFORD-SCHLÄFLI functions of order $\nu$ (see Section 6.9 below). TAYLOR has also shown that when $h(t) = C_1\{g(t)\}^\nu$ with $C_1$ an arbitrary constant [recall (6.5.2)], then the modified parity-condition parameter $Z^*$ can be chosen to be independent of the combat-intensity parameter $\lambda_I$ if and only if either $g(t) = (t-t_0)^\mu$ or $g(t) = e^{\lambda_a t}$. This latter result also implies that the same mathematical functions may be used to analyze "aimed-fire" combat modelled by (6.2.4) with both the power attrition-rate coefficients with "no offset" (6.6.10) [i.e. set $D = 0$ in (6.2.9)] and also the exponential attrition-rate coefficients (6.2.12).

Theorems 6.7.2 and 6.7.3 show how the parity-condition parameter $Q^*$ depends on the combat-intensity parameter $\lambda_I$ and the relative-fire-effectiveness parameter $\lambda_R$. In contrast to the classic constant-coefficient results, we saw that battle outcome (i.e. force annihilation through $Q^*$) depends on $\lambda_I$ unless the ratio of attrition-rate coefficients is constant, i.e., $a(t)/b(t) =$ constant. It is doubtful that one would ever have learned about such dependence merely by numerically determining the parity-condition parameter (see the next section). Thus, our theoretical investigation here has yielded some important insights into the dynamics of combat that would be otherwise difficult to perceive.

## 6.8. Numerically Determining the Parity-Condition Parameter

The result (6.7.3) suggests a numerical procedure for approximately determining the parity-condition parameter $Q^*$ in those cases for which explicit analytical results are not available: we may approximate the parity-condition parameter $Q^*$ by $\hat{Q} = 1/\{S_X(\hat{t})/C_X(\hat{t})\}$, where $\hat{t}$ is a "suitably large" value of $t$. In other words, we may estimate $Q^*$ simply by picking a large value for $t$ (we denote this selected large value by $\hat{t}$), computing $S_X(\hat{t})$ and $C_X(\hat{t})$, and then forming their ratio. Our estimate of $Q^*$ is then given by $\hat{Q} = 1/\{S_X(\hat{t})/C_X(\hat{t})\}$. The only problem is that we do not know right now how large to take $\hat{t}$ for "satisfactory" estimation of $Q^*$: there is an estimation error $Q^* - \hat{Q}(\hat{t})$, which depends monotonically on $\hat{t}$, and a priori we do not know how large this error is. In this section we give a bound on the magnitude of this error, and this error estimate allows the goodness of approximation to be easily evaluated in many cases of interest.

In actual practice we have found it more convenient to numerically determine the modified parity-condition parameter $Z^*$ defined by (6.7.12). Our idea is to use knowledge about the modified parity-condition parameter $Z^*$ corresponding to one pair of attrition-rate coefficients, denoted as $a(t)$ and $b_1(t)$, to numerically determine $Z^*$ for a related pair, $a(t)$ and $b(t)$. With this in mind, let us denote $c_X(s)$ corresponding to $a(t)$ and $b(t)$ as $c_X(s;a,b)$, and similarly for $s_X$ and $n_X = s_X/c_X$. In other words, we will now write

$$n_X(s;a,b) = s_X(s;a,b)/c_X(s;a,b) \quad . \tag{6.8.1}$$

In this notation, we may write (6.7.12) as

$$\lim_{s \to +\infty} \eta_X(s;a,b) = \frac{1}{Z^*[a,b]} , \tag{6.8.2}$$

where $Z^*[a,b]$ denotes that the modified parity-condition parameter is a functional (i.e. a function for which the independent variables themselves are functions), depending on only the attrition-rate coefficients $a(t)$ and $b(t)$.

The relation (6.8.2) suggests that we should estimate $Z^*[a,b]$ with $Z$ defined by

$$\hat{Z}(\hat{s};a,b) = 1/\eta_X(\hat{s};a,b) , \tag{6.8.3}$$

where $\hat{s}$ denotes a suitably chosen value for $s$. It may be shown that $\eta_X(s;a,b)$ in a strictly increasing function of $s$ so that the larger we take $\hat{s}$ in (6.8.3), the better our approximation becomes. How large should we take $\hat{s}$ for "satisfactory" estimation of $Z^*$? What is the error made by taking $\hat{Z}(\hat{s};a,b)$ as an estimate of $Z^*[a,b]$? The answer to this latter question involves comparison with known results for $Z^*$ and helps us to determine how large to take $\hat{s}$. Theorem 6.8.1 (an error estimate for our approximation) tells us exactly how large to take $s$.

THEOREM 6.8.1 (TAYLOR and BROWN [51]): Assume that $b_1(t) < b(t)$ for all finite $t < t_0$. Let $f_E(\hat{s})$ denote the fractional error made in the estimation of $Z^*[a,b]$ by $Z(\hat{s};a,b)$, i.e.

$$f_E(\hat{s}) = \frac{\hat{Z}(s;a,b) - Z^*[a,b]}{Z^*[a,b]} . \tag{6.8.4}$$

Then

$$0 < f_E(\hat{s}) < \{1/Z^*[a,b_1] - \eta_X(\hat{s};a,b_1)\}\ \hat{Z}(\hat{s};a,b) . \tag{6.8.5}$$

Thus, we have presented a method for numerically determining $Z^*[a,b]$: we simply pick a large value for $\hat{s}$ (and denote the selected value as $\hat{s}$), compute $s_X(\hat{s})$ and $c_X(\hat{s})$, and then compute the estimate $Z(\hat{s};a,b)$ according to (6.8.3). Theorem 6.8.1 allows us to know the accuracy of our approximation, which can be improved by taking $\hat{s}$ larger. Accordingly, we can numerically determine $Z^*[a,b]$ to any specified degree of accuracy once $Z^*[a,b_1]$ is known. Moreover, exact analytical results for the modified parity-condition parameter $Z^*$ have been obtained for only the two cases of attrition-rate coefficients considered in Section 6.5 above: namely, (I) a constant ratio of attrition-rate coefficients, and (II) power attrition-rate coefficients with "no offset." We will now show how to use the latter known results to numerically determine (by comparison with the known results via Theorem 6.8.1) the parity condition parameter in a very important related case.

We will now apply the above theory to the analysis of battles modelled by LANCHESTER-type equations of modern warfare (6.5.1) with power attrition-rate coefficients with "positive offset," i.e.

$$a(t) = k_a(t + C)^{\mu} \qquad \text{and} \qquad b(t) = k_b(t + C + D)^{\nu} , \tag{6.8.6}$$

with $C \geq 0$ and $D > 0$ [cf. (6.2.9)]. In order that $a(t) \in L(t_0,T)$ for any finite $T \geq t_0$ we must have $\mu > -1$, and hence $a(t) \notin L(0,+\infty)$ so that Theorem (6.7.1) holds. If we choose $K = [\lambda_I/(\mu + 1)]^{2p-1}$, then it follows from (6.7.5) that the modified time variable $s$ is given by

$$s(t) = \left(\frac{\lambda_I}{\mu + 1}\right)^{2p} (t + C)^{\mu+1} , \tag{6.8.7}$$

and the invariant $J(s)$ of the normal form (6.7.6) simplifies to

$$J(s;a,b) = J(s;\gamma,\mu,\nu) = s^{\beta} \left(1 + \frac{\gamma}{s^{\alpha}}\right)^{\nu} , \tag{6.8.8}$$

where $p = (\mu+1)/(\mu + \nu + 2)$, $\alpha = 1/(\mu+1)$, $\beta = (\nu-\mu)/(\mu+1)$, and $\gamma = D\,[\lambda_I/(\mu+1)]^{2/(\mu+\nu+2)}$. Here we have denoted the invariant corresponding to the attrition-rate coefficients $a(t)$ and $b(t)$ as $J(s;\gamma,\mu,\nu)$, since we may take $\gamma$, $\mu$, and $\nu$ as a basis for generating the four parameters $\alpha$, $\beta$, $\gamma$, and $\nu$ that explicitly appear in the right-hand side of (6.8.8). Furthermore, we will denote the normal-form hyperbolic-like GLF that correspond to $J(s;\gamma,\mu,\nu)$ as $c_X(s;\gamma,\mu,\nu)$ and $s_X(s;\gamma,\mu,\nu)$.

We can now use the known results for the power attrition-rate coefficients with "no offset" (6.6.10) to assure that $Z^*[a,b] = Z^*(\gamma,\mu,\nu)$ is numerically determined to within any specified degree of accuracy. Let $T_\alpha = F_\alpha/H_{1-\alpha}$ denote the quotient of two LANCHESTER-CLIFFORD-SCHLÄFLI (LCS) functions (see the next section). Then the following theorem tells us exactly how large to take $\hat{s}$ for the estimation of $Z^*(\gamma > 0,\mu,\nu)$ by $\hat{Z}(\hat{s};\gamma,\mu,\nu)$ to any desired degree of accuracy.

THEOREM 6.8.2 (TAYLOR and BROWN [51]): For a battle modified by LANCHESTER-type equations of modern warfare (6.5.1) with power attrition-rate coefficients with "positive offset" (6.8.6), if we estimate $Z^*(\gamma,\mu,\nu)$ with $\hat{Z}(\hat{s};\gamma,\mu,\nu)$ defined by

$$\hat{Z}(\hat{s};\gamma,\mu,\nu) = 1/\eta_X(\hat{s};\gamma,\mu,\nu) \ , \tag{6.8.9}$$

then bounds on the fractional error made in this approximation are given by

$$0 < f_E(\hat{s}) < p^{q-p} \left\{ \frac{\Gamma(p)}{\Gamma(q)} - T_q(\hat{S}) \right\} \hat{Z}(\hat{s};\gamma,\mu,\nu) \ , \tag{6.8.10}$$

where $q = 1-p$, $S(s) = 2ps^{1/(2p)}$, and $\eta_X(s;\gamma,\mu,\nu)$ denotes the quotient of two normal-form hyperbolic like GLF for the attrition-rate coefficients (6.8.6), i.e. $\eta_X(s;\gamma,\mu,\nu) = s_X(s;\gamma,\mu,\nu)/c_X(s;\gamma,\mu,\nu)$. Also, $\hat{S}$ denotes $S(\hat{s})$, and $f_E(\hat{s})$ denotes the fractional error defined by (6.8.4).

In order to numerically determine the modified parity-condition parameter for the offset power attrition-rate coefficients (6.8.6), we must use knowledge about how quickly the limiting value (i.e. $Z^*[a,b_1]$) of a hyperbolic-tangent-like function of a related pair of coefficients [denoted as $a(t)$ and $b_1(t)$], power attrition-rate coefficients with "no offset" (6.6.10), is reached as its argument increases without bound. In Figure 6.13 we see that this limiting value, denoted as $Z^*(\mu,\nu) = Z^*[a,b_1]$,

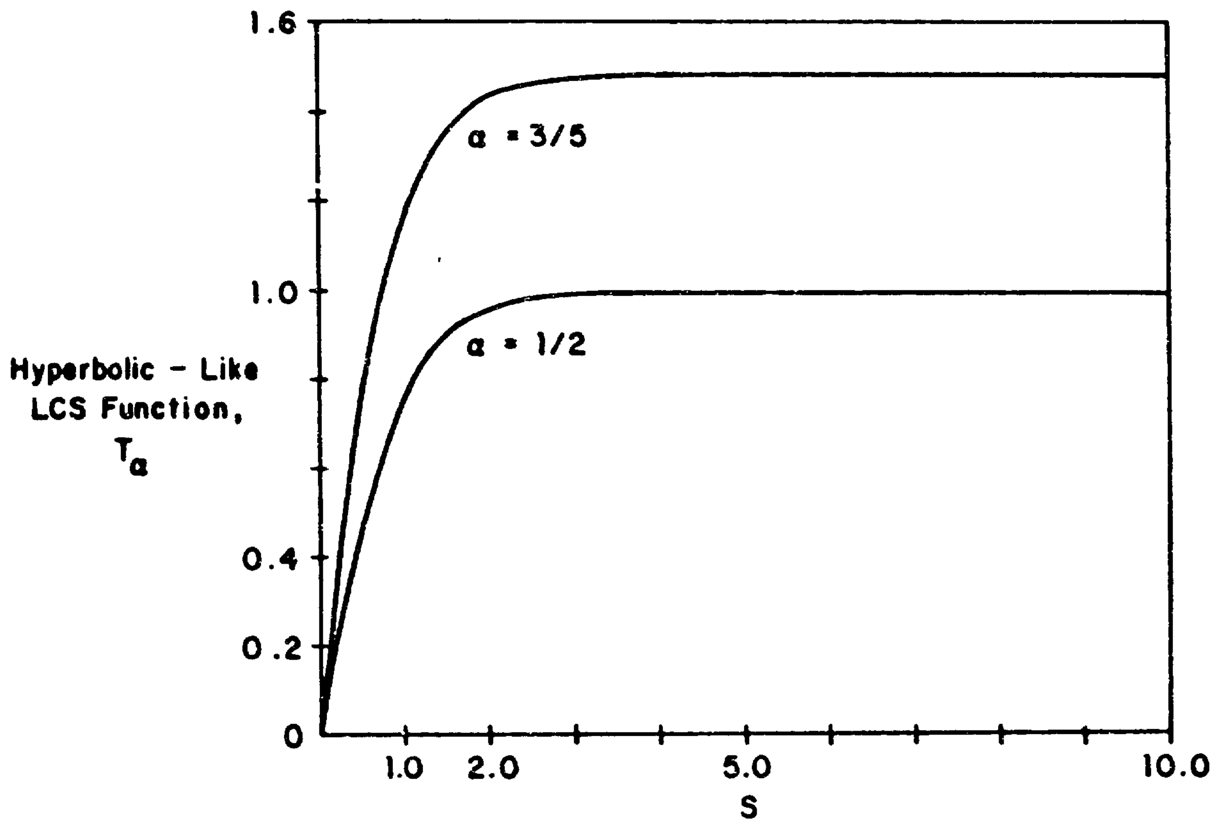


Figure 6.13. Rapidity with which limiting value of hyperbolic-tangent-like LCS function $T_\alpha(S)$ is reached as $S \to +\infty$. Note: $T_\alpha(S) = \tanh s$ for $\alpha = 1/2$, which corresponds to $\mu = \nu$ in (6.6.10).

is quite quickly reached, and consequently (recall Theorem 6.8.2) $\hat{Z}(s;\gamma,\mu,\nu)$ has essentially converged to $Z^*(\gamma,\mu,\nu)$ when $\hat{s} = 10.0$ (see TAYLOR and BROWN [51] for further details). Results generated by this numerical procedure for the power attrition-rate coefficients with "positive offset" (6.8.6) with $\mu = 1$ and $\nu = 2$ are shown in Figure 6.14.

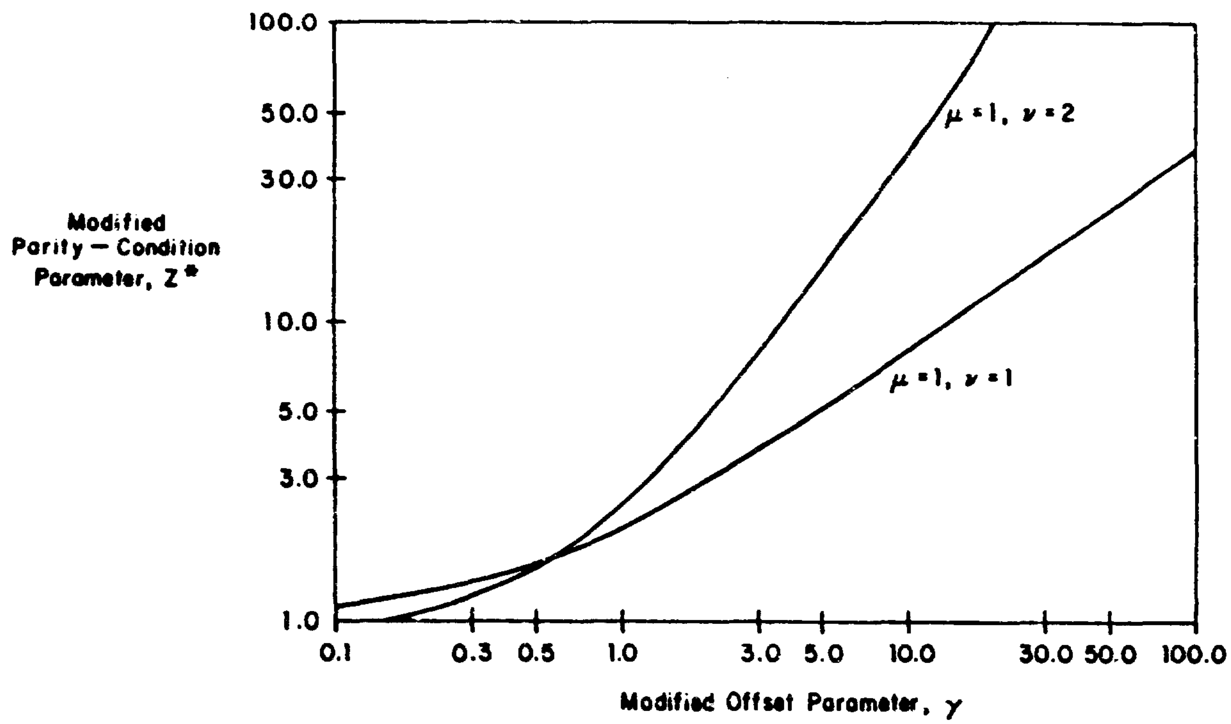


Figure 6.14. Dependence of the modified parity-condition parameter $Z^*$ on the modified offset parameter $\gamma$ for the offset power attrition-rate coefficients. The modified offset parameter is given by $\gamma = D[\lambda_I/(\mu+1)]^{2/(\mu+\nu+2)}$, where $D$ is the offset parameter in (6.8.6).

## 6.9. Application to General Power Attrition-Rate Coefficients

In this section we will give analytical results for combat modelled by variable-coefficient LANCHESTER-type equations for modern warfare (6.2.4) with the general power attrition-rate coefficients (6.2.9), which we rewrite here as

$$a(t) = k_a(t + C)^{\mu}, \quad \text{and} \quad b(t) = k_b(t + C + D)^{\nu}. \tag{6.9.1}$$

Physical motivation for the use of these coefficients as well as the relation between their parameters $k_a$, $k_b$, $C$, and $D$ and those of the range-dependent attrition-rate coefficients $\alpha(r)$ and $\beta(r)$ in BONDER's constant-speed-attack model) may be found in Section 6.2 above. Thus, the parameters $k_a$, $k_b$, $C$, and $D$ may ultimately be related to the performance and operational characteristics of the two opposing weapon-system types.

Within the context of BONDER's constant-speed attack considered in Section 6.2, both $C$ and $D \geq 0$ if and only if $r_\alpha \geq r_\beta \geq r_0$, the maximum effective range of X's weapon-system type is greater than that of Y which is in turn greater than the opening range of battle $r_0$. Also, on physical grounds we should have $\mu$ and $\nu \geq 0$, i.e. the weapon-system kill rates increase with decreasing force separation. The only restrictions (besides the general ones discussed in Section 6.5) that we place on these parameters, however, is that both $C$ and $D \geq 0$, since it makes more physical sense to consider a slightly different form for the coefficients in other cases. Formally all our mathematical results hold in these other cases, though.

Analytical results have been developed for the following two special cases of general power attrition-rate coefficients (6.9.1):

(C1) power attrition-rate coefficients with no offset, i.e.

$D = 0$ in (6.9.1),

and

(C2) power attrition-rate coefficients with positive offset and a nonnegative integral exponent for $X$'s kill rate, i.e.

$D > 0$ and $\nu = n$ (a nonnegative integer) in (6.9.1).

Although general analytical results have not been obtained for the attrition-rate coefficients (6.9.1), the above two special cases may be used for many such battles of tactical interest. Within the context of BONDER's constant-speed attack, power attrition-rate coefficients with "no offset" allow one to model combat between two weapon-system types with same minimum effective range but different range dependencies for each system's fire effectiveness, while power attrition-rate coefficients with "positive offset and integral $X$ exponent" allow one to model such combat between two weapon-system types with different maximum effective ranges for a mildly restrictive case of range dependencies for $X$'s weapon-system type.

Let us first consider the case (C1) of <u>power attrition-rate coefficients with no offset</u>, i.e.

$$a(t) = k_a(t + C)^{\mu}, \quad \text{and} \quad b(t) = k_b(t + C)^{\nu}, \tag{6.9.2}$$

with $C \geq 0$. In order that both $a(t)$ and $b(t) \in L(t_0,T)$ for any finite $T \geq t_0$, we must have both $\mu$ and $\nu > -1$, and then both $a(t)$ and $b(t) \notin L(0,+\infty)$. As we saw in Section 6.5, the $X$ and $Y$ force levels $x(t)$ and $y(t)$ may be expressed in terms of hyperbolic-like GLF, which for the above coefficients (6.9.2) are given by (TAYLOR and BROWN [53; 54])

$$C_X(t) = F_q(\tau) , \qquad S_X(t) = (\lambda_I/\sigma)^{1-2p} H_p(\tau) ,$$
$$C_Y(t) = F_p(\tau) , \qquad S_Y(t) = (\lambda_I/\sigma)^{2p-1} H_q(\tau) , \tag{6.9.3}$$

where $\sigma = \mu + \nu + 2$, $p = (\mu+1)/\sigma$, $q = 1-p$, and

$$\tau(t) = (2\lambda_I/\sigma)(t + C)^{\sigma/2} . \tag{6.9.4}$$

Here $F_\alpha(\xi)$ and $H_\alpha(\xi)$ denote LANCHESTER-CLIFFORD-SCHLÄFLI (LCS) functions[12] of order $\alpha$ and may be represented for $\alpha \neq 0, -1, -2,...$ as the infinite series

$$F_\alpha(\xi) = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k}}{\{k!\ \Gamma(k+\alpha)\}} ,$$

and (6.9.5)

$$H_\alpha(\xi) = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(\xi/2)^{2(k+\alpha)}}{\{k!\ \Gamma(k+\alpha+1)\}} .$$

In other words, the X force level $x(t)$ is given by[13]

$$x(t) = x_0\{F_p(\tau_0)\ F_q(\tau) - H_q(\tau_0)\ H_p(\tau)\}$$
$$- y_0\sqrt{\lambda_R} \left(\frac{\lambda_I}{\sigma}\right)^{q-p} \{F_q(\tau_0)\ H_p(\tau) - H_p(\tau_0)\ F_q(\tau)\}, \tag{6.9.6}$$

where $\tau_0$ denotes $\tau(0)$. We finally observe that for both $\mu$ and $\nu > -1$ it follows that both $p$ and $q \in (0,1)$.

The LCS functions $F_\alpha$ and $H_{1-\alpha}$ form a fundamental system of solutions to

$$\frac{d^2F}{d\xi^2} + \left(\frac{2\alpha - 1}{\xi}\right)\frac{dF}{d\xi} - F = 0 , \qquad (6.9.7)$$

with Wronskian $W(F_\alpha, H_{1-\alpha}) = (\xi/2)^{1-2\alpha}$. Further mathematical properties are given in Table 6.II, and the reader is directed to TAYLOR and BROWN [54] for further details. It is convenient to introduce an additional LCS function $T_\alpha$ analogous to the hyperbolic tangent and defined by

$$T_\alpha(\xi) = H_{1-\alpha}(\xi)/F_\alpha(\xi) . \qquad (6.9.8)$$

It follows that $T_\alpha(\xi)$ is a strictly increasing function of $\xi$ on $[0,+\infty)$ with $T_\alpha(0) = 0$ and

$$\lim_{\xi \to +\infty} T_\alpha(\xi) = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} . \qquad (6.9.9)$$

Tabulations of these LCS functions are given in Appendix D for cases corresponding to a wide variety of tactical situations[14] (<u>see</u> also TAYLOR and BROWN [55; 56]. A representative tabulation of the hyperbolic-like LCS functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ is shown in Tables 6.III and 6.IV. We observe from Table 6.IV and (6.9.9) that the limiting value of $T_{3/5}(x)$ as $x \to +\infty$ is quickly reached, with three-decimal-place agreement by $x = 4.5$.

The X force will be annihilated in finite time if and only if[15]

$$\frac{x_0}{y_0} < \frac{\Gamma(p)}{\Gamma(q)} \sqrt{\lambda_R}\left(\frac{\lambda_I}{\sigma}\right)^{q-p} \frac{\{F_q(\tau_0) - H_p(\tau_0)\,\Gamma(q)/\Gamma(p)\}}{\{F_p(\tau_0) - H_q(\tau_0)\,\Gamma(p)/\Gamma(q)\}} . \qquad (6.9.10)$$

It is readily shown that $F_\alpha(\xi) - H_{1-\alpha}(\xi)\,\Gamma(\alpha)/\Gamma(1-\alpha) > 0$ for all

TABLE 6.II. Properties of the LCS Functions $F_\alpha(\xi)$ and $H_\alpha(\xi)$.

1. $dF_\alpha/d\xi = (\xi/2)^{1-2} \; H_\alpha(\xi)$

2. $dH_\alpha/d\xi = (\xi/2)^{2\alpha-1} \; F_\alpha(\xi)$

3. $F_\alpha(\xi) \; F_{1-\alpha}(\xi) - H_\alpha(\xi) \; H_{1-\alpha}(\xi) = 1$ for all $\xi$,

   where $\alpha$ is not an integer (including zero)

4. $F_\alpha(0) = 1$

5. $H_\alpha(0) = 0$ for $\alpha > 0$

6. $dF_\alpha/dx(0) = 0$

7. $\{(\xi/2)^{1-2\alpha} \; dH_\alpha/d\xi\}_{\xi=0} = 1$

8. $F_{1/2}(\xi) = \cosh \xi$

9. $H_{1/2}(\xi) = \sinh \xi$

$\alpha = 3/5$

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 0.0 | [illegible] | 0.0 | 0.0 |
| 0.01 | [illegible] | [illegible] | [illegible] |
| 0.02 | [illegible] | [illegible] | [illegible] |
| 0.03 | [illegible] | [illegible] | [illegible] |
| 0.04 | [illegible] | [illegible] | [illegible] |
| 0.05 | [illegible] | [illegible] | [illegible] |
| 0.06 | [illegible] | [illegible] | [illegible] |
| 0.07 | [illegible] | [illegible] | [illegible] |
| 0.08 | [illegible] | [illegible] | [illegible] |
| 0.09 | [illegible] | [illegible] | [illegible] |
| 0.10 | [illegible] | [illegible] | [illegible] |
| 0.11 | [illegible] | [illegible] | [illegible] |
| 0.12 | [illegible] | [illegible] | [illegible] |
| 0.13 | [illegible] | [illegible] | [illegible] |
| 0.14 | [illegible] | [illegible] | [illegible] |
| 0.15 | [illegible] | [illegible] | [illegible] |
| 0.16 | [illegible] | [illegible] | [illegible] |
| 0.17 | [illegible] | [illegible] | [illegible] |
| 0.18 | [illegible] | [illegible] | [illegible] |
| 0.19 | [illegible] | 0.38275 | 0.37706 |
| 0.20 | 1.01672 | [illegible] | [illegible] |
| 0.21 | [illegible] | [illegible] | [illegible] |
| 0.22 | [illegible] | [illegible] | [illegible] |
| 0.23 | [illegible] | [illegible] | [illegible] |
| 0.24 | [illegible] | 0.46317 | 0.45227 |
| 0.25 | [illegible] | [illegible] | [illegible] |
| 0.26 | [illegible] | [illegible] | [illegible] |
| 0.27 | [illegible] | [illegible] | [illegible] |
| 0.28 | [illegible] | [illegible] | [illegible] |
| 0.29 | [illegible] | 0.54142 | [illegible] |
| 0.30 | 1.03776 | [illegible] | [illegible] |
| 0.31 | [illegible] | [illegible] | [illegible] |
| 0.32 | [illegible] | [illegible] | [illegible] |
| 0.33 | [illegible] | [illegible] | [illegible] |
| 0.34 | [illegible] | 0.61834 | [illegible] |
| 0.35 | [illegible] | [illegible] | [illegible] |
| 0.36 | [illegible] | [illegible] | [illegible] |
| 0.37 | [illegible] | [illegible] | [illegible] |
| 0.38 | [illegible] | [illegible] | [illegible] |
| 0.39 | [illegible] | [illegible] | [illegible] |
| 0.40 | [illegible] | [illegible] | [illegible] |
| 0.41 | [illegible] | [illegible] | [illegible] |
| 0.42 | [illegible] | [illegible] | [illegible] |
| 0.43 | [illegible] | [illegible] | [illegible] |
| 0.44 | [illegible] | [illegible] | 0.71220 |
| 0.45 | [illegible] | [illegible] | 0.72370 |
| 0.46 | [illegible] | [illegible] | [illegible] |
| 0.47 | [illegible] | [illegible] | [illegible] |
| 0.48 | [illegible] | [illegible] | [illegible] |
| 0.49 | [illegible] | 0.84670 | [illegible] |
| 0.5C | 1.10622 | 0.86199 | 0.77922 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 0.50 | 1.10622 | 0.86199 | 0.77922 |
| 0.51 | 1.11069 | 0.87731 | 0.78994 |
| 0.52 | 1.11507 | 0.89265 | 0.80054 |
| 0.53 | 1.11963 | 0.90803 | 0.81100 |
| C.54 | 1.12429 | 0.92343 | 0.82134 |
| 0.55 | 1.12905 | 0.93887 | 0.83156 |
| 0.56 | 1.13390 | 0.95435 | 0.84164 |
| 0.57 | 1.13885 | 0.96987 | 0.85163 |
| 0.58 | 1.14389 | 0.98544 | 0.86150 |
| 0.59 | 1.14903 | 1.00104 | 0.87121 |
| 0.60 | 1.15427 | 1.01670 | 0.88081 |
| 0.61 | 1.15960 | 1.03240 | 0.89030 |
| 0.62 | 1.16504 | 1.04815 | 0.89968 |
| 0.63 | 1.17057 | 1.06396 | 0.90893 |
| 0.64 | 1.17620 | 1.07983 | 0.91807 |
| 0.65 | 1.18193 | 1.09575 | 0.92709 |
| 0.66 | 1.18776 | 1.11174 | 0.93599 |
| 0.67 | 1.19370 | 1.12778 | 0.94478 |
| 0.68 | 1.19973 | 1.14389 | 0.95346 |
| 0.69 | 1.20587 | 1.16007 | 0.96202 |
| 0.70 | 1.21211 | 1.17632 | 0.97048 |
| 0.71 | 1.21845 | 1.19264 | 0.97882 |
| 0.72 | 1.22489 | 1.20903 | 0.98705 |
| 0.73 | 1.23145 | 1.22550 | 0.99517 |
| 0.74 | 1.23810 | 1.24204 | 1.00318 |
| C.75 | 1.24486 | 1.25864 | 1.01109 |
| 0.76 | 1.25173 | 1.27537 | 1.01889 |
| 0.77 | 1.25870 | 1.29216 | 1.02658 |
| 0.78 | 1.26579 | 1.30903 | 1.03416 |
| 0.79 | 1.27298 | 1.32599 | 1.04165 |
| 0.80 | 1.28028 | 1.34304 | 1.04903 |
| 0.81 | 1.28769 | 1.36019 | 1.05630 |
| 0.82 | 1.29521 | 1.37742 | 1.06348 |
| 0.83 | 1.30284 | 1.39475 | 1.07055 |
| 0.84 | 1.31058 | 1.41218 | 1.07753 |
| 0.85 | 1.31843 | 1.42971 | 1.08440 |
| 0.86 | 1.32640 | 1.44734 | 1.09118 |
| 0.87 | 1.33448 | 1.46508 | 1.09786 |
| 0.88 | 1.34268 | 1.48292 | 1.10445 |
| 0.89 | 1.35099 | 1.50086 | 1.11094 |
| 0.90 | 1.35942 | 1.51892 | 1.11733 |
| 0.91 | 1.36797 | 1.53709 | 1.12363 |
| 0.92 | 1.37663 | 1.55537 | 1.12984 |
| 0.93 | 1.38541 | 1.57377 | 1.13596 |
| 0.94 | 1.39431 | 1.59229 | 1.14199 |
| 0.95 | 1.40333 | 1.61092 | 1.14792 |
| 0.96 | 1.41248 | 1.62968 | 1.15377 |
| 0.97 | 1.42176 | 1.64856 | 1.15953 |
| C.98 | 1.43113 | 1.66757 | 1.16521 |
| 0.99 | 1.44064 | 1.68670 | 1.17080 |
| 1.00 | 1.45028 | 1.70597 | 1.17630 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 1.00 | 1.45028 | 1.70597 | 1.17630 |
| 1.01 | 1.46004 | 1.72536 | 1.18172 |
| 1.02 | 1.46993 | 1.74489 | 1.18706 |
| 1.03 | 1.47995 | 1.76456 | 1.19231 |
| 1.04 | 1.49009 | 1.78436 | 1.19748 |
| 1.05 | 1.50037 | 1.80431 | 1.20258 |
| 1.06 | 1.51077 | 1.82439 | 1.20759 |
| 1.07 | 1.52131 | 1.84463 | 1.21253 |
| 1.08 | 1.53198 | 1.86500 | 1.21738 |
| 1.09 | 1.54278 | 1.88553 | 1.22217 |
| 1.10 | 1.55371 | 1.90621 | 1.22687 |
| 1.11 | 1.56476 | 1.92704 | 1.23150 |
| 1.12 | 1.57599 | 1.94802 | 1.23606 |
| 1.13 | 1.58736 | 1.96917 | 1.24055 |
| 1.14 | 1.59882 | 1.99047 | 1.24496 |
| 1.15 | 1.61044 | 2.01193 | 1.24930 |
| 1.16 | 1.62221 | 2.03356 | 1.25357 |
| 1.17 | 1.63411 | 2.05535 | 1.25778 |
| 1.18 | 1.64616 | 2.07731 | 1.26191 |
| 1.19 | 1.65836 | 2.09944 | 1.26598 |
| 1.20 | 1.67069 | 2.12175 | 1.26998 |
| 1.21 | 1.68318 | 2.14423 | 1.27392 |
| 1.22 | 1.69581 | 2.16688 | 1.27779 |
| 1.23 | 1.70859 | 2.18972 | 1.28159 |
| 1.24 | 1.72152 | 2.21274 | 1.28534 |
| 1.25 | 1.73460 | 2.23594 | 1.28902 |
| 1.26 | 1.74784 | 2.25932 | 1.29264 |
| 1.27 | 1.76123 | 2.28290 | 1.29620 |
| 1.28 | 1.77477 | 2.30667 | 1.29970 |
| 1.29 | 1.78847 | 2.33063 | 1.30314 |
| 1.30 | 1.80233 | 2.35479 | 1.30653 |
| 1.31 | 1.81634 | 2.37914 | 1.30985 |
| 1.32 | 1.83052 | 2.40369 | 1.31312 |
| 1.33 | 1.84485 | 2.42845 | 1.31634 |
| 1.34 | 1.85935 | 2.45342 | 1.31950 |
| 1.35 | 1.87402 | 2.47859 | 1.32261 |
| 1.36 | 1.88885 | 2.50397 | 1.32566 |
| 1.37 | 1.90384 | 2.52956 | 1.32866 |
| 1.38 | 1.91901 | 2.55537 | 1.33161 |
| 1.39 | 1.93434 | 2.58140 | 1.33451 |
| 1.40 | 1.94985 | 2.60765 | 1.33736 |
| 1.41 | 1.96553 | 2.63412 | 1.34016 |
| 1.42 | 1.98138 | 2.66082 | 1.34291 |
| 1.43 | 1.99741 | 2.68775 | 1.34562 |
| 1.44 | 2.01361 | 2.71490 | 1.34828 |
| 1.45 | 2.03000 | 2.74229 | 1.35089 |
| 1.46 | 2.04656 | 2.76992 | 1.35345 |
| 1.47 | 2.06331 | 2.79779 | 1.35597 |
| 1.48 | 2.08024 | 2.82589 | 1.35845 |
| 1.49 | 2.09735 | 2.85425 | 1.36088 |
| 1.5C | 2.11465 | 2.88285 | 1.36327 |

TABLE 6.III. Lanchester-Clifford-Schläfli Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and $x$ from 0.00 to 1.50.

$\alpha = 3/5$

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 1.50 | 2.11465 | 2.88285 | 1.36327 |
| 1.51 | [illegible] | [illegible] | [illegible] |
| 1.52 | [illegible] | [illegible] | [illegible] |
| 1.53 | [illegible] | [illegible] | [illegible] |
| 1.54 | 2.18576 | [illegible] | 1.37241 |
| 1.55 | 2.20402 | [illegible] | [illegible] |
| 1.56 | [illegible] | [illegible] | [illegible] |
| 1.57 | [illegible] | [illegible] | [illegible] |
| 1.58 | [illegible] | [illegible] | [illegible] |
| 1.59 | [illegible] | [illegible] | [illegible] |
| 1.60 | [illegible] | [illegible] | [illegible] |
| 1.61 | [illegible] | [illegible] | [illegible] |
| 1.62 | [illegible] | [illegible] | [illegible] |
| 1.63 | [illegible] | [illegible] | [illegible] |
| 1.64 | [illegible] | 3.31076 | [illegible] |
| 1.65 | [illegible] | [illegible] | [illegible] |
| 1.66 | [illegible] | [illegible] | [illegible] |
| 1.67 | [illegible] | [illegible] | [illegible] |
| 1.68 | [illegible] | [illegible] | 1.39970 |
| 1.69 | 2.40113 | 3.47704 | 1.40139 |
| 1.70 | [illegible] | [illegible] | [illegible] |
| 1.71 | [illegible] | [illegible] | [illegible] |
| 1.72 | [illegible] | [illegible] | [illegible] |
| 1.73 | [illegible] | [illegible] | [illegible] |
| 1.74 | [illegible] | 3.65098 | 1.40942 |
| 1.75 | [illegible] | [illegible] | [illegible] |
| 1.76 | [illegible] | [illegible] | [illegible] |
| 1.77 | [illegible] | [illegible] | [illegible] |
| 1.78 | [illegible] | [illegible] | [illegible] |
| 1.79 | [illegible] | 3.83302 | 1.41674 |
| 1.80 | 2.72927 | [illegible] | 1.41812 |
| 1.81 | [illegible] | [illegible] | [illegible] |
| 1.82 | [illegible] | [illegible] | [illegible] |
| 1.83 | [illegible] | [illegible] | [illegible] |
| 1.84 | 2.82675 | 4.02361 | 1.42341 |
| 1.85 | [illegible] | [illegible] | [illegible] |
| 1.86 | [illegible] | [illegible] | [illegible] |
| 1.87 | [illegible] | [illegible] | [illegible] |
| 1.88 | [illegible] | [illegible] | [illegible] |
| 1.89 | 2.95436 | 4.22320 | [illegible] |
| 1.90 | [illegible] | [illegible] | [illegible] |
| 1.91 | [illegible] | [illegible] | [illegible] |
| 1.92 | [illegible] | [illegible] | [illegible] |
| 1.93 | [illegible] | [illegible] | [illegible] |
| 1.94 | [illegible] | 4.45226 | 1.43501 |
| 1.95 | [illegible] | [illegible] | [illegible] |
| 1.96 | [illegible] | [illegible] | [illegible] |
| 1.97 | [illegible] | [illegible] | [illegible] |
| 1.98 | [illegible] | [illegible] | [illegible] |
| 1.99 | [illegible] | 4.65131 | 1.44004 |
| 2.00 | 3.25912 | 4.69636 | 1.44099 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 2.0 | 3.25912 | 4.69636 | 1.44099 |
| 2.1 | 3.56725 | 5.17046 | 1.44957 |
| 2.2 | 3.90799 | 5.69251 | 1.45665 |
| 2.3 | 4.28459 | 6.26617 | 1.46269 |
| 2.4 | 4.70063 | 6.89726 | 1.46731 |
| 2.5 | 5.16004 | 7.59185 | 1.47127 |
| 2.6 | 5.66727 | 8.35659 | 1.47453 |
| 2.7 | 6.22710 | 9.19878 | 1.47722 |
| 2.8 | 6.84499 | 10.12649 | 1.47943 |
| 2.9 | 7.52655 | 11.14858 | 1.48123 |
| 3.0 | 8.27860 | 12.27485 | 1.48272 |
| 3.1 | 9.10823 | 13.51608 | 1.48394 |
| 3.2 | 10.01539 | 14.86417 | 1.48494 |
| 3.3 | 11.05285 | 16.39226 | 1.48577 |
| 3.4 | 12.14631 | 18.05476 | 1.48644 |
| 3.5 | 13.37464 | 19.88770 | 1.48699 |
| 3.6 | 14.72906 | 21.90848 | 1.48745 |
| 3.7 | 16.22317 | 24.13713 | 1.48782 |
| 3.8 | 17.87117 | 26.59449 | 1.48812 |
| 3.9 | 19.68890 | 29.30442 | 1.48837 |
| 4.0 | 21.69389 | 32.29303 | 1.48858 |
| 4.1 | 23.90546 | 35.58913 | 1.48874 |
| 4.2 | 26.34195 | 39.22452 | 1.48888 |
| 4.3 | 29.03589 | 43.23426 | 1.48899 |
| 4.4 | 32.00427 | 47.65715 | 1.48909 |
| 4.5 | 35.27878 | 52.53584 | 1.48916 |
| 4.6 | 38.89107 | 57.91754 | 1.48922 |
| 4.7 | 42.87609 | 63.85430 | 1.48928 |
| 4.8 | 47.27239 | 70.40357 | 1.48932 |
| 4.9 | 52.12255 | 77.62878 | 1.48935 |
| 5.0 | 57.47356 | 85.59991 | 1.48938 |
| 5.1 | 63.37728 | 94.39423 | 1.48940 |
| 5.2 | 69.89096 | 104.09703 | 1.48942 |
| 5.3 | 77.07779 | 114.80242 | 1.48944 |
| 5.4 | 85.00753 | 126.61432 | 1.48945 |
| 5.5 | 93.75716 | 139.64742 | 1.48946 |
| 5.6 | 103.41170 | 154.02832 | 1.48947 |
| 5.7 | 114.06496 | 169.89679 | 1.48947 |
| 5.8 | 125.82052 | 187.40709 | 1.48948 |
| 5.9 | 138.79270 | 206.72954 | 1.48948 |
| 6.0 | 153.10773 | 228.05212 | 1.48949 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 6.0 | 153.10773 | 228.05212 | 1.48949 |
| 6.1 | 168.90493 | 251.58239 | 1.48949 |
| 6.2 | 186.33813 | 277.54946 | 1.48949 |
| 6.3 | 205.57712 | 306.20625 | 1.48950 |
| 6.4 | 226.80937 | 337.83198 | 1.48950 |
| 6.5 | 250.24183 | 372.73492 | 1.48950 |
| 6.6 | 276.10298 | 411.25538 | 1.48950 |
| 6.7 | 304.64504 | 453.76907 | 1.48950 |
| 6.8 | 336.14649 | 500.69077 | 1.48950 |
| 6.9 | 370.91477 | 552.47842 | 1.48950 |
| 7.0 | 409.28931 | 609.63759 | 1.48950 |
| 7.1 | 451.64484 | 672.72646 | 1.48950 |
| 7.2 | 498.39511 | 742.36131 | 1.48950 |
| 7.3 | 549.99691 | 819.22253 | 1.48950 |
| 7.4 | 606.95457 | 904.06134 | 1.48950 |
| 7.5 | 669.82494 | 997.70714 | 1.48950 |
| 7.6 | 739.22282 | 1101.07568 | 1.48950 |
| 7.7 | 815.82701 | 1215.17807 | 1.48950 |
| 7.8 | 900.38700 | 1341.13066 | 1.48950 |
| 7.9 | 993.73032 | 1480.16606 | 1.48950 |
| 8.0 | 1096.77069 | 1633.64526 | 1.48950 |
| 8.1 | 1210.51696 | 1803.07095 | 1.48950 |
| 8.2 | 1336.08310 | 1990.10239 | 1.48950 |
| 8.3 | 1474.69908 | 2196.57164 | 1.48950 |
| 8.4 | 1627.72303 | 2424.50162 | 1.48951 |
| 8.5 | 1796.65455 | 2676.12602 | 1.48951 |
| 8.6 | 1983.14952 | 2953.91126 | 1.48951 |
| 8.7 | 2189.03632 | 3260.58073 | 1.48951 |
| 8.8 | 2416.33384 | 3599.14166 | 1.48951 |
| 8.9 | 2667.27158 | 3972.91466 | 1.48951 |
| 9.0 | 2944.31108 | 4385.56643 | 1.48951 |
| 9.1 | 3250.17066 | 4841.14586 | 1.48951 |
| 9.2 | 3587.85195 | 5344.12390 | 1.48951 |
| 9.3 | 3960.46950 | 5899.43758 | 1.48951 |
| 9.4 | 4372.28341 | 6512.53663 | 1.48951 |
| 9.5 | 4826.73538 | 7189.44720 | 1.48951 |
| 9.6 | 5328.48855 | 7936.81115 | 1.48951 |
| 9.7 | 5882.47166 | 8761.91155 | 1.48951 |
| 9.8 | 6494.12659 | 9673.03503 | 1.48951 |
| 9.9 | 7169.46404 | 10678.95366 | 1.48951 |
| 10.0 | 7915.12075 | 11789.61318 | 1.48951 |

TABLE 6.IV. Lanchester-Clifford-Schläfli Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and $x$ from 1.50 to 10.0.

$\alpha \in (0,1)$ when $\xi \geq 0$ is finite. Also, neither side will be annihilated in finite time if and only if the inequality sign in (6.9.10) is replaced by an equlaity sign. When $C = 0$, (6.9.10) reduces to

$$\frac{x_0}{y_0} < \frac{\Gamma(p)}{\Gamma(q)} \sqrt{\lambda_R} \left( \frac{\lambda_I}{\sigma} \right)^{q-p} . \tag{6.9.11}$$

The time to annihilate the $X$ force, denoted as $t_a^X$, is determined by $x(t_a^X) = 0$, and it follows that

$$T_q(\tau(t_a^X)) = \frac{x_0 F_p(\tau_0) + y_0\sqrt{\lambda_R}\,(\lambda_I/\sigma)^{q-p}\, H_p(\tau_0)}{x_0 H_q(\tau_0) + y_0\sqrt{\lambda_R}\,(\lambda_I/\sigma)^{q-p}\, F_a(\tau_0)} , \tag{6.9.12}$$

or

$$t_a^X = \left\{ \left( \frac{\sigma}{2\lambda_I} \right) T_q^{-1} \left[ \frac{x_0 F_p(\tau_0) + y_0\sqrt{\lambda_R}\,(\lambda_I/\sigma)^{q-p} H_p(\tau_0)}{x_0 H_q(\tau_0) + y_0\sqrt{\lambda_R}\,(\lambda_I/\sigma)^{q-p} F_q(\tau_0)} \right] \right\}^{2/\sigma} - C. \tag{6.9.13}$$

We will now examine a couple of numerical examples to show the use of the above analytical results for developing insights into the dynamics of combat. These examples illustrate the use of the LCS functions $F_\alpha$, $H_{1-\alpha}$, and $T_\alpha$ for analyzing "aimed-fire" combat modelled by the power attrition-rate coefficients with "no offset" (6.9.2). Consider BONDER's constant-speed-attack model, which we have examined in Section 6.2 above. All the force-level trajectories shown in Section 6.2 for battles in which the two opposing weapon-system types have the same maximum effective range (i.e. Figures 6.5 through 6.9) were developed by using (6.9.6) or the

analogous result for $y(t)$. Let us now focus on the prediction of battle outcome from initial conditions without explicitly computing the force-level trajectories (cf. questions (Q1), (Q4), and (Q5) of Table 6.I). We will consider combat situations modelled by the input data and computed parameter values shown in Table 6.V. The reader should observe from Tables 6.IV and 6.V the predicted agreement between $\Gamma(1-\alpha)/\Gamma(\alpha)$ and the limiting value of $T_\alpha(x)$ as $x \to +\infty$ [recall (6.9.9)] for $\alpha = q = 3/5$. We will now consider two cases: (I) $r_0 = 2000$ meters, and (II) $r_0 = 1250$ meters.

When $r_0 = 2000$ meters (see Figure 6.5 above), we have $C = 0$ and $\tau_0 = 0$. The maximum time that the battle can last is $t_{max} = 14.91$ minutes, since at this time the attacking $Y$ force reaches its final objective (i.e. the defensive position of the $X$ force). We will now consider the qualitative behavior of the $\mu = 1$, $\nu = 2$ X-force-level trajectory denoted as curve (C) linear-quadratic in Figure 6.5. The inequality (6.9.11) tells us that the $X$ force can be annihilated if and only if $x_0/y_0 < 0.420$. By (6.9.12) the annihilation time of the $X$ force is given by $T_q(\tau(t_a^X)) = 3.544\ x_0/y_0$. For $x_0 = 10$, $y_0 = 30$, we have $T_q(\tau_a^X) = 1.18122$ so that from Table 6.III (using linear interpolation) we obtain $\tau_a^X = 1.009$. Hence, (6.9.4) yields $t_a^X = 14.24$ minutes and $r_a^X = 89.8$ meters. Further results are given in Table 6.VI.

When $r_0 = 1250$ meters (see Figure 6.6 above), we have $C = 5.5923$ minutes, $\tau_0 = 0.0975$, and $t_{max} = 9.32$ minutes. In this case (again, for $\mu = 1$, $\nu = 2$), $X$ can be annihilated if and only if $x_0/y_0 < 0.382$. with from (6.9.12) the annihilation time of the $X$ force given by $T_q(\tau_a^X) = (3.5656\mu_0 + 0.223)/(0.156\mu_0 + 1.004)$, where $\mu_0 = x_0/y_0$. Some

TABLE 6.V. Particulars for the Numerical Examples for Combat Modelled by the Power Attrition-Rate Coefficients with No Offset (6.9.2).

1. <u>Input Data</u>

$\mu = 1$, $\nu = 2$

$\alpha_0 = 0.06$ X casualties/minute/(a single Y firer)

$\beta_0 = 0.6$ Y casualties/minute/(a single X firer)

$r_\alpha = r_\beta = 2000$ meters

$v = 5$ miles/hour

2. <u>Computed Parameter Values</u>

$k_a = 4.023 \times 10^{-3}$ X casualties/(minute)$^\mu$/(a single Y firer)

$k_b = 2.698 \times 10^{-3}$ Y casualties/(minute)$^\nu$/(a single X firer)

$p = 2/5$, $q = 3/5$

$\Gamma(p)/\Gamma(q) = 1.48951$

$D = 0$

further numerical results are given in Table 6.VIII. Again, these parametric results should be contrasted with the single $\mu = 1$, $\nu = 2$ X-force-level trajectory [denoted as curve (C) linear-quadratic] shown in Figure 6.6.

Let us next consider case (C2) of power attrition-rate coefficients with positive offset and integral X exponent, i.e.[16]

$$a(t) = k_a(t + C) , \quad \text{and} \quad b(t) = k_b(t + C + D)^n, \tag{6.9.14}$$

with $C \geq 0$, $D > 0$, and $n$ a nonnegative integer. We also assume that $\mu > -1$, and then both $a(t)$ and $b(t) \notin L(0, +\infty)$. As we developed in Section 6.5, the X and Y force levels $x(t)$ and $y(t)$ may be expressed in terms of hyperbolic-like GLF so that once we have determined the latter, we can compute the force-level trajectories. Using the method of successive approximations (see Section 6.5), one can compute that for the above coefficients (6.9.14) we have the following offset power LANCHESTER functions

$$C_X(t) = \Gamma(q) \sum_{k=0}^{\infty} \left\{ \frac{(\tau/2)^{2k}}{k!\ \Gamma(k + q)} \sum_{j=0}^{nk} A_k^j\, \delta^j \right\}, \tag{6.9.15}$$

$$S_X(t) = \left(\frac{\lambda_I}{\sigma}\right)^{1-2p} \Gamma(p) \sum_{k=0}^{\infty} \left\{ \frac{(\tau/2)^{2(k+p)}}{k!\ \Gamma(k + p + 1)} \sum_{j=0}^{nk} B_k^j \delta^j \right\}, \tag{6.9.16}$$

$$C_Y(t) = \Gamma(p) \sum_{k=0}^{\infty} \left\{ \frac{(\tau/2)^{2k}}{k!\ \Gamma(k + p)} \sum_{j=0}^{nk} C_k^j\, \delta^j \right\}, \tag{6.9.17}$$

and

$$S_Y(t) = \left(\frac{\lambda_I}{\sigma}\right)^{1-2q} \Gamma(q) \sum_{k=0}^{\infty} \left\{ \frac{(\tau/2)^{2(k+q)}}{k!\ \Gamma(k + q + 1)} \sum_{j=0}^{n(k+1)} D_k^j \delta^j \right\}, \tag{6.9.18}$$

TABLE 6.VI. Annihilation of the X Force as a Function of the Initial Force Ratio for the Coefficients with No Offset (6.9.2) with $r_0 = 2000$ Meters.

| $(x_0/y_0)$ | $t_a^X$(minutes) | $r_a^X$(meters) |
|---|---|---|
| 0.333 | 14.24 | 89.8 |
| 0.250 | 11.61 | 443.2 |
| 0.200 | 10.19 | 633.2 |

TABLE 6.VII. Annihilation of the X force as a Function of the Initial Force Ratio for the Coefficients with No Offset (6.9.2) with $r_0 = 1250$ Meters.

| $(x_0/y_0)$ | $t_a^X$(minutes) | $r_a^X$(meters) |
|---|---|---|
| 0.333 | 10.63 | ——† |
| 0.250 | 7.56 | 235.9 |
| 0.200 | 6.17 | 422.8 |

† $t_{max} = 9.32$ minutes and $x_f = x(r = 0) = 1.35$.

where $\sigma = \mu + n + 2$, $p = (\mu+1)/\sigma$, $q = 1-p$, $\delta(t) = D/(t + C)$, $\tau(t)$ is again given by (6.9.4), and the offset coefficients $A_k^j$, $B_k^j$, $C_k^j$, and $D_k^j$ are given in Table 6.VIII. In this table

$$\binom{n}{\ell} = \frac{n!}{\ell!\,(n-\ell)!}$$

denotes the usual binomial coefficient. We observe that $\mu > -1$ and $n \geq 0$ imply that both $p$ and $q \in (0,1)$.

We may use Theorem 6.7.1 (which is a special case of Theorem 6.6.1) to predict force annihilation. Unfortunately, we have not been able to analytically compute the parity-condition parameter $Q^* = Q^*(D,\mu,n)$ for the offset power attrition-rate coefficients (6.9.14), but it may be numerically determined by the method given in Section 6.8. For such determinations as well as for analyzing force annihilation, though, we have found it more convenient to use the normal-form GLF [e.g. see (6.7.9)] than to use $C_X(t)$, $S_X(t)$, $C_Y(t)$ and $S_Y(t)$. Thus, we introduce the modified time variable $s$ defined by (6.8.7) which we rewrite as

$$s(t) = [\lambda_I/(\mu+1)]^{2p}\,(t + C)^{\mu+1} \tag{6.9.19}$$

with $s_0 = s(0) = [\lambda_I/(\mu+1)]^{2p}\,C^{(\mu+1)}$, and obtain [cf. (6.7.9)] the normal-form hyperbolic-like GLF. Thus, we obtain the normal form offset power LANCHESTER functions, for example

$$c_X(s) = \Gamma(q) \sum_{k=0}^{\infty} \left\{ \frac{(S/2)^{2k}}{k!\,\Gamma(k+q)} \sum_{j=0}^{nk} A_k^j \Delta^j \right\}, \tag{6.9.20}$$

and

TABLE 6.VIII. The Offset Coefficients for the Offset Power LANCHESTER Functions (6.9.15) through (6.9.18).

$A_0^0 = 1$, and for $k \geq 1$

$$A_k^j = \frac{k(k-p)}{(k-j/\sigma)(k-p-j/\sigma)} \left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} A_{k-1}^{j-\ell} \right\} \qquad \text{for } 0 \leq j \leq nk$$

$B_0^0 = 1$, and for $k \geq 1$

$$B_k^j = \frac{k(k+p)}{(k-j/\sigma)(k+p-j/\sigma)} \left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} B_{k-1}^{j-\ell} \right\} \qquad \text{for } 0 \leq j \leq nk$$

$C_0^0 = 1$, and for $k \geq 1$

$$C_k^j = \frac{k(k+p-1)}{(k-j/\sigma)} \left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{C_{k-1}^{j-\ell}}{(k+p-1+(\ell-j)/\sigma)} \right\} \qquad \text{for } 0 \leq j \leq nk$$

$$D_0^j = \binom{n}{j} \left(\frac{+1}{n+1-j}\right)$$

$$D_k^j = \frac{k(k-p+1)}{(k-p+1-j/\sigma)} \left\{ \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{D_{k-1}^{j-\ell}}{(k+(\ell-j)/\sigma)} \right\} \qquad \text{for } 0 \leq j \leq n(k+1)$$

NOTES: We have adopted here the convention that $A_k^j$, $B_k^j$, and $C_k^j = 0$ for $j < 0$ or $j > nk$. Also, $D_k^j = 0$ for $j < 0$ or $j > n(k+1)$.

$$s_X(s) = p^{(1-2p)}\,\Gamma(p) \sum_{k=0}^{\infty} \left\{ \frac{(S/2)^{2(k+p)}}{k!\,\Gamma(k+p+1)} \sum_{j=0}^{nk} B_k^j \Delta^j \right\}, \qquad (6.9.21)$$

where $S(s) = 2ps^{1/(2p)}$, $\Delta(s) = \gamma/s^{\alpha}$, $\alpha = 1/(\mu+1)$, and the offset parameter $\gamma$ is given by $\gamma = [\lambda_I/(\mu+1)]^{2/\sigma}D$. We may use these normal-form power LANCHESTER functions to predict force annihilation by means of (6.7.11) after the modified parity-condition parameter $Z^* = Z^*(\gamma,\mu,n)$ has been determined. Numerical results (see TAYLOR and BROWN [57] for $Z^*$) are shown in Figure 6.14 for two sets of values for the exponents in the coefficients (6.9.14): (I) $\mu = 1$, $n = 1$, and (II) $\mu = 1$, $n = 2$. The time to annihilate the X force, denoted as $t_a^X$, is determined by $x(t_a^X) = 0$, and hence

$$\eta(s(t_a^X)) = \frac{\{x_0 c_Y(s_0) + y_0(\sqrt{\lambda_R}/K)s_X(s_0)\}}{\{x_0 s_Y(s_0) + y_0(\sqrt{\lambda_R}/K)c_X(s_0)\}}, \qquad (6.9.22)$$

where $\eta_X(s) = c_X(s)/s_X(s)$ and $K = [\lambda_I/(\mu+1)]^{2p-1}$.

We will now consider a couple of numerical examples for analyzing "aimed-fire" combat modelled by the power attrition-rate coefficients with "positive offset and integral X exponent" (6.9.14). As above, we will consider BONDER's constant-speed-attack model. All the force-level trajectories shown in Section 6.2 for battles in which the two opposing weapon-system types have different maximum effective ranges (i.e. Figure 6.10) were developed by using the above analytical results. Focusing now on the prediction of battle outcome, we will consider combat situations

modelled by the input data and computed parameter values shown in Table 6.IX. We will now consider two cases: (I) $r_0 = 1500$ meters, and (II) $r_0 = 1250$ meters.

When $r_0 = 1500$ meters, we have $C = 0$ and $s_0 = 0$. The maximum time that the battle can last is $t_{max} = 11.18$ minutes, since at this time the advancing attackers (i.e. the Y force) overrun the defensive position of the X force. In this case $Z^*(\gamma,\mu,n) = Z^*(0.32,1,1) = 1.381$, so that (6.7.10) tells us that the X force can be annihilated if and only if $x_0/y_0 < 0.264$. By (6.9.22) the X-force annihilation time is given by $\eta_X(s(t_a^X)) = 2.739\ x_0/y_0$. For $x_0 = 10$ and $y_0 = 50$, we have $\eta_X(s_a^X) = 0.54772$ so that by techniques similar to those used above for the previous examples, we find that $s_a^X = 0.771$. These computations for determining $s_a^X$ involve generation of tables of $s_X$, $c_X$, and $\eta_X$ for $\gamma = 0.32$ and $\mu = n = 1$. Hence, (6.9.19) yields that $t_a^X = 10.25$ minutes and $r_a^X = 125.7$ meters. Further results are given in Table 6.X.

When $r_0 = 1250$ meters (<u>see</u> Figure 6.10 above), we have $C = 1.864$ minutes, $s_0 = 0.0255$, and $t_{max} = 9.32$ minutes. In this case X can be annihilated if and only if $x_0/y_0 < 0.281$, with the X-force annihilation time given by $\eta_X(s_a^X) = (1.001\mu_0 + 0.009)/(0.127\mu_0 + 0.366)$, where $\mu_0 = x_0/y_0$. Numerical results are given in Table 6.XI. Finally, these parametric results should be contrasted with merely computing a force-level curve for a particular set of values for battle parameters (e.g. compare them with, for example, the single X-force-level trajectory for $r_\beta = 2000$ meters shown in Figure 6.10).

A few final remarks about the results of this section seem to be in order. We have given results that allow one in principle to study the variable-coefficient model (6.2.4) with the general power attrition-rate

TABLE 6.IX. Particulars for the Numerical Examples for Combat Modelled by the Power Attrition-Rate Coefficients with Positive Offset and Integral X Exponent (6.9.14).

1. Input Data

$\mu = \nu = 1$

$\alpha_0 = 0.006$ X casualties/minute/(a single Y firer)

$\beta_0 = 0.6$ Y casualties/minute/(a single X firer)

$r_\alpha = 1500$ meters, $\qquad r_\beta = 2000$ meters

$v = 5$ miles/hour

2. Parameter Values

$k_a = 5.364 \times 1)^{-3}$ X casualties/minute/(a single Y firer)

$k_b = 4.023 \times 10^{-3}$ Y casualties/minute/(a single X firer)

$p = q = 1/2$

$D = 3.728$ minutes, $\qquad \gamma = 0.320$ (casualties minutes)$^{1/2}$

TABLE 6.X. Annihilation of the X Force as a Function of the Initial Force Ratio for the Coefficients with Positive Offset (6.9.14) with $r_0 = 1500$ Meters.

| $(x_0/y_0)$ | $t_a^X$(minutes) | $r_a^X$(meters) |
|---|---|---|
| 0.250 | 14.09 | ——† |
| 0.200 | 10.25 | 125.7 |
| 0.167 | 8.80 | 319.4 |

† $t_{max} = 11.18$ minutes and $x_f = x(r = 0) = 2.48$.

TABLE 6.XI. Annihilation of the X Force as a Function of the Initial Force Ratio for the Coefficients with Positive Offset (6.9.14) with $r_0 = 1250$ Meters.

| $(x_0/y_0)$ | $t_a^X$(minutes) | $r_a^X$(meters) |
|---|---|---|
| 0.250 | 10.87 | ——† |
| 0.200 | 8.17 | 154.4 |
| 0.167 | 6.93 | 320.4 |

† $t_{max} = 9.32$ minutes and $x_f = x(r = 0) = 1.74$.

coefficients (6.9.1) almost as easily and thoroughly as one can study LANCHESTER's classic constant-coefficient model (2.2.1). In practice, though, the details for such variable-coefficient combat models are generally rather complicated as we have seen above. Furthermore, except in special cases (e.g. a constant ratio of attrition-rate coefficients) the solution to such variable-coefficient LANCHESTER-type equations for modern warfare, unfortunately, apparently cannot be represented in terms of any of the "elementary" functions of analysis but requires the introduction of new transcendents defined by infinite series. Moreover, such infinite-series solutions by themselves provide little insight into the dynamics of combat and, in fact, as we have seen above require a fairly high degree of mathematical proficiency just to understand, let alone to use. In the next section we will therefore give a simple approximation to such solutions.

Finally, we note that the above results for power attrition-rate coefficients with no offset (6.9.2) may be used to analyze "aimed-fire" combat modelled by (6.2.4) with exponential attrition-rate coefficients (6.2.12). This may be seen by observing that the substitution $s = \int_{-\infty}^{t} a(\sigma)d\sigma = (k_a/\lambda_a)e^{\lambda_a t}$ transforms the X force-level equation (6.5.7) into the normal form (6.7.6) with invariante $J(s) = Ks^{\nu}$, where $K = (k_b/k_a)(\lambda_a/k_a)^{\nu}$ and $\nu = (\lambda_b/\lambda_a) - 1$.

## 6.10. The LIOUVILLE-GREEN-LANCHESTER Approximation.

As we have seen above, the analytical solution to variable-coefficient LANCHESTER-type equations of modern warfare generally involves so-called higher transcendental functions with which most OR workers are quite unfamiliar. In this section we will give a simple approximation that involves only "elementary" functions and requires no advanced mathematical theory to apply. We call our approximation (6.10.1) to the solution of LANCHESTER-type equations for modern warfare (6.5.1) the LIOUVILLE-GREEN-LANCHESTER (LGL) approximation.[17] Error bounds, i.e. bounds for the errors in the approximate solutions, are given in terms of simple a priori estimates that are both realistic and also easy to evaluate. These error bounds are based on new theoretical results by the author (see TAYLOR [47]) for the theory of the LIOUVILLE-GREEN (LG) approximation[18] and do not require knowledge of the exact solution.

Let us make the additional assumption that the attrition-rate coefficients $a(t)$ and $b(t)$ are twice differentiable for $t_0 < t < +\infty$. Then our approximation to the solution of the $X$ force-level equation (6.5.7) is given by

$$\hat{x}(t) = \left[\frac{R(t)}{R_0}\right]^{1/4} \{x_0 \cosh(\tau-\tau_0) - (y_0\sqrt{R_0} + x_0\varepsilon_0) \sinh(\tau-\tau_0)\}, \tag{6.10.1}$$

where $\hat{x}(t)$ denotes the LGL approximation, $R_0$ denotes $R(0)$, $\varepsilon_0$ denotes $\varepsilon(0)$, $\varepsilon(t) = \{1/[4I(t)]\} d\,\ln R/dt$, $\tau_0$ denotes $\tau(0)$, and $\tau(t) = \int_{t_0}^{t} \sqrt{a(s)b(s)}\, ds$. This approximation was developed by the author (see TAYLOR [47]) by transforming the $X$ force-level equation (6.5.7) into LIOUVILLE's normal form (see INCE [23, p. 271]) with the first derivative of the dependent variable removed

$$\frac{d^2X}{d\tau^2} - \{1 + F(\tau)\}X = 0 \tag{6.10.2}$$

by means of the substitution $\tau = \int_{t_0}^{t} \sqrt{a(s)\, b(s)}\, ds$ and $x(\tau) = X(\tau)[R(t)/R_0]^{1/4}$. In (6.10.2) we have that

$$F(\tau) = P''(\tau)/P(\tau) \ , \tag{6.10.3}$$

where $P(\tau) = [R(t)]^{-1/4}$ and $P'(\tau)$ denotes $dP/d\tau$. Heuristically, if the appropriate fractional power of the relative fire effectiveness $R(t) = a(t)/b(t)$ is "slowly varying," then from (6.10.3) we would expect that $|F(\tau)| << 1$ so that the term $F(\tau)$ is "negligible" in (6.10.2). The LGL approximation (6.10.1) comes dropping this term, and Theorem 6.10.1 gives us bounds on how "negligible" it is.

What is the error made in using (6.10.1)? This is an important question for any OR analyst who wishes to use such an approximation. It is important for him to know the accuracy of the approximation (6.10.1) and especially to know when it is particularly accurate or inaccurate. The following theorem gives a priori error bounds for the LGL approximation.

THEOREM 6.10.1 (TAYLOR [47]): Error bounds for the LIOUVILLE-GREEN-LANCHESTER (LGL) approximation (6.10.1) to the solution of LANCHESTER-type equations of modern warfare (6.5.1) are given by

$$|x(t) - \hat{x}(t)| \leq x_0 K_J e(t) < x_0 K_U e(t) \ , \tag{6.10.4}$$

where

$$K_U = 2\{(1 + |\varepsilon_0|) + (y_0/x_0)\sqrt{R_0}\ , \qquad (6.10.5)$$

$J = I$ for $1 - (y_0/x_0)\sqrt{R_0} \le \varepsilon_0$

$$\text{and then}\quad K_I = 1 + \varepsilon_0 + (y_0/x_0)\sqrt{R_0}\ , \qquad (6.10.6)$$

$J = II$ for $-1 - (y_0/x_0)\sqrt{R_0} < \varepsilon_0 < 1 - (y_0/x_0)\sqrt{R_0}$

$$\text{and then}\quad K_{II} = 2\ , \qquad (6.10.7)$$

$J = III$ for $\varepsilon_0 \le -1 - (y_0/x_0)\sqrt{R_0}$

$$\text{and then}\quad K_{III} = 1 - \varepsilon_0 - (y_0/x_0)\sqrt{R_0} > 0\ , \qquad (6.10.8)$$

and

$$e(t) = \left[\frac{R(t)}{R_0}\right]^{1/4} \{\exp(\tfrac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)|d\sigma) - 1\} \sinh(\tau-\tau_0)\ . \qquad (6.10.9)$$

The sign of the error is determined by the sign of $F(\tau)$. As long as $x(t) \ge 0$, it follows that

$F(\tau) > 0$ fpr all $\tau \ge \tau_0$ implies that $x(t) \ge \hat{x}(t)$ ,

with the last inequality being reversed when $F(\tau) \le 0$ always.

Example 6.10.1. For combat modelled by (6.5.1) with the power attrition-rate coefficients with no offset (6.9.2), the LGL approximation to the X force level is given by

$$\hat{x}(t) = (1 + t/C)^{(\mu-\nu)/4} \Big\{ x_0 \cosh(\tau-\tau_0)$$

$$- [y_0\sqrt{\lambda_R}\, C^{(\mu-\nu)/2} + \{x_0(\mu-\nu)/(4\lambda_I)\}\, C^{-\delta}] \sinh(\tau-\tau_0) \Big\}, \qquad (6.10.10)$$

where

$$\tau(t) = (1/\delta)\, \lambda_I(t + C)^{\delta}, \qquad (6.10.11)$$

and $\delta = (\mu + \nu + 2)/2$. For the error estimate (6.10.4) of Theorem 6.10.1, we have

$$\frac{1}{2}\int_{\tau_0}^{\tau} |F(\sigma)|\,d\sigma = \frac{|\mu-\nu|\,(3\mu + \nu + 4)}{32\lambda_I\delta}\,\{C^{-\delta} - (t + C)^{-\delta}\}.$$

Also, it may be shown that $F(\tau) \geq 0$ for all $\tau \geq \tau_0 > 0$ if and only if $\mu \geq \nu$.

## 6.11. HELMBOLD's Modification of LANCHESTER's Equations

Based on consideration of historical combat data, HELMBOLD [18] has proposed a modification of LANCHESTER's equation for "modern warfare" to account for inefficiencies of scale for the larger force when force sizes are grossly unequal (see Section 2.12 for further details). His basic idea is to modify relative force-attrition (or fire-effectiveness) capability by a multiplicative factor depending on only the force ratio, and for temporal variations in fire effectiveness, his proposed modification would read

$$\begin{cases} \dfrac{dx}{dt} = -a(t) \cdot E_Y(\frac{x}{y}) \cdot y & \text{with} \quad x(0) = x_0, \\ \\ \dfrac{dy}{dt} = -b(t) \cdot E_X(\frac{y}{x}) \cdot x & \text{with} \quad y(0) = y_0 , \end{cases} \tag{6.11.1}$$

where $E_X$ and $E_Y$ denote the fire-effectiveness-modification factors that model the inefficiencies of scale. HELMBOLD argued that these fire-effectivenss-modification factors should satisfy the following three requirements:

(R1) $E_X(u) = E_Y(u) = E(u)$ (i.e. the same inefficiencies of scale for each side),

(R2) $E(u)$ is an increasing function of its argument,

(R3) $E(1) = 1$.

HELMBOLD then considered the special case in which $E(u)$ is a power function, i.e. $E(u) = u^c$ with $c \geq 0$. In this case, (6.11.1) becomes

$$\begin{cases} \frac{dx}{dt} = -a(t) \cdot \left(\frac{x}{y}\right)^{1-W} \cdot y & \text{with } x(0) = x_0, \\ \frac{dy}{dt} = -b(t) \cdot \left(\frac{y}{x}\right)^{1-W} \cdot x & \text{with } y(0) = y_0, \end{cases} \tag{6.11.2}$$

where we will call $W$ the "WEISS parameter" (<u>see</u> Section 2.12). It follows that $W = 1 - c$. We will refer to (6.11.2) as the <u>equations for HELMBOLD-type combat</u>. These equations are particularly significant because a simple generalization of them gives a much better fit to casualty-rate curves used in several important contemporary large-scale combat models than does LANCHESTER's classic model of modern warfare (2.2.1) (<u>see</u> Section 7.11 below). As for the case of constant attrition-rate coefficients (<u>see</u> Section 2.12 above), the substitution $p = x^W$ and $q = y^W$ transforms the nonlinear combat model (6.11.2) into a linear one, namely

$$\begin{cases} \frac{dp}{dt} = -W\, a(t)\, q & \text{with } p(0) = x_0^W, \\ \frac{dq}{dt} = -W\, b(t)\, p & \text{with } q(0) = y_0^W. \end{cases} \tag{6.11.3}$$

Hence, all the results for variable-coefficient LANCHESTER-type equations of modern warfare (<u>see</u> Sections 6.5 through 6.10 above) also apply to the

equations for HELMBOLD-type combat (6.11.2). Moreover, it may be shown that for $E_X(u) = E_Y(u) = E(u)$ if $x$ and $y$ are "separated" in $E(x/y)$, i.e. if $E(x/y) = F(x)/G(y)$, then the only form for $E(u)$ satisfying (R2) and (R3) above such that we can obtain a linear model, i.e. the attrition rates proportional to only the "numbers" of firers, by a transformation of only the dependent variables is given by $E(u) = u^c$ with $c \geq 0$. Thus, the only combat model of the form (6.11.1) [with $E_X$ and $E_Y$ satisfying (R1) through (R3)] transformable into a linear model like (6.11.3) is given by (6.11.2) when $E(x/y) = F(x)/G(y)$.

In the case of constant coefficients, (6.11.2) becomes

$$\begin{cases} \dfrac{dx}{dt} = -a \cdot \left(\dfrac{x}{y}\right)^{1-W} \cdot y & \text{with } x(0) = y_0 , \\ \\ \dfrac{dy}{dt} = -b \cdot \left(\dfrac{y}{x}\right)^{1-W} \cdot x & \text{with } y(0) = y_0, \end{cases} \tag{6.11.4}$$

where $a$ and $b$ denote constant attrition-rate coefficients. The state equation for (6.11.4) is given by (see Section 2.12 for details)

$$b(x_0^{2W} - x^{2W}) = a(y_0^{2W} - y^{W}) \qquad \text{for } W \neq 0$$

and (6.11.5)

$$b \ln(x_0/x) = a \ln(y_0/y) \qquad \text{for } W = 0 .$$

Thus, for the case of constant attrition-rate coefficients, the equations for HELMBOLD-type combat yield the square law when $W = 1$, the linear law when $W = 1/2$, and the logarithmic law when $W = 0$. Hence, we should think of

(6.11.4) as a general combat model which contains many of the classic homogeneous-force combat models as special cases (*see* Section 2.12 for further details).

We will find it very instructive for future developments (*see* Section 7.11 below) to examine casualty rates (expressed as a fraction of each side's current strength) for the above model of HELMBOLD-type combat (6.11.4). Considering X's fractional casualties per unit time, we obtain from the first of equations (6.11.4)

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \begin{pmatrix} \text{X's fractional casualties} \\ \text{per unit time} \end{pmatrix} = \frac{a}{u^W} = av^W , \qquad (6.11.6)$$

where $u$ denotes the X-to-Y force ratio, i.e. $u = x/y$, and $v$ denotes its reciprocal (*cf*. Section 5.2).

In Figure 6.15 (*cf*. Figure 5.3) we have plotted X's fractional casualties per unit time versus the force ratio $v = y/x$ (denoted in the figure as A/D) for the case in which Y attacks and X defends. As in Section 5.2 above, for the force ratio we have used the quotient of the attacker's strength (here, force level) divided by that of the defender (denoted as A/D), since most combat analyses use this ratio A/D and consequently we will be able to more easily relate such LANCHESTER-type models to them.

In Figure 6.15, $W = 1$ corresponds to the case in which X's casualty rate is proportional to only the number of enemy firers, and (in the symmetric case in which Y's casualty rate has the same functional form) consequently the corresponding attrition model is given by LANCHESTER's equations for modern warfare (2.2.1), which yield the square law. We observe (*see* also

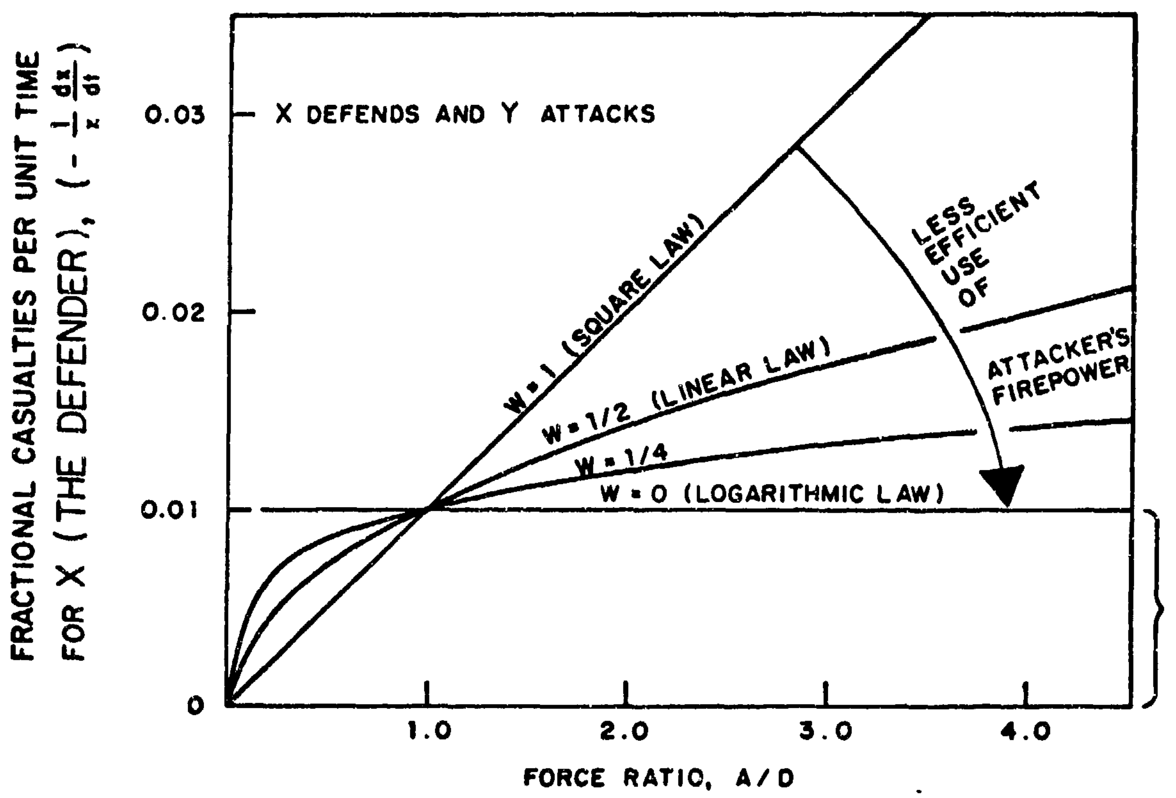


Figure 6.15. Relation between the defender's casualty rate [expressed as a fraction of his current force level $x(t)$] and the attacker/defender force ratio for the model $\frac{dx}{dt} = -a \cdot \left(\frac{x}{y}\right)^{1-W} \cdot y$ with X defending. [NOTE: In the legend of the above figure, A denotes the attacker's force level, and D denotes that of the defender.]

Section 5.2) that in this case (i.e. $W = 1$) X's fractional casualties per unit time are directly proportional to the force ratio A/D when Y attacks and X defends. Referring back to the first of equations (6.11.4), we see that $W = W_1$ corresponds to a more efficient use of the attacker's firepower for force ratios $v = A/D = y/x > 1$ than does $W = W_2$ when $1 \geq W_1 > W_2$, since the attacker's fire-effectiveness-modification factor for $W = W_1$ [i.e. $E_Y(x/y) = (x/y)^{1-W_1}$] is greater than that for $W = W_2$ when $y/x > 1$. Figure 6.16 shows the same type of plot when X is the attacker and Y the defender. In this case, the casualty-rate curve corresponding to the square law is a hyperbola (<u>see</u> also Section 5.2).

Similar curves for daily casualty rates (but not expressed in terms of differential equations) are commonly used to assess casualties in currently operational large-scale ground-combat models (<u>see</u> Section 7.11). Consequently, by studying analytical representations of these curves, we can obtain some valuable insights into the dynamics of combat as portrayed by such models (e.g. <u>see</u> Section 7.14) below.

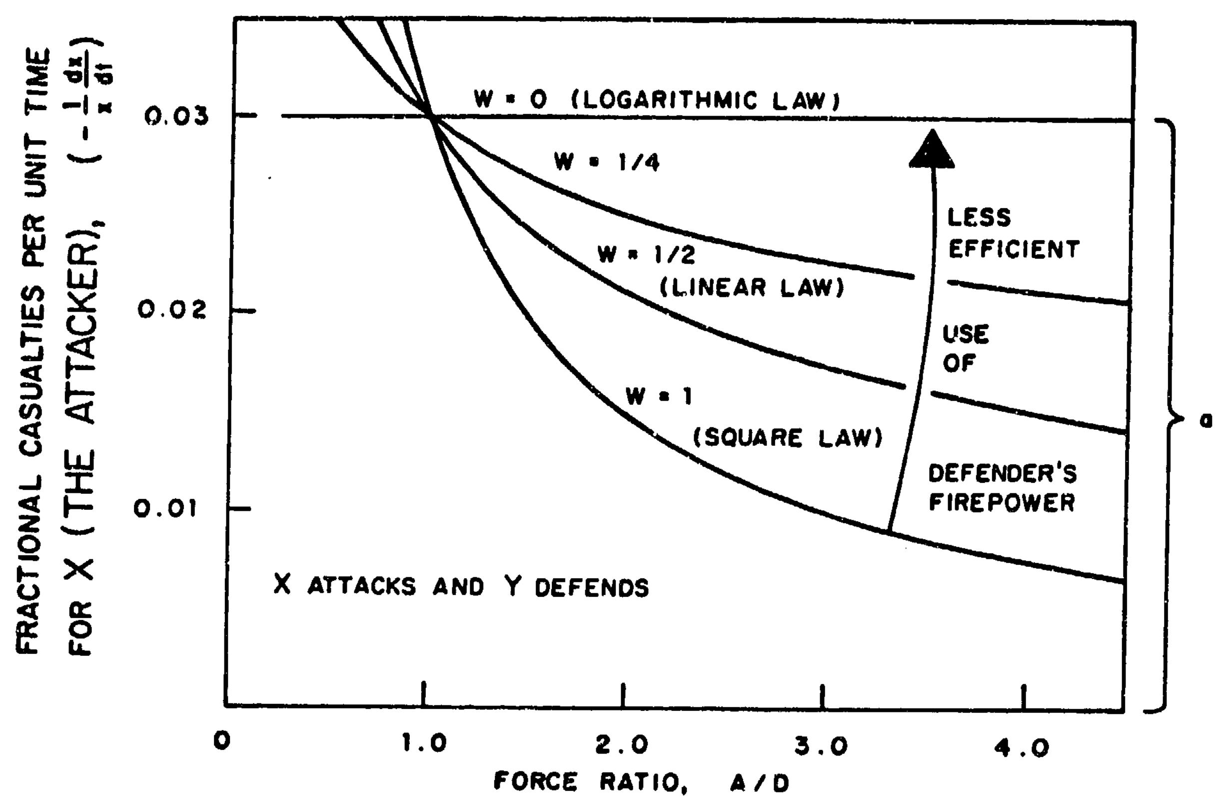


Figure 6.16. Relation between the attacker's casualty rate [expressed as a fraction of his current force level $x(t)$] and the attacker/defender force ratio for the model $\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-W}\cdot y$ with X attacking. [NOTE: In the legend of the above figure, A denotes the attacker's force level, and D denotes that of the defender.]

## 6.12. The General Linear Model for Combat Between Two Homogeneous Forces

In this section we will briefly examine the general linear-differential-equation model for combat between two homogeneous forces. Special cases of this general model will be examined in more detail in subsequent sections of this chapter.

Thus, we consider the following LANCHESTER-type equations for $x$ and $y > 0$

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y - \beta(t)x + r(t) & \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t)x - \alpha(t)y + s(t) & \text{with} \quad y(0) = y_0 , \end{cases} \tag{6.12.1}$$

where $x(t)$ and $y(t)$ denote the $X$ and $Y$ force levels at time $t$, and $a(t)$ and $b(t)$ denote LANCHESTER attrition-rate coefficients, which represent the fire effectiveness of a single firer on each side. The coefficients $\alpha(t)$, $\beta(t)$, $r(t)$, and $s(t)$ have different physical interpretations, depending upon the context in which the model (6.12.1) is viewed. Thus, there are several different sets of physical circumstances to which the model (6.12.1) may be hypothesized to apply, and we will now discuss several possibilities.

The term $r(t)$ in the first of equations (6.12.1) can model either (A) the replacement rate of the $X$ force (with a negative value representing a net continuous withdrawal of the $X$ force), or (B) the attrition [with $r(t) < 0$] of the $X$ force from exogenous fires (not subject to attrition) at a rate not dependent on $X$'s force level. Similar remarks apply to $s(t)$. For simplicity, however, we will consider

only the first possibility here, and we will consequently refer to $r(t)$ and $s(t)$ as replacement rates. Within this context, two different tactical situations may again be hypothesized to yield the above equations (6.12.1) (_cf_. Figure 2.15 of Chapter 2):

either (S1) "aimed-fire" combat between two homogeneous forces with "operational" losses and with continuous replacements,

or (S2) "aimed-fire" combat between two homogeneous primary forces (or infantries) with superimposed effects of supporting fires not subject to attrition and with continuous replacements for the primary forces (_see_ Figure 6.17).

In the second case (S2), it is assumed that each side uses "aimed" fire and that target-acquisition times do not depend on the number of enemy targets (_see_ Section 6.5 for a further discussion). The supporting weapons are assumed to employ "area" fire against enemy infantry (_see_ WEISS [61] for a more thorough discussion of assumptions). In this case, determination of numerical values for the attrition-rate coefficients $\alpha(t)$ and $\beta(t)$, modelling the supporting fires, follows along the lines discussed in Section 5.7. In the simplest instance we then have that, for example, $\alpha(t) = a_{L_U} \cdot v_U u_0/A_Y$, where the $X$ force's artillery is denoted as the $U$ force with force level $u(t)$, $a_{L_U}$ denotes the lethal area of a single $U$ artillery round, $v_U$ denotes the $U$ firing rate per tube, $u_0$ denotes the $U$ force level (which is constant because

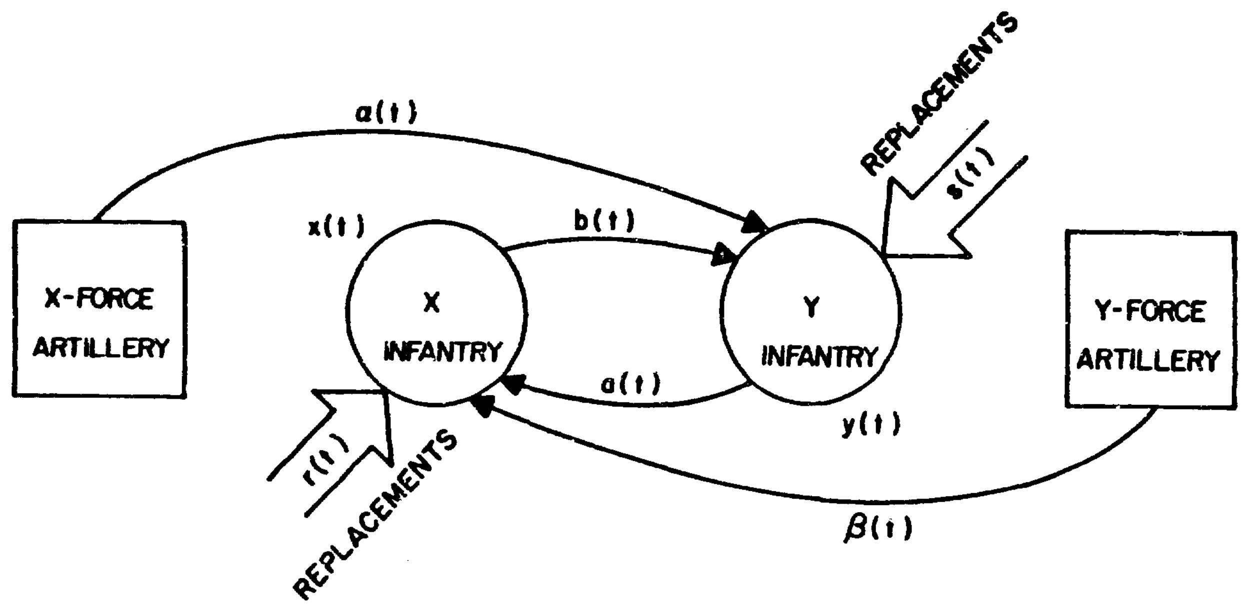


Figure 6.17. "Aimed-fire" combat between two homogeneous primary forces (infantries) with superimposed effects of supporting fires (here, from artillery) not subject to attrition and with continuous replacements for the primary forces.

the $U$ force suffers no losses), and $A_Y$ denotes the area of the region occupied by the $Y$ force.

Mathematically, we make the following assumptions about the attrition-rate coefficients and replacement rates in the model (6.12.1):

(A1) $a(t)$ and $b(t)$ are defined, positive, and continuous for $t_0 < t < +\infty$ with $t_0 \leq 0$ ,

(A2) $\alpha(t)$ and $\beta(t) \geq 0$ for $t_0 \leq t < +\infty$,

(A3) $a(t)$, $b(t)$, $\alpha(t)$, $\beta(t)$, $r(t)$, and $s(t) < L(t_0,T)$ for any finite $T$.

We place no further restrictions on the replacement rates $r(t)$ and $s(t)$, and thus negative values are possible for them. We further assume that $a(t)$ and $b(t)$ are given in the form (6.5.2), and we then introduce for the primary weapon systems the combat-intensity parameter $\lambda_I$ and the relative-fire-effectiveness parameter $\lambda_R$ defined by (6.5.4).

No results have previously appeared in the literature for the general model (6.12.1) with variable attrition-rate coefficients. We will now show that (6.12.1) may be transformed into a simpler canonical form to which results for variable-coefficient LANCHESTER-type equations of modern warfare (6.5.1) may be applied. Thus, the model (6.5.1) is basic for studying a wide variety of combat situations (cf. also Section 6.11 above). The substitution

$$p(t) = x(t)\exp\{\int_0^t \beta(s)ds\}\ , \qquad q(t) = y(t)\exp\{\int_0^t \alpha(s)ds\} \tag{6.12.2}$$

transforms (6.12.1) into

$$\begin{cases} \dfrac{dp}{dt} = -A(t)q + R(t) & \text{with } p(0) = x_0\ , \\ \\ \dfrac{dq}{dt} = -B(t)p + S(t) & \text{with } q(0) = y_0\ , \end{cases} \tag{6.12.3}$$

where

$$A(t) = a(t)\exp\{\int_0^t [\beta(s) - \alpha(s)]ds\},$$

and

$$B(t) = b(t)\exp\{-\int_0^t [\beta(s) - \alpha(s)]ds\}, \tag{6.12.4}$$

$$R(t) = r(t)\exp\{\int_0^t \beta(s)ds\}$$

and

$$S(t) = s(t)\exp\{\int_0^t \alpha(s)ds\}\ . \tag{6.12.5}$$

The transformation (6.12.2) is motivated by looking for an "integrating factor" for, for example, the first equation of (6.12.1), as writing $dx/dt + \beta(t)x = -a(t)y + r(t)$ suggests to us.

As we have seen above, we may consider equations (6.12.3) to model "aimed-fire" combat between two homogeneous forces with continuous replacements. However, there is another very important set of circumstances that leads to similar equations of this form. Consider aimed-fire

combat between two homogeneous forces modelled by LANCHESTER's equations of modern warfare (6.5.1). In this model the state variables $x(t)$ and $y(t)$ are the numbers of combatants that are effective on each side. Furthermore, consider now a fixed-force-level-breakpoint battle. If we introduce new state variables $X(t)$ and $Y(t)$ defined by

$$X(t) = x(t) - x_{BP} \quad \text{and} \quad Y(t) = y(t) - y_{BP}, \tag{6.12.6}$$

where $x_{BP}$ and $y_{BP}$ denote the $X$ and $Y$ force-level breakpoints, then (6.5.1) is transformed into (for $X(t)$ and $Y(t) > 0$)

$$\begin{cases} \dfrac{dX}{dt} = -a(t)Y - w(t) & \text{with } X(0) = x_0 - x_{BP}, \\ \\ \dfrac{dY}{dt} = -b(t)X - v(t) & \text{with } Y(0) = y_0 - y_{BP}, \end{cases} \tag{6.12.7}$$

where $w(t) = a(t)y_{BP}$ and $v(t) = b(t)x_{BP}$. These equations (6.12.7) are of the same form as (6.12.3), and thus we see that the equations (6.12.3) may also be taken to model force attrition "above a unit's breakpoint." We observe that for the transformed force-level variable $X$, $X = 0$ corresponds to the $X$ force reaching its breakpoint.

The force-level trajectories $x(t)$ and $y(t)$ for the model (6.12.1) [equivalently, (6.12.3) or (6.12.7)], moreover, no longer possess a very important mathematical property that is possessed by all solutions to (6.5.1) with $a(t)$ and $b(t) \geq 0$ for all $t \geq 0$ and $x_0$ and $y_0 > 0$: namely, all solutions to (6.12.1) are no longer nonoscillatory

in the strict sense that $x(t)$ and $y(t)$ can now have more than one zero. This mathematical property is troublesome and makes analysis of battles modelled with (6.12.1) much more difficult than analysis of those modelled with (6.5.1). This nonoscillatory property is further discussed in Section 6.15 below.

The $X$ force level as a function of time, $x(t)$, for the general model (6.12.1) may be represented as

$$x(t) = \left[\exp\left\{-\int_0^t \beta(s)ds\right\}\right]$$

$$\times \left[x_0\{C_Q(0)\ C_P(t) - S_Q(0)\ S_P(t)\} - y_0\sqrt{\lambda_R}\ \{C_P(0)\ S_P(t) - S_P(0)\ C_P(t)\}\right.$$

$$\left. + \sqrt{\lambda_R}\ \int_0^t \frac{Z(s)}{a(s)}\ \{C_P(s)\ S_P(t) - S_P(s)\ C_P(t)\}ds\right], \qquad (6.12.8)$$

where $Z(t) = -A(t)\ S(t) + dR/dt - \{R(t)/A(t)\}dA/dt$, and the hyperbolic-like GLF $C_P(t)$ and $S_P(t)$ are linearly-independent solutions to the $P$ force-level equation (6.13.3) that satisfy the initial conditions (6.13.4). The GLF $C_Q(t)$ and $S_Q(t)$ are similarly defined. The above result is readily developed by considering (6.12.3) and applying well-known results for inhomogeneous ordinary differential equations (e.g. see HILDEBRAND [19, pp. 29-30]). Further analysis of the general model (6.12.1) is beyond the scope of our present investigation, but we will now consider some important special cases.

## 6.13. Combat with Supporting Fires

An important special case of the general linear combat model (6.12.1) is that in which there are no replacements, i.e. $r(t)$ and $s(t) \equiv 0$, and in this case our combat model becomes (again, for $x$ and $y > 0$)

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y - \beta(t)x & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t)x - \alpha(t)y & \text{with } y(0) = y_0 . \end{cases} \tag{6.13.1}$$

As discussed in the previous section, two different tactical situations that may be hypothesized to yield the above equations (6.13.1) are (cf. Figure 2.15 of Chapter 2):

either (S1) "aimed-fire" combat between two homogeneous forces with "operational" losses (see BACH et al. [1])

or (S2) "aimed-fire" combat between two homogeneous primary forces (or infantries) with superimposed effects of supporting fires not subject to attrition (see TAYLOR and PARRY [59]) (see Figure 6.18).

For convenience, we will refer to (6.13.1) simply as modelling combat with supporting fires and hence follows the name of this section. The modelling of the attrition-rate coeffcients in (6.13.1) is discussed in Section 6.12 above, with further details to be found in Chapter 5.

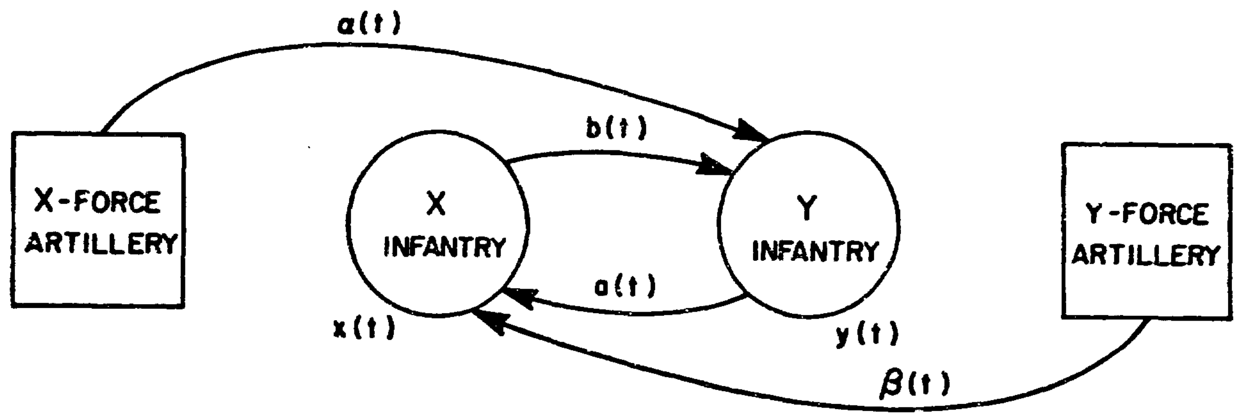


Figure 6.18. Combat between two homogeneous primary forces (infantries) with supporting weapons (artillery) not subject to attrition.

For our analysis of the LANCHESTER-type model (6.13.1) of combat with supporting fires, we make the following mathematical assumptions about the attrition-rate coefficients

(A1) $a(t)$ and $b(t)$ are defined, positive, and continuous for $t_0 < t < +\infty$ with $t_0 \leq 0$ ,

(A2) $\alpha(t)$ and $\beta(t) \geq 0$ for $t_0 \leq t < +\infty$,

(A3) $a(t)$, $b(t)$, $\alpha(t)$, and $\beta(t) \in L(t_0,T)$ for any finite $T$.

We further assume that $a(t)$ and $b(t)$ are given in the form (6.5.2), and we then introduce for the primary weapon systems the combat-intensity parameter $\lambda_I$ and the relative-force-effectiveness parameter $\lambda_R$ defined by (6.5.4).

The $X$ force level as a function of time, $x(t)$, for the model (6.13.1) may be written as (see TAYLOR [49])

$$x(t) = [\exp\{-\int_0^t \beta(s)ds\}]$$

$$\times [x_0\{C_Q(0)C_P(t) - S_Q(0)S_P(t)\} - y_0\sqrt{\lambda_R}\{C_P(0)S_P(t) - S_P(0)C_P(t)\}], \qquad (6.13.2)$$

where the hyperbolic-like GLF $C_P(t)$ and $S_P(t)$ are linearly-independent solutions to the $P$ force-level equation

$$\frac{d^2p}{dt^2} - \left\{\beta(t) - \alpha(t) + \frac{1}{a(t)}\frac{da}{dt}\right\}\frac{dp}{dt} - a(t)\ b(t)p = 0 , \qquad (6.13.3)$$

with initial conditions

$$C_P(t_0) = 1 , \qquad\qquad S_P(t_0) = 0 ,$$
$$\{1/a(t_0)\}\, dC_P/dt(t_0) = 0, \qquad \{1/a(t_0)\} dS_P/dt(t_0) = 1/\sqrt{\lambda_R} \quad . \tag{6.13.4}$$

The GLF $C_Q(t)$ and $S_Q(t)$ are similarly defined (see TAYLOR [49] for further details). Finally, we observe that the above result (6.13.2) is a special case of (6.12.8).

The above force-level results are readily developed by observing that the substitution (6.12.2) transforms (6.13.1) into

$$\begin{cases} \frac{dp}{dt} = -A(t)q & \text{with } p(0) = x_0 , \\ \frac{dq}{dt} = -B(t)p & \text{with } q(0) = y_0 , \end{cases} \tag{6.13.5}$$

with

$$A(t) = a(t) \exp\{\int_0^t [\beta(s) - \alpha(s)]ds\}$$

and

$$B(t) = b(t) \exp\{-\int_0^t [\beta(s) - \alpha(a)]ds\} \ . \tag{6.13.6}$$

From (6.13.5) we see that all the results for LANCHESTER's equations of modern warfare (6.5.1) may be used in our study of combat with supporting fires as modelled by (6.13.1). Then, for example, the X force level $x(t)$ as given by (6.13.2) follows from this observation. Let us also observe that from (6.13.3) the transformed "force-level" variable $p(t)$ satisfies

$$\frac{d^2p}{dt^2} - \left\{\frac{1}{A(t)}\frac{dA}{dt}\right\}\frac{dp}{dt} - A(t)\ B(t)p = 0\ , \qquad (6.13.7)$$

which may be written in the equivalent form (6.13.3). In a similar vein, TAYLOR [49] has developed the following results that describe the behavior of the model (6.13.1):

RESULT 1: At most one of the two force levels $x(t)$ and $y(t)$ can ever vanish in finite time.

RESULT 2: If either $A(t) \notin L(0,+\infty)$ or $B(t) \notin L(0,+\infty)$, then the $X$ force (with supporting fires) will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \left\{\frac{C_P(0) - \Lambda^* S_P(0)}{\Lambda^* C_Q(0) - S_Q(0)}\right\},$$

where $\lim_{t\to+\infty}\{S_P(t)/C_P(t)\} = 1/\Lambda^*$. Also, neither side will be annihilated in finite time if and only if the above inequality sign is replaced by an equality sign.

RESULT 3: If $\alpha(t) \equiv \beta(t)$, then

$$x(t) = \left[\exp\{-\int_0^t \beta(s)ds\}\right]$$
$$\times\ [x_0\{C_Y(0)C_X(t) - S_Y(0)S_X(t)\} - y_0\sqrt{\lambda_R}\ \{C_X(0)S_X(t) - S_X(0)C_X(t)\}\ ],$$

and the $X$ force (with supporting fires) will be annihilated in finite time if and only if (6.6.1) holds.

Further results and a discussion of their significance is to be found in TAYLOR [49]. In particular, Result 3 says that when each side's supporting fires are always equally effective [i.e. $\alpha(t) \equiv \beta(t)$], their effects cancel out and the battle's outcome in a fight-to-the-finish is the same (although the victor suffers greater losses) as when they are not present.

Thus, we see that the combat model with supporting fires (6.13.1) may be transformed into LANCHESTER's equations for modern warfare (6.5.1) so that all the results for the latter (see Sections 6.5 through 6.10 above) may be invoked. In particular, one is interested in developing battle-outcome-prediction conditions (recall Section 6.6). Exact force-annihilation-prediction conditions for the model (6.13.1) are readily developed by a translation of Theorem 6.6.1 to the transformed equation (6.13.5), and a special case of such conditions appears as Result 2 above. We will now consider simple approximate battle-outcome-prediction conditions for this model.

Example 6.13.1. For constant coefficients in the model (6.13.1), we have $C_p(t) = \exp[t(\beta-\alpha)/2]\,\{\cosh \theta t + [(\alpha-\beta)/2\theta] \sinh \theta t\}$, and $S_p(t) = (\sqrt{ab}/\theta) \exp[t(\beta-\alpha)/2] \sinh \theta t$, where $\theta = \sqrt{ab + [(\alpha-\beta)/2]^2}$. If follows that

$$\frac{1}{\Lambda^*} = \frac{\theta - (\alpha-\beta)/2}{\sqrt{ab}} .$$

Hence, Result 2 yields that the $X$ force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{R}\left\{\frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + 1}\right\}, \tag{6.13.8}$$

where $R = a/b$ denotes the relative fire effectiveness of the two opposing primary weapon-system types, and $S = (\beta-\alpha)/\sqrt{ab}$ denotes the net effectiveness of $Y$'s supporting units normalized by the "intensity" of combat between the primary units. Moreover, when each side's supporting fires are equally effective, i.e. $\alpha = \beta$ or $S = 0$, then the $X$ force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}} \quad ,$$

which is the same as LANCHESTER's classic model (2.2.1) without the supporting fires. Finally, we observe that the $X$ force level $x(t)$ is given by

$$x(t) = \{x_0 \cosh \theta t - \frac{1}{\theta} [ay_0 + (\frac{\alpha - \beta}{2}) x_0] \sinh \theta t\} \exp[-t(a + b)/2] \quad .$$

<u>Simple approximate battle-outcome-prediction conditions</u> for a <u>fixed-force-ratio-breakpoint battle</u> may be developed by considering the RICCATI equation satisfied by the force ratio $u = x/y$, namely

$$\frac{du}{dt} = b(t)u^2 + \{\alpha(t) - \beta(t)\}u - a(t) \quad \text{with} \quad u(0) = u_0 = \frac{x_0}{y_0} \,. \qquad (6.13.9)$$

This observation was apparently first made by TAYLOR and PARRY [59]. Before developing simple approximate victory-prediction conditions with (6.13.9), we will develop some "local" conditions of force superiority which will motivate subsequent developments.

For a fixed-force-ratio-breakpoint battle, it seems appropriate to say that "the course of battle is moving towards a $Y$ victory" when $du/dt < 0$. Moreover, $du/dt < 0$ if and only if

$$b(t)\ x^2(t) + \{\alpha(t) - \beta(t)\}\ x(t)\ y(t) < a(t)\ y^2(t)\ , \tag{6.13.10}$$

which may be rearranged to yield that for nonnegative force ratios

$$\text{"Y is winning" if and only if} \quad \frac{x(t)}{y(t)} < \sqrt{R(t)} \left\{ \frac{S(t)}{2} + \sqrt{\left[\frac{S(t)}{2}\right]^2 + 1} \right\}\ , \tag{6.13.11}$$

where

$$R(t) = \frac{a(t)}{b(t)}\ , \qquad \text{and} \qquad S(t) = \frac{\beta(t) - \alpha(t)}{\sqrt{a(t)\ b(t)}}\ . \tag{6.13.12}$$

Here $R(t)$ represents the relative fire effectiveness ($Y$ to $X$) of the primary units, while $S(t)$ represents the net effectiveness of $Y$'s supporting units normalized by the "intensity" of combat between the primary units. The "local" condition of force superiority (6.13.11) then says that the force ratio $x/y$ will continue to decrease (to $Y$'s favor) when it is below a certain (time-varying) critical "threshold" value. This threshold value depends on only the weapon-system-performance parameters (i.e. the attrition-rate coefficients) through the model parameter $R(t)$ and $S(t)$. In a sense, we have decoupled the quantity and quality of weapon systems in the "local" condition of force superiority (6.13.11).

In a moment we will extend the above "local" condition to be a "global" one of force superiority, but let us first consider a very important special case. When the supporting weapon systems are equally effective, i.e., $\alpha(t) \equiv \beta(t)$, (6.13.10) reduces to the "instantaneous" square law

$$b(t)\ x^2(t) < a(t)\ y^2(t)\ , \tag{6.13.13}$$

which may be considered to be a "local" condition for $Y$ to win. In other words, when the supporting weapon systems are equally effective, their effects cancel out. Furthermore, if $R(t) = a(t)/b(t)$ is a nondecreasing function of time and a certain technical condition is satisfied then (6.13.13) holding at $t = 0$ is sufficient for $Y$ to win (recall Theorem 6.6.2). It is also necessary when $R(t)$ is constant. Similar statements may be made about (6.13.11) in those cases for which $\alpha(t) \not\equiv \beta(t)$, and we will now develop such simple approximate battle-outcome-prediction conditions.

Thus, we will now develop a simple approximate battle-outcome-prediction condition for combat with supporting fires not subject to attrition (see Theorem 6.13.3 below). First, we must attend to some preliminaries. Let us denote the right-hand side of the inequality (6.13.11) as $u_+(t)$. More precisely, let $u_+(t)$ and $u_-(t)$ denote, respectively, the positive root and the negative root of $b(t)u^2 + \{\alpha(t) - \beta(t)\}u - a(t) = 0$. It follows that

$$u_{\pm}(t) = \sqrt{R(t)} \left\{ \frac{S(t)}{2} \pm \sqrt{\left[\frac{S(t)}{2}\right]^2 + 1} \right\}, \tag{6.13.14}$$

so that $u_-(t) < 0 < u_+(t)$ and (see Figure 6.19)

$$\frac{du}{dt} \begin{cases} < 0 & \text{for } u_-(t) < u < u_+(t) , \\ > 0 & \text{for } u_+(t) < u . \end{cases} \tag{6.13.15}$$

We then have

THEOREM 6.13.1 (TAYLOR and PARRY [59]): If $du/du(0) < 0$ and $u_+(t)$ is a nondecreasing function of time, then $du/dt(t) < 0$ for all $t \geq 0$.

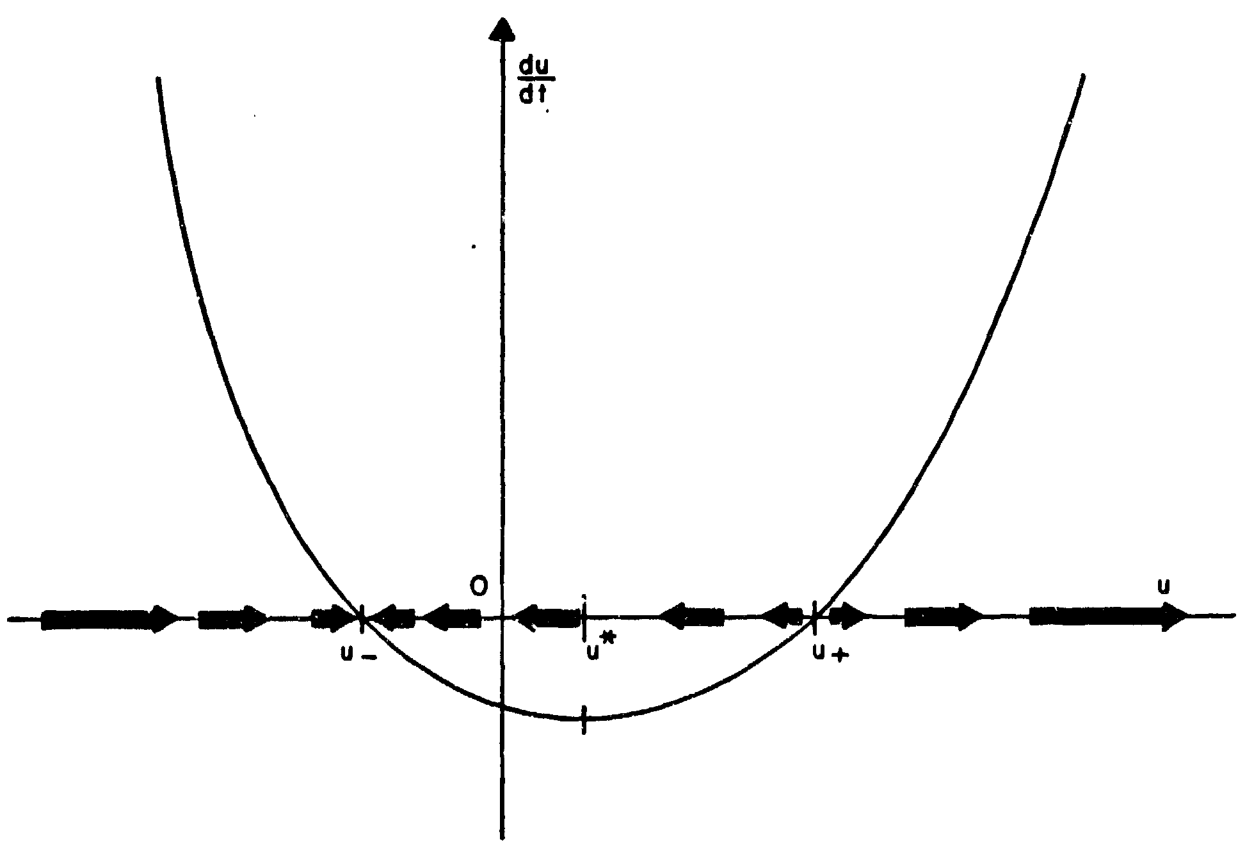


Figure 6.19. Force-ratio velocity as a function of the force ratio for combat modelled by LANCHESTER-type equations for an $(F+T)|(F+T)$ attrition process [see equations (6.13.1) in the text]. Here the length of the arrow drawn on the u-axis is in proportion to the magnitude of du/dt corresponding to that force ratio u, and the direction in which the arrow points corresponds to the sign of du/dt, e.g. an arrow pointing to the left corresponds to a minus sign for du/dt (cf. Figure 2.7).

PROOF. The basic idea behind this proof is that $u(t)$ and $u_+(t)$ "move in opposite directions." The hypothesis that $du/dt(0) < 0$ yields that $0 < u_0 < u_+(0)$ by (6.13.15). The assumption that $u_+(t)$ is nondecreasing then yields that $u_0 < u_+(0) \leq u_+(t)$ for all $t \geq 0$. It follows that $u(t)$ is a strictly decreasing function of time, since for $t$ near zero we have $u(t) \leq u_0 < u_+(0) \leq u_+(t)$ and consequently (6.13.15) yields that $du/dt(t) < 0$ always. Q.E.D.

Theorem 6.13.2 then tells us when $u_+(t)$ is nondecreasing.

THEOREM 6.13.2 (TAYLOR and PARRY [59]): If $R(t)$ and $S(t)$ are both nondecreasing functions of time, then $u_+(t)$ is a nondecreasing function of time.

We now make the following additional assumptions.

(A4) $R(t)$ and $S(t)$ are nondecreasing functions of time,

(A5) $b(t) \notin L(0, +\infty)$

(A6) $R(t)$ is not identically equal to zero.

Let $R_0$ denote $R(0)$ and similarly for $S_0$. Then a simple approximate battle-outcome-prediction condition is given by the following theorem.

THEOREM 6.13.3 (TAYLOR[50]): Assume that (A4) through (A6) hold. Then $Y$ will win a fixed-force-ratio-breakpoint battle in finite time if

$$\frac{x_0}{y_0} < \sqrt{R_0}\left\{\frac{S_0}{2} + \sqrt{\left(\frac{S_0}{2}\right)^2 + 1}\right\}. \tag{6.13.16}$$

<u>PROOF</u> (sketch; <u>see</u> TAYLOR [50] for complete details). The initial-condition inequality (6.13.16) implies that $du/dt(0) < 0$ so that Theorem 6.13.1 tells us that $du/dt(t) < 0$ for all $t \geq 0$. It remains to show that $u(t) \to u_{BP}^X < u_0$ in finite time, where $u_{BP}^X > 0$ denotes X's "breakpoint" force ratio. The latter result may be proven by showing that $u(t) \leq u_0 - K_1 \int_{t_1}^t b(s)ds$ with $K_1 > 0$, since $\lim_{t\to+\infty} \int_0^t b(s)ds = +\infty$. There are now two cases to be considered: (C1) $S(t) < 0$ for all $t \geq 0$, and (C2) there exists $t_1 \geq 0$ such that $R(t_1) > 0$ and $S(t_1) \geq 0$. In the first case (C1) it may be shown that $du/dt(t) \leq \{b(t)/b_0\}du/dt(0)$, whence $u(t) \leq u_0 + (1/b_0)\, du/dt(0) \int_0^t b(s)ds$, and the theorem follows in this case. In the second case (C2) it may be shown that

$$\frac{du}{dt}(t) \leq \begin{cases} -b(t)\, R(t_1) & \text{for } 0 \leq u \leq \{S(t)/2\}\sqrt{R(t)}\,, \\ -b(t)\, [-1/b(t_1)]du/dt(t_1) & \text{for } 0 \leq \{S(t)/2\}\sqrt{R(t)} \leq u \leq u_+(t), \end{cases}$$

whence $u(t) \leq u_0 - K_1 \int_{t_1}^t b(s)ds$ with $K_1 = \text{minimum } [R(t_1),(-1/b(t_1))du/dt(t_1)]$. <u>Q.E.D.</u>

The assumption that $\lim_{T\to+\infty}\int_0^T b(t)dt = +\infty$ means that an $X$ primary weapon system [and, by implication from assumption (A4), a $Y$ primary weapon system also] has unlimited firepower, i.e. there are no logistics constraints on the

battle. Theorem 6.13.3's proof, which we have sketched above, is particularly significant because it allows several important extensions: (1) cumulative firepower need not be unlimited, and (2) conditions for Y to achieve a given force ratio <u>within</u> a specified time.

Let us now make a few observations about the simple approximate battle-outcome-prediction condition (6.13.16).

<u>Comment 1</u>. Although there are six absolute quantities (i.e. two force levels and four attrition-rate coefficients) in our model of combat with supporting fires (6.13.1), there are only three independent <u>relative-capability parameters</u> (one relative-initial-primary-force-size parameter and two relative-fire-effectiveness parameters) involved in victory prediction: (1) the initial force ratio of the primary systems $u_0 = x_0/y_0$, (2) the initial relative fire effectiveness of the primary weapon systems $R_0$, and (3) the initial net fire effectiveness of the supporting weapons normalized by the intensity of combat between the primary weapon systems $S_0$.

<u>Comment 2</u>. When the supporting fires are always equally effective, i.e. $\alpha(t) \equiv \beta(t)$, their effects "cancel out," and (in terms of the force ratio) the battle's outcome is the same as though they were not present.

Although highly idealized, the model (6.13.1) is significant because of the insights that it provides into the dynamics of combat. As we discussed above, we may consider (6.13.1) to model combat between two homogeneous forces (primary weapon systems) with superimposed effects of supporting fires not subject to attrition. F. W. LANCHESTER [26] apparently believed

that before 1914 the "modern" trend in warfare had been towards greater concentration of forces (i.e. higher troop densities in combat area) and formulated his now classic model of combat (without supporting fires) in order to quantitatively justify the principle of concentration. It is significant to note (e.g. see HERO [20-22], however, that the actual trend in combat operations over the past two thousand years of military history has been towards greater dispersion of forces (i.e. lower troop densities in combat areas). Some figures for the last hundred years are shown in Table 6.XII (see STEWART [41]).

Furthermore, the model (6.13.1) may be used to gain important insights into whether or not it is "beneficial" to concentrate forces, i.e. whether or not a side should make its initial commitment of forces as large as possible (e.g. see Section 2.9 above). Results show that if the "intensity" of the supporting-fire combat exceeds that of the primary systems [i.e. $\alpha(t)\ \beta(t) > a(t)\ b(t)$],[19] then the victor should not concentrate his forces (see TAYLOR [48] for a detailed analysis of the decision of whether or not to concentrate forces; also see Section 8.10 below). Considering the past increases [20-22] in the fire effectiveness of supporting weapons relative to that for primary weapon systems (e.g. small arms), we would expect that in general $\alpha(t)\ \beta(t) > a(t)\ b(t)$ on the modern battlefield. Consequently, the victor should not concentrate his forces according to the above. Thus, the model (6.13.1) yields a theoretical result (about optimal military tactics) that is in better agreement with the historical trend in military operations than is that yielded by LANCHESTER's original model (2.2.1) without supporting fires (i.e. the victor should always concentrate forces [see Section 2.9 above]).

TABLE 6.XII. Increase in the Dispersion of Troops from the U. S. Civil War to World War II (from STEWART [41]).

| ITEM | CIVIL WAR | WORLD WAR I | WORLD WAR II |
|---|---|---|---|
| Area of 100,000 men (in square miles) | 26.8 | 140 | 1727 |
| Average frontage of 100,000 men (miles) | 8.0 | 11 | 38.4 |
| Average depth of 100,000 men (miles) | 3.3 | 13 | 45 |

It will be instructive for us to consider a more concrete case and examine more closely this question about the optimal initial commitment of forces. Hence, let us consider the constant-coefficient model of combat with supporting fires

$$\begin{cases} \dfrac{dx}{dt} = -ay - \beta x & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -bx - \alpha y & \text{with } y(0) = y_0 , \end{cases} \qquad (6.13.17)$$

where $a$, $b$, $\alpha$, and $\beta$ now denote constant attrition-rate coefficients. Returning to first principles, to determine the optimal initial commitment of forces, we must consider a "combat-optimization" problem as we have done in Section 2.9 above (<u>see</u> also Section 8.10 below). Consider now a battle in which Y has more than enough troops to win. Will Y be "better off" by initially committing all his forces to battle? Should he hold some of them in reserve? We assume that this initial-commitment decision is to be made (only) once before the battle begins. If we take the overall casualty-exchange ratio $R_c$ ($= y_c/x_c$, where $y_c$ denotes Y's casualties and similarly for $x_c$) as Y's decision criterion, then Y should initially commit more forces to battle as long as $\partial R_c/\partial y_0 < 0$. Then for either a fixed-force-level-breakpoint battle or a fixed-force-ratio-breakpoint one, it may be shown (<u>see</u> TAYLOR [49]) that $\partial R_c/\partial y_0 < 0$ if and only if $\partial(dy/dx)/\partial u > 0$. This if-and-only-if statement holds because $\partial(dy/dx)/\partial u$ always has the same sign (<u>see</u> below) and the attrition-rate coefficients are constant.

For the model (6.13.17) we have

$$\frac{dy}{dx} = \frac{\alpha + bu}{a + \beta u},$$

and a straightforward computation yields

$$\frac{\partial}{\partial u}\left(\frac{dy}{dx}\right) = \frac{ab - \alpha\beta}{(a + \beta u)^2}. \tag{6.13.18}$$

Thus, we see that $\partial(dy/dx)/\partial u > 0$ always if and only if $ab > \alpha\beta$. Hence the prospective victor should initially commit as many primary-system forces (e.g. infantry forces) as possible to battle when the intensity of combat between the primary forces exceeds the "intensity" of the supporting fires, i.e. when $ab > \alpha\beta$. When $\alpha\beta > ab$, more forces than are required to "just" assure victory should not be initially committed because they are more vulnerable to supporting fires (<u>see</u> TAYLOR [48] and Section 8.10 for further details).

As discussed in Section 2.9, there is a very simple and intuitively appealing interpretation of the above optimal force-commitment decision rule. The instantaneous casualty-exchange ratio $dy/dx$ represents the "cost" to $Y$ of reducing the $X$ force level a unit amount. The partial derivative $\partial(dy/dx)/\partial u$ represents the variation in this cost to changes in the force ratio $u = x/y$. When $\partial(dy/dx)/\partial u > 0$ always, then $Y$'s instantaneous cost of doing battle is always reduced when the battle is fought at lower force ratios $u = x/y$. If $Y$ initially commits more forces to battle (i.e. $Y$ makes $y_0$ larger, then the battle is fought at lower force ratios, and $Y$ is cumulatively better off according to this decision criterion. Hence, $ab > \alpha\beta$ yields that $Y$ is better off by initially committing more forces to battle. Moreover, this decision rule is surprisingly robust and holds for other decision criteria (<u>see</u> TAYLOR [48]). Finally, this heuristic reasoning is shown to be mathematically precise in Section 8.10 below.

## 6.14. HELMBOLD-Type Combat with Supporting Fires

If we assume that attrition between the two primary weapon systems (e.g. infantries, see Figure 6.18) follows HELMBOLD's modification of LANCHESTER's equations of "modern warfare" to account for inefficiencies of scale when infantry-force sizes are grossly unequal (see Section 6.11), our model of combat with supporting fires (6.13.1) becomes (see Figure 6.20)

$$\begin{cases} \dfrac{dx}{dt} = -a(t) \cdot \left(\dfrac{x}{y}\right)^{1-W} \cdot y - \beta(t)x \qquad \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t) \cdot \left(\dfrac{y}{x}\right)^{1-W} \cdot x - \alpha(t)y \qquad \text{with} \quad y(0) = y_0 , \end{cases} \tag{6.14.1}$$

where $\alpha(t)$ and $\beta(t)$ again represent the effectivenesses of the supporting fires, and $W$ denotes the "WEISS parameter" of the battle.

More formally, we will call (6.14.1) the equations for HELMBOLD-type combat with supporting fires not subject to attrition, although (of course) we know that other interpretations are possible (see Sections 2.12 and 6.13 above). Here, we have assumed that both sides suffer the same inefficiencies of scale. This nonlinear combat model (6.14.1) reduces to the above studied linear model (6.13.1) when $W = 1$. In analyzing this model we will again assume that assumptions (A1) through (A6) of Section 6.13 hold. Finally, let us note that the above nonlinear combat model (6.14.1) is highly operationally significant, since it provides an excellent fit to large-unit (i.e. division-level and larger) casualty-rate curves currently used in several of the principal large-scale ground-combat models used in the United States (see Section 7.11 below for further details).

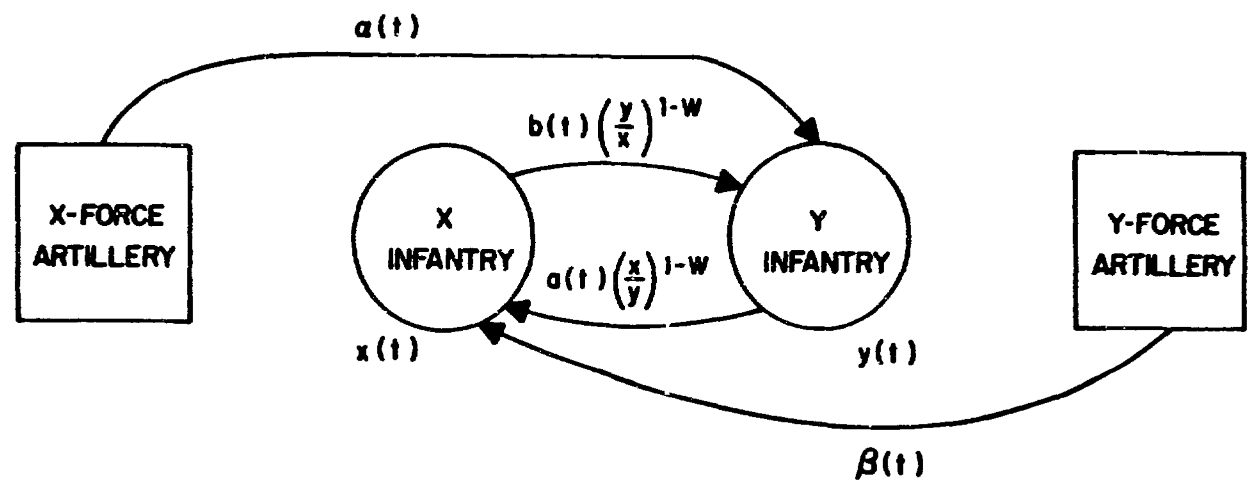


Figure 6.20. HELMBOLD-type combat between two homogeneous primary forces (infantries) with supporting weapons (artillery) not subject to attrition. HELMBOLD's modification of the primary-force-mutual-attrition process models inefficiences of scale when primary-force sizes are grossly unequal (see Section 6.11).

Again (<u>see</u> Section 6.11), this nonlinear HELMBOLD-type combat model may be transformed into a linear combat model by the appropriate transformation of the dependent variable. Thus, the substitution $p = x^W$ and $q = y^W$ transforms (6.14.1) into

$$\begin{cases} \dfrac{dp}{dt} = -W\{a(t)q + \beta(t)p\} & \text{with } p(0) = x_0^W , \\ \\ \dfrac{dq}{dt} = -W\{b(t)p + \alpha(t)q\} & \text{with } q(0) = y_0^W . \end{cases} \tag{6.14.2}$$

Hence, all the results (<u>see</u> Section 6.13 above) for the linear model with supporting fires not subject to attrition (6.13.1) apply the the nonlinear HELMBOLD-type combat model (6.14.1). For example, when assumptions (A4) through (A6) of Section 6.13 are satisfied, then the $Y$ force will win a fixed-force-ratio-breakpoint battle in finite time if

$$\left(\frac{x_0}{y_0}\right)^W < \sqrt{R_0}\left\{\frac{S_0}{2} + \sqrt{\left(\frac{S_0}{2}\right)^2 + 1}\right\}, \tag{6.14.3}$$

where $R(t)$ and $S(t)$ are given by (6.13.12), $R_0$ denotes $R(0)$, and similarly for $S_0$.

## 6.15. The General Linear Model with Replacements (Constant Attrition-Rate Coefficients).

In the case of constant attrition-rate coefficients, the general linear model (6.12.1) reads

$$\frac{dx}{dt} = -ay - \beta x + r \qquad \text{with} \quad x(0) = x_0 ,$$
$$\frac{dy}{dt} = -bx - \alpha y + s \qquad \text{with} \quad y(0) = y_0 , \tag{6.15.1}$$

where $a$, $b$, $\alpha$, $\beta$, $r$, and $s$ denote quantities that remain constant during a particular battle, and we assume that $a$ and $b > 0$, while $\alpha$ and $\beta \geq 0$. Although there are several different sets of physical circumstances that may be hypothesized to yield (6.15.1) (see Section 6.12 above), we will consider (6.15.1) to model "aimed-fire" combat between two homogeneous forces with supporting fires not subject to attrition and continuous replacements/withdrawals. In this case we should consider $r$ and $s$ to be replacement rates, with a negative value denoting a net rate of withdr.wal of forces. Accordingly, we will place no restrictions on the replacement rates $r$ and $s$, i.e. $r$ and $s$ are unrestricted in sign.

The model (6.15.1) is of interest because it provides insights into the consequences of additional troops (continuously) committed to battle. We may consider a term like, for example, $r$ to represent the rate at which additional $X$ forces are committed to battle. Another related interpretation is that $r$ represents the net rate at which the $X$ force enters the fields of fire of the $Y$ force. Such interpretations essentially apply

to small-unit combat in fire fights. We may also (see Section 6.12 above), however, consider (6.15.1) to model combat with operational losses and continuous replacements. In this case we may consider (6.15.1) to apply to large-scale combat over a sustained period of time, and then $r$ and $s$ represent the rates at which additional resources are committed to the theater of operations (see MORSE and KIMBALL [31, pp. 71-73]). In this light, analysis of this combat model will provide important insights into the nature of tradeoffs among (1) direct combat capability, (2) "build-up" capability, and (3) operational losses. In terms of the NATO scenario, the model (6.15.1) provides rough insights into the structure of tradeoffs among the quality of weapon systems, the quantity of weapon systems, and the "build-up" rates at which new systems are introduced into the theater of operations.

Unlike the previous variable-coefficient versions considered above, the constant-coefficient model (6.15.1) yields an analytical solution that is simple enough to provide some important insights into the dynamics of combat through direct analysis. When $ab \neq \alpha\beta$, the $X$ and $Y$ force levels $x(t)$ and $y(t)$ for the model (6.15.1) are given by[20]

$$x(t) = \xi + Ae^{(\theta-\sigma)t} + \left(\frac{\theta + \delta}{b}\right) Be^{-(\theta+\sigma)t} ,$$

and (6.15.2)

$$y(t) = \eta - \left(\frac{\theta + \delta}{a}\right) Ae^{(\theta-\sigma)t} + Be^{-(\theta+\sigma)t} ,$$

where

$$A = \frac{ab}{2\theta(\theta + \delta)} \left\{ (x_0 - \xi) - \left(\frac{\theta + \delta}{b}\right) (y_0 - \eta) \right\}, \qquad (6.15.3)$$

$$B = \frac{ab}{2\theta(\theta + \delta)} \left\{ \left(\frac{\theta + \delta}{a}\right) (x_0 - \xi) + (y_0 - \eta) \right\}, \qquad (6.15.4)$$

$$\xi = \frac{as - \alpha r}{\Delta}, \qquad \eta = \frac{br - \beta s}{\Delta}, \qquad \Delta = ab - \alpha\beta, \qquad (6.15.5)$$

$$\theta = \sqrt{ab + \delta^2}, \qquad \delta = \frac{\beta - \alpha}{2}, \qquad \text{and} \quad \sigma = \frac{\alpha + \beta}{2}. \qquad (6.15.6)$$

Let us also note the following identity

$$\frac{\theta + \delta}{b} = \sqrt{R} \left\{ \frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + 1} \right\}, \qquad (6.15.7)$$

where $R = a/b$ and $S = (\beta - \alpha)/\sqrt{ab}$ (see Section 6.13 fcr a discussion of the military interpretations of these parameters $R$ and $S$).

<u>When</u> $ab = \alpha\beta$, <u>the</u> $X$ <u>and</u> $Y$ <u>force levels</u> $x(t)$ <u>and</u> $y(t)$ <u>for the model</u> (6.15.1) <u>are given by</u>

$$x(t) = x_0\, e^{-(\alpha+\beta)t} + \left(\frac{\alpha r - as}{\alpha + \beta}\right) t$$

$$+ \left\{ \frac{(\beta r + as)}{(\alpha + \beta)^2} + \left( \frac{\alpha x_0 - \beta y_0}{\alpha + \beta} \right) \right\} \{1 - e^{-(\alpha+\beta)t}\}, \qquad (6.15.8)$$

and

$$y(t) = y_0\, e^{(\alpha+\beta)t} - \left(\frac{b}{\alpha}\right)\left(\frac{\alpha r - as}{\alpha + \beta}\right) t$$

$$+ \frac{(\alpha s + br)}{(\alpha + \beta)^2} - \left(\frac{b}{\alpha}\right) \left(\frac{\alpha x_0 - ay_0}{\alpha + \beta}\right) \{1 - e^{-(\alpha+\beta)t}\}. \qquad (6.15.9)$$

In this latter case, i.e. when $ab = \alpha\beta$, the constant-coefficient combat model (6.15.1) possesses the state equation

$$b(x_0 - x) = \beta(y_0 - y) + (\beta s - br)t, \tag{6.15.10}$$

which yields that the overall casualty-exchange ratio is constant, i.e.

$$\frac{x_c}{y_c} = \frac{\beta}{b}, \tag{6.15.11}$$

where the $X$ and $Y$ casualties are given by

$$x_c = x_0 + rt - x, \quad \text{and} \quad y_c = y_0 + st - y. \tag{6.15.12}$$

Let us observe that in all cases the instantaneous casualty-exchange ratio $dx/dy$ is given by

$$\frac{dx}{dy} = \frac{\beta}{b} + \left\{ \frac{r - \frac{\beta}{b} s - \frac{\Delta}{b} y}{s - bx - \alpha y} \right\}, \tag{6.15.13}$$

which for $ab = \alpha\beta$ becomes

$$\frac{dx}{dy} = \frac{\beta}{b} + \left\{ \frac{r - (\beta/b)s}{s - bx - \alpha y} \right\}. \tag{6.15.14}$$

In particular, for $br = \beta s$ and $ab = \alpha\beta$ we have the linear law

$$b(x_0 - x) = \beta(y_0 - y). \tag{6.15.15}$$

Determination of the qualitative behavior, e.g. battle-outcome-prediction conditions, for the linear combat model with replacements (6.15.1) is much more difficult than we have heretofore encountered because the force levels $x(t)$ and $y(t)$ no longer possess a very important mathematical property that facilitated analysis of combat modelled with LANCHESTER's equations for modern warfare (6.5.1): namely, all solutions to (6.15.1) are no longer nonoscillatory in the strict sense that $x(t)$ and $y(t)$ can have more than one zero. We will give an example of such solution behavior below. However, analysis of the qualitative behavior of the model (6.15.1) is relatively straightforward when $ab > \alpha\beta$, i.e. the intensity of combat between the primary systems exceeds the "intensity" of the supporting fires, and we will now develop <u>force-annihilation-prediction conditions</u> for this case. Let us first observe that $\theta - \sigma > 0$ if and only if $ab > \alpha\beta$. Hence, in this case the exponential $e^{(\theta-\sigma)t}$ in (6.15.2) is a strictly increasing function that grows without bound. Furthermore, the signs of $x(t)$ and $y(t)$ for large $t$ are opposite and determined by the sign of $A$. For $A = 0$, i.e. $(x_0 - \xi) = (y_0 - \eta)(\theta+\delta)/b$, (6.15.2) reduces to

$$x(t) = x_0\, e^{-(\theta+\sigma)t} + \xi\{1 - e^{-(\theta+\sigma)t}\},$$

and (6.15.16)

$$y(t) = y_0\, e^{-(\theta+\sigma)t} + \eta\{1 - e^{-(\theta+\sigma)}\}.$$

We observe that $\theta + \delta > 0$. It follows that for $ab > \alpha\beta$, and $\xi$ and $\eta \geq 0$

$$\begin{pmatrix} \text{X will be annihilated} \\ \text{in finite time if} \\ \text{and only if} \end{pmatrix} \quad (x_0 - \xi) < \left(\frac{\theta + \delta}{b}\right) (y_0 - \eta), \qquad (6.15.17)$$

which may also be written in the equivalent form

$$(x_0 - \xi) < \sqrt{R} \left\{\frac{S}{2} + \sqrt{\left(\frac{S}{2}\right)^2 + 1}\right\} (y_0 - \eta) . \qquad (6.15.18)$$

The Y force will be annihilated (and only then) in finite time when the above inequality (6.15.18) is reversed. Moreover, from (6.15.2) we see that $y(t) > 0$ for all $t \geq 0$ when (6.15.18) holds with $\xi$ and $\eta \geq 0$. The requirement that $\xi$ and $\eta \geq 0$ in the force-annihilation-prediction condition (6.15.18) is absolutely essential as the example depicted in Table 6.XIII shows. In other words, (6.15.18) [equivalently, (6.15.17)] is satisfied for the battle depicted in Table 6.XIII, but the Y force is actually annihilated before the X force is. The reason why (6.15.18) fails to correctly predict force annihilation is that $\eta < 0$. This example should alert the reader to the fact that determination of the qualitative behavior, e.g. force-annihilation prediction, for the constant-coefficient model with replacements/withdrawals (6.15.1) is much trickier than that for the variable-coefficient model (6.5.1) with no placements/withdrawals.

Let us finally sketch the development of the above expressions for the force levels $x(t)$ and $y(t)$. When $ab \neq \alpha\beta$, we may write (6.15.1) as

$$\frac{dx}{dt} = -a(y - \eta) - \beta(x - \xi) \quad \text{and} \quad \frac{dy}{dt} = -b(x - \xi) - \alpha(y - \eta) , \qquad (6.15.19)$$

whence the substitution $X = x - \xi$ and $Y = y - \eta$ transforms (6.15.19) into

TABLE 6.XIII. Example That Shows That One Must have Both $\xi$ and $\eta \geq 0$ in Order for the Inequality (6.15.18) to Correctly Predict a Y Victory in a Fight-to-the Finish.

NOTE: In this battle we have taken (in compatible units) $a = b = 2$, $\alpha = \beta = 1$, $r = 0$, and $s = 150$. It follows that (6.15.18) is satisfied but with $\xi = 100$ and $\eta = -50$.

| t | x(t) | y(t) |
|---|---|---|
| 0.00 | 200.00 | 60.00 |
| 0.1 | 172.26 | 33.31 |
| 0.2 | 151.52 | 13.73 |
| 0.3 | 135.94 | -0.56 |
| 0.4 | 124.17 | -10.92 |
| 0.5 | 115.19 | -18.33 |
| 0.6 | 108.25 | -23.53 |
| 0.7 | 102.79 | -27.07 |
| 0.8 | 98.40 | -29.35 |
| 0.9 | 94.76 | -30.65 |
| 1.0 | 91.64 | -31.18 |
| 1.1 | 88.85 | -31.11 |
| 1.2 | 86.27 | -30.53 |
| 1.3 | 83.78 | -29.53 |
| 1.4 | 81.30 | -28.15 |
| 1.5 | 78.76 | -26.43 |
| 1.6 | 76.10 | -24.37 |
| 1.7 | 73.27 | -21.99 |
| 1.8 | 70.23 | -19.28 |
| 1.9 | 66.92 | -16.22 |
| 2.0 | 63.31 | -12.79 |
| 2.1 | 59.36 | - 8.98 |
| 2.2 | 55.02 | - 4.73 |
| 2.3 | 50.23 | - 0.02 |
| 2.4 | 44.96 | 5.19 |
| 2.5 | 39.15 | 10.97 |
| 2.6 | 32.72 | 17.36 |
| 2.7 | 25.63 | 24.43 |
| 2.8 | 17.80 | 32.35 |
| 2.9 | 9.15 | 40.89 |
| 3.0 | -0.41 | 50.44 |
| 3.1 | -10.98 | 61.00 |
| 3.2 | -22.66 | 72.67 |
| 3.3 | -35.56 | 85.57 |
| 3.4 | -49.82 | 99.82 |
| 3.5 | -65.57 | 115.58 |

$$\frac{dX}{dt} = -aY - \beta X \qquad \text{and} \qquad \frac{dY}{dt} = -bX - \alpha Y ,$$

for which we have given a solution in Section 6.13 above. When $ab = \alpha\beta$, we may write (6.15.1) as

$$\frac{dx}{dt} = r - \beta(x + \frac{\alpha}{b} y) \quad \text{and} \quad \frac{dy}{dt} = s - b(x + \frac{\alpha}{b} y) ,$$

whence follow the above results.

### 6.16. Variable-Coefficient Equations for FT|FT Attrition Process

As emphasized above, S. BONDER [5;10] has stressed the importance for weapon-system evaluations of using time-dependent attrition-rate coefficients in LANCHESTER-type combat models to represent temporal variations in firepower on the battlefield (e.g. see the battle trajectories given in Section 6.2 above). We have considered various aspects of such variable-coefficient generalizations of LANCHESTER's equations for modern warfare in several of the above sections. Let us now, however, consider the following LANCHESTER-type equations for a FT|FT attrition process with time-dependent attrition-rate coefficients

$$\begin{cases} \dfrac{dx}{dt} = -a(t)xy & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t)xy & \text{with } y(0) = y_0 . \end{cases} \tag{6.16.1}$$

These equations may be hypothesized to model combat under either of the following two sets of circumstances (cf. Sections 2.4 and 2.11 above):

either (S1) both sides use "area" fire and a constant-area defense [12; 61],

or (S2) both sides use "aimed" fire with the rate of target acquisition being inversely proportional to the number of enemy targets and also being the controlling factor in the attrition process [12].

The modelling of the attrition-rate coefficients $a(t)$ and $b(t)$ is discussed in Sections 5.4 and 5.7 above. Mathematically, we assume that the attrition-rate coefficients $a(t)$ and $b(t)$ are positive and piecewise differentiable. We further assume that both $a(t)$ and $b(t) \in L(0,T)$ for any finite $T \geq 0$ and similarly for $d/dt\{b(t)/a(t)\}$.

The development of analytical results for the X and Y force levels $x(t)$ and $y(t)$ is very much more difficult for time-dependent attrition-rate coefficients than it was for constant coefficients (see Section 2.4). Since no relation like LANCHESTER's linear law (2.4.3) generally holds for the variable-coefficient combat model (6.16.1), we are led to a nonlinear second-order differential equation in order to analytically determine, for example, $x(t)$. Accordingly, we may use differentiation and algebraic elimination to obtain from (6.16.1) the X <u>force-level equation</u>

$$\frac{d^2x}{dt^2} - \frac{1}{x}\left(\frac{dx}{dt}\right)^2 + b(x)x\,\frac{dx}{dt} - \left\{\frac{1}{a(t)}\,\frac{da}{dt}\right\}\frac{dx}{dt} = 0\ , \tag{6.16.2}$$

with initial conditions

$$x(0) = x_0\ , \qquad \text{and} \qquad \frac{dx}{dt}(0) = -a_0x_0y_0\ ,$$

where $a_0$ denotes $a(0)$ and similarly for $b_0$. Unfortunately, this second-order nonlinear differential equation is apparently not equivalent to any standard equation solvable in terms of "elementary" functions, e.g. see INCE [23] or DAVIS [16]. However, we will give some simple approximations to the solution of this nonlinear differential equation.

TAYLOR [46] has developed the following two simple approximations to the solution of (6.16.2), denoted as $\hat{x}_i(t)$ for $i = 1$ and 2, namely

$$\hat{x}_i(t) = \frac{x_0}{[\exp\{-\int_0^t G_i(s)ds\} + x_0 \int_0^t b(s)\,(\exp\{-\int_s^t G_i(r)dr\})ds]}, \tag{6.16.3}$$

where

$$G_i(t) = \begin{cases} \dfrac{a(t)}{a_0}(b_0x_0 - a_0y_0) & \text{for } i = 1, \\ b(t)x_0 - a(t)y_0 & \text{for } i = 2. \end{cases} \tag{6.16.4}$$

What is the error made in using the above approximations? How "good" are they? To answer these important questions, TAYLOR [46] has developed a bound for the error made in using either of the two approximations $\hat{x}_1(t)$ and $\hat{x}_2(t)$. This bound is easy to evaluate and does not require knowledge of the exact solution $x(t)$. His result is as follows.

THEOREM 6.16.1 (TAYLOR [46]): A bound on the error made in the approximation (6.16.3) $\hat{x}_i(t)$ (for $i = 1,2$) to the exact solution $x(t)$ of (6.16.1) is given by

$$x_2(t) - x_1(t) \geq |x(t) - \hat{x}_i(t)| \quad \text{for } i = 1,2, \tag{6.16.5}$$

where

$$x_j(t) = \frac{x_0}{[\exp\{-\int_0^t H_j(s)ds\} + x_0 \int_0^t b(s)(\exp\{-\int_s^t H_j(r)dr\})ds]},$$

and

$$H_j(t) = a(t)\left\{\frac{1}{a_0}(b_0x_0 - a_0y_0) + (-1)^j\, V_{0,t}\left(\frac{b}{a}\right)\right\}$$

for $j = 1,2$.

In Theorem 6.16.1 $V$ denotes the variational operator defined and discussed in OLVER [34, pp. 27-29], i.e.

$$V_{0,t}\left(\frac{b}{a}\right) = \int_0^t \left|\frac{d}{ds}\left\{\frac{b(s)}{a(s)}\right\}\right| ds\ .$$

When $b(t)/a(t)$ is monotonic, however, this bound simplifies and becomes tighter. Thus, we have

THEOREM 6.16.2 (TAYLOR [46]): If $d/dt\{b(t)/a(t)\} \geq 0$ for all $t \in [0,T]$, then a bound on the error made in the approximation (6.16.3) $\hat{x}_i(t)$ (for $i = 1,2$) to the exact solution of (6.16.1) is given by

$$\hat{x}_2(t) - \hat{x}_1(t) \geq (-1)^{i+1}\{x(t) - \hat{x}_i(t)\} \geq 0 \quad \text{for } i = 1,2.$$

The above are the only analytical results known to the author for the nonlinear combat model with temporal variations in fire effectiveness (6.16.1).

Let us finally observe that all the above results apply to a more general nonlinear combat model. When each side has supporting weapons not subject to attrition (cf. Section 6.13 above), our model becomes

$$\begin{cases} \dfrac{dx}{dt} = -a(t)xy - \beta(t)x & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t)xy - \alpha(t)y & \text{with } y(0) = y_0 , \end{cases} \tag{6.16.6}$$

where $\alpha(t)$ and $\beta(t)$ are nonnegative and represent the effectiveness of supporting fires. However, the substitution (6.12.2) transforms (6.16.6) into

$$\begin{cases} \dfrac{dp}{dt} = -A(t)pq & \text{with } p(0) = x_0 , \\ \\ \dfrac{dq}{dt} = -B(t)pq & \text{with } q(0) = y_0 , \end{cases} \tag{6.16.7}$$

with $A(t) = a(t) \exp\{-\int_0^t \alpha(s)ds\}$ and $B(t) = b(t) \exp\{-\int_0^t \beta(s)ds\}$.
Thus, all the above results for the model (6.16.1) may be applied to the more general model of combat with supporting fires not subject to attrition (6.16.6).

## *6.17. A Result for the General Model with Temporal Variations in Fire Effectiveness

Two quantities of fundamental interest to the military OR worker are (1) the force ratio, and (2) the casualty-exchange ratio. In this section we will show that for the general case of combat between two homogeneous forces, the difference between these two fundamental quantities provides a simple (but yet very basic) "local" condition of force superiority that sometimes allows one to determine that the force ratio is a monotonic function of time. Such a result is not only of intrinsic interest but also important for understanding the dynamics of FEBA movement (Forward Edge of the Battle Area, which is the contact zone between opposing forces) when combined with a rate-of-advance equation for FEBA motion. In large-scale combat models for a given engagement, the motion of the FEBA is usually taken to depend monotonically on the force ratio so that monotonic behavior of the force ratio over time can be translated into qualitative statements about cumulative FEBA movement (*see* Sections 7.13 and 7.14 for further details). Thus, the results of this section may be used to develop fundamental qualitative insights into the dynamics of combat.

As we saw in Section 6.1 above, we may generally model combat between two homogeneous forces with the following deterministic LANCHESTER-type equations for $x$ and $y \geq 0$

$$\begin{cases} \dfrac{dx}{dt} = -G(t,x,y) & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -H(t,x,y) & \text{with } y(0) = y_0 , \end{cases} \qquad (6.17.1)$$

---

*Starred sections are not required for the understanding of the sequel and should be omitted at first reading. They usually require more mathematical sophistication to be understood.

where $x(t)$ and $y(t)$ denote the X and Y force levels at time t, and G and H denote force-change rates (with a negative force-change rate signifying a net influx of replacements). When there are no replacements and withdrawals, G and H are simply casualty rates. To insure the existence of partial derivatives needed in subsequent analysis, we assume that G and H are differentiable.

It is of interest to be able to determine in whose favor the course of battle is progressing without solving the equations (6.17.1) in detail. If we consider a fixed-force-ratio-breakpoint battle (a special case of which is a fight to the finish in which one side or the other is annihilated), then the rate of change of the force ratio is an appropriate measure of the direction in which the course of battle is moving, since we can then identify towards which combatant's force-ratio breakpoint the battle is being "steered." Then according to this criterion, there is a simple criterion (with a rich military interpretation) for a force to be "winning": namely, a force is "winning" when the force ratio exceeds the casualty-exchange ratio.[21] This "local" condition of force superiority applies to <u>all</u> LANCHESTER-type models with two force-level variables and yields a "global" condition of force superiority (i.e. the force ratio monotonically changes to the advantage of one side) when certain trends over time hold.

Let us now develop our local condition of force superiority. Accordingly, we introduce the force ratio $u = x/y$. As pointed out by TAYLOR and PARRY [59], for a fixed-force-ratio-breakpoint battle it seems appropriate to say that "the course of battle is moving towards an X victory" when $du/dt > 0$ (or, simply, that "X is winning"). Our "local" condition of force superiority is developed by determining the sign of

du/dt at a point in time. We will do this without solving the equations (6.17.1) in detail. Considering the force ratio $u = x/y$, we find after some straightforward manipulations that

$$u - \frac{dx}{dy} = \frac{du/dt}{\left\{-\frac{1}{y}\frac{dy}{dt}\right\}} . \tag{6.17.2}$$

This result (6.17.2) is the key result from which all subsequent developments in this section follow. We assume for simplicity that we always have $dy/dt < 0$, with other cases being handled in a straightfoward manner. When $dy/dt < 0$, then $du/dt$ and $(u - dx/dy)$ have the same sign. Thus, for $dy/dt < 0$ we see from (6.17.2) that a "local" condition of X-force superiority (i.e. X is "winning" a fixed-force-ratio-breakpoint battle) is

$$u > \frac{dx}{dy}(t,x,y) . \tag{7.17.3}$$

The inequality (6.17.3) has a very important military interpretation. In general, the quantity $dx/dy$ is the instantaneous (or differential) force-change ratio, which for cases of no replacements and withdrawals becomes the instantaneous (or differential) casualty-exchange ratio. Consequently, in such cases, (6.17.3) says that X is "winning" when the force ratio exceeds the instantaneous casualty-exchange ratio. In other words, the relative size of the force ratio and the casualty-exchange ratio determine the direction of the course of battle. Such a rule of thumb may be very useful in such an interpretative sense when the exact dynamics of combat are not known, i.e. one can still determine in whose favor the direction of battle is moving.

It is of particular interest to be able to predict when (6.17.3) will hold throughout a battle (i.e. to determine a "global" condition of force superiority). Although we have not succeeded in developing such conditions in general, we will now give results for a special case of fairly wide applicability. Thus, for many LANCHESTER-type combat models of interest, the instantaneous force-change ratio $dx/dy$ depends on only $t$ and the force ratio $x/y$, i.e. $dx/dy$ is a homogeneous function of degree zero in the force-level variables $x$ and $y$ (see COURANT [15, pp. 108-110]). When this is true, we will say that Condition (HO) holds and will denote $dx/dy$ as $\rho = \rho(t,x/y)$, i.e.

$$\text{Condition (HO)}: \frac{dx}{dy}(t,x,y) = \rho(t,u), \quad \text{with} \quad u = x/y \ . \tag{6.17.4}$$

In this case, we may write

$$\frac{du}{dt} = \left\{-\frac{1}{y}\frac{dy}{dt}\right\} E(t,u) \ , \tag{6.17.5}$$

where

$$E(t,u) = u - \rho(t,u) \ . \tag{6.17.6}$$

We will call $E(t,u)$ the <u>excess function</u>, since it represents by how much the force ratio $u = x/y$ exceeds the force-change ratio $dx/dy$. Motivated by consideration of a number of specific LANCHESTER-type models, we assume that $E(t,u) = 0$ has a unique positive root, which we will denote as $u_+$, for each finite value of $t$ and that $E$ is positive for $u > u_+$ but negative for $u < u_+$. In order to assure that $u_+(t)$

"behaves properly" over time, we assume that $\partial E/\partial u$ is nonpositive for $u = u_+$. More precisely, we assume

$$\text{(A1)} \quad E(t,u) \begin{cases} < 0 & \text{for } 0 \leq u < u_+ , \\ > 0 & \text{for } u_+ < u , \end{cases}$$

and

$$\text{(A2)} \quad \frac{\partial E}{\partial u}(t,u_+) \leq 0 \qquad \text{for all } t \geq 0 ,$$

where $u_+$ denotes the unique positive root of $E(t,u_+) = 0$ for any fixed value of $t$.

Let us now consider combat modelled by LANCHESTER-type equations for which Condition (H0) holds. Then $X$ is "winning" a fixed-force-ratio-break-point battle when (6.17.3) holds. This is a "local" condition of force superiority. As discussed above, one can specify certain trends over time to in some sense strengthen (6.17.3) into a "global" condition of force superiority (cf. developments in Section 6.13 above). In particular, when $u_+(t)$ is nonincreasing over time, then (6.17.3) holding at only $t = 0$ guarantees that $du/dt(t)$ is always positive,[22] i.e. the force ratio $u = x/y$ continuously changes to the favor of $X$.

THEOREM 6.17.1 (TAYLOR [44]): Assume that Condition (H0) and Assumption (A1) hold and that $u_+(t)$ is a nonincreasing function of time. It follows that

$$u_0 > \left(\frac{dx}{dy}\right)_0 , \tag{6.17.7}$$

implies that $u(t) = x(t)/y(t)$ is a strictly increasing function of time $t$.

PROOF. From (6.17.5) we see that $du/dt$ and $E$ have the same sign when $dy/dt < 0$. We assume that this latter condition holds. Hence (6.17.7) and Assumption (A1) imply that $u_+(0) < u_0$. The assumption that $u_+(t)$ is nonincreasing then yields that $u_+(t) \leq u_+(0) < u_0$ for all $t \geq 0$. It follows that $u_+(t)$ is a strictly increasing function of time, since for $t$ near zero we have $u_+(t) \leq u_+(0) < u_0 \leq u(t)$ and consequently Assumption (A1) implies that $E(t,u(t)) > 0$ for all $t \geq 0$. Q.E.D.

We now establish a necessary and sufficient condition for $u_+(t)$ to be nonincreasing.

THEOREM 6.17.2 (TAYLOR [44]): Assume that Condition (H0) holds. Then $u_+(t)$ is a nonincreasing function of time if and only if $\partial\rho/\partial t(t,u_+) \leq 0$ for all $t \geq 0$, i.e., Assumption (A2) holds.

PROOF. Differentiating the identity $E(t,u_+) = 0 = u_+ - \rho(t,u_+)$, we obtain

$$\frac{du_+}{dt} = \frac{\frac{\partial\rho}{\partial t}(t,u_+)}{\frac{\partial E}{\partial u}(t,u_+)}, \tag{6.17.8}$$

whence follows the theorem by (A2). Q.E.D.

We will now briefly consider several concrete examples in order to illustrate the above general theory.

Example 6.17.1. For LANCHESTER's equations of modern warfare (6.5.1), we have $dx/dy = (1/u)\ a(t)/b(t) = (1/u)\ R(t) = \rho(t,u)$ so that Condition (HO) is satisfied. We also then have that $E(t,u) = u - (1/u)\ R(t)$ so that Assumption (A1) is satisfied with $u_+(t) = \sqrt{R(t)}$. Computing $\partial\rho/\rho t = (1/u)\ dR/dt$, we see from Theorem 6.17.2 that $u_+(t)$ is nonincreasing if and only if $R(t)$ is. We leave it as an exercise for the reader to show that (6.17.8) yields the same result as direct computation of $du_+/dt$. Theorem 6.17.1 then yields that the force ratio $u = x/y$ is a strictly increasing function of time when $u_0 > \sqrt{R_0}$ and $R(t)$ is nonincreasing.

Example 6.17.2. For the equations of HELMBOLD-type combat with supporting fires (6.14.1) with $W \in (0,1]$, we have $dx/dy = u^{1-W}\{a(t) + \beta(t)u^W\}/\{\alpha(t) + b(t)u^W\}$ $= \rho(t,u)$ so that Condition (HO) is satisfied. We also then have that $E(t,u) = \{u^{1-W}/(\alpha(t) + b(t)u^W)\}\ F(t,u)$ where $F(t,u) = b(t)u^{2W} + \{\alpha(t) - \beta(t)\}\ u^W - a(t)$ so that Assumption (A1) is satisfied with

$$u_+(t) = \sqrt{R(t)}\ \left\{\frac{S(t)}{2} + \sqrt{\left[\frac{S(t)}{2}\right]^2 + 1}\right\}^{1/W}, \qquad (6.17.9)$$

where the normalized net effectiveness of supporting fires $S(t)$ is given by (6.13.11). It may be shown (cf. Theorem 6.13.2 above) by direct computation using (6.17.9) that $R(t)$ and $S(t)$ nonincreasing implies that $u_+(r)$ is nonincreasing. Applying Theorem 6.17.1, we find that the force ratio $u = x/y$ is a strictly increasing function of time when $(x_0/y_0)^W > \sqrt{R_0}\ \left\{S_0/2 + \sqrt{1 + (S_0/2)^2}\ \right\}$ and $R(t)$ and $S(t)$ are nonincreasing.

A more thorough analysis of the force-ratio equation, however, is required to develop a battle-outcome-prediction condition analogous to (6.13.16) (cf. the proof of Theorem 6.13.3).

Example 6.17.3. Consider combat modelled with

$$\frac{dx}{dt} = -a(t)\ g(t,x,y) \qquad \text{and} \qquad \frac{dy}{dt} = -b(t)\ g(t,x,y)\ ,$$

where $a(t)$, $b(t)$, and $g(t,x,y) > 0$. It follows that $\rho(t,u) = R(t)$ so that our results yield the "instantaneous" linear law $b(t)x < a(t)y$ for $Y$ to be winning a fixed-force-ratio-breakpoint battle. When $g(t,x,y) = xy$ [i.e. combat is modelled with (6.16.1)], further analysis of (6.17.5) yields that

$$u(t) \leq u_0 \exp\{-\ (R_0 - u_0)y_f \int_0^t b(s)ds\}\ , \tag{6.17.10}$$

where $y_f$ denotes Y's (final) force level when $X$ is annihilated and we have assumed that $u_0 < R_0$ and $R(t)$ is nondecreasing. If we assume that $b(t) \notin L(0, +\infty)$, then (6.17.10) only guarantees that $X$ will lose any fixed-force-ratio-breakpoint battle with $u_{BP}^X > 0$ in finite time. It does not guarantee that $X$ will be annihilated in finite time (and, indeed, $X$ will not be). Furthermore, this annihilation-time bound (i.e. infinite time being required to annihilate the $X$ force) cannot be improved upon.

Every military man intuitively knows that the force ratio and the (instantaneous) casualty-exchange ratio influence the outcome of battle. In this section we have shown that these two ratios may be quantitatively related to develop battle-trend predictions, e.g. the force ratio will always change to the advantage of one of the combatants, without having to solve the LANCHESTER-type equations in detail. In particular, we showed that a general "local" condition of force superiority which applies to all deterministic LANCHESTER-type models with two force-level variables may be based on comparing the force ratio with the instantaneous casualty-exchange ratio. When appropriate temporal trends are satisfied, "global" conditions of force superiority may be developed from these "local" ones.

FOOTNOTES for Chapter 6

1. By the classic LANCHESTER theory of combat (i.e. its classic developments) we mean developments in the differential-equation modelling of combat before the publication of DOLANSKY's [17] 1964 survey article. Constant attrition-rate coefficients were assumed for reasons of simplicity and lack of methodology and data for their prediction [17].

2. S. BONDER (see BONDER and FARRELL [10, pp. 30-31]) has stressed the importance of analytical solutions to such models for developing insights into the dynamics of combat by portraying the relation between various factors in the combat attrition process and the surviving numbers of forces and for facilitating sensitivity and other parametric analysis (see BONDER [9]). Furthermore, finite-difference methods for developing numerical approximate solutions to such equations are discussed in Chapter 7 below.

3. Other significant work appears in BARFOOT [2], BONDER and FARRELL [10], and KIMBLETON [25].

4. Here we would like to mention the work of RUSTAGI and SRIVASTAVA [39] and RUSTAGI and LAITINEN [38] on the estimation of the Markov-dependent-fire parameters in BONDER's [6;8] expression for the LANCHESTER attrition-rate coefficeints (see also Footnote 1 for Chapter 5).

5. To be precise, we only conjecture that this statement is true. It is, of course, a very difficult task (and one well beyond the scope of this book) to prove that the solution to a differential equation cannot be expressed in terms of "elementary" functions (e.g. see RITT [37] or RISCH [36]). Based on our work in this field, however, we feel that the statement is probably true for combat modelled with many (if not most) time-dependent attrition-rate coefficients of tactical interest.

6. See Footnote 13 of Chapter 3.

7. See Footnote 14 of Chapter 3.

8. In other words, both $x(t)$ and $y(t) > 0$ for all finite $t \geq 0$.

9. It seems appropriate to delineate a set of physical circumstances that may be hypothesized to yield a battle with attrition-rate coefficients such that $h(t) \in L(0,+\infty)$. For example, consider a fire fight in which the combatants take cover and continue to reduce their vulnerability so that enemy fire effectiveness decays exponentially over time, i.e. $a(t) = k_a e^{-\gamma t}$ and $b(t) = k_b e^{-\gamma t}$ with $\gamma \geq 0$. In this case, $M = \lambda_1/\gamma$, and $M$ is finite when $h(t) \in L(0,+\infty)$.

10. This point was not noted by TAYLOR and PARRY [59].

11. See TAYLOR [50] for an example that shows that such a battle need not ever end when $b(t) \in L(0,+\infty)$, i.e. limited cumulative firepower is available to the $X$ force.

12. The naming of our LCS functions is based on the facts that a function similar to $F_\alpha(\xi)$ was introduced by LUDWIG SCHLÄFLI (1814-1895) in 1867 (see [40]) and that another related one appears in a posthumous fragment of the great English geometer WILLIAM KINGDON CLIFFORD (1845-1879) (see [14, pp. 343-348]). Although the GLF given by (6.9.3) may be expressed in terms of modified BESSEL functions of the first kind of fractional order (i.e. $I_\alpha$ for $0 < \alpha < 1$) [see (6.6.1) through (6.6.14) above], we have introduced the LCS functions because too few of such BESSEL functions $I_\alpha$ are tabulated (i.e. tabulations apparently only exist for $\alpha = \pm 1/4, \pm 1/3, \pm 1/2, \pm 2/3, \pm 3/4$, and these do not correspond to cases of interest). Observing that we may write

$$I_\alpha(\xi) \doteq \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k+\alpha}}{\{k!\ \Gamma(k+\alpha+1)\}},$$

the reader may find it instructive to show that the results given in Example 6.5.2 are equivalent to (6.6.11) and (6.6.12) and also to (6.9.3) above.

13. Equation (6.9.6) follows directly from substituting (6.9.3) into (6.5.6).

14. The tabulations provided in Appendix D are taken from the longer (i.e. [55]) of the two reports by TAYLOR and BROWN [55; 56] (also available from the National Technical Information Service) which contain five-decimal-place tables of the hyperbolic-like LCS functions tions $F_{\alpha}(\xi)$, $H_{1-\alpha}(\xi)$, and $T_{\alpha}(\xi)$ for values of the argument $\xi = 0.00(0.01)\ 2.00(0.0)\ 10.0$ and various values of the order $\alpha$. The short table [56] contains tabulations for $\alpha = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 3/7$, and $4/7$ corresponding to $\mu, \nu = 0,1,2,3$ for the attrition-rate coefficients (6.9.2); while the longer table [55] contains tabulations for $\alpha = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 2/7, 3/7, 4/7, 5/7, 4/9, 5/9, 3/11, 5/11, 6/11, 8/11, 5/13, 8/13, 5/17, 12/17, 5/21$, and $16/21$ corresponding to $\mu, \nu = 0, 1/4, 1/2, 1, 1\ 1/2, 2, 3$. As we have seen above in Section 6.2 [see (6.2.1), (6.2.5), (6.2.6), and Figure 6.2], such values for $\mu$ and $\nu$ allow one to analyze, for example, a wide variety of range capabilities for weapon systems in BONDER's constant-speed-attack model (6.2.1).

15. These force-annihilation-prediction results may be obtained by substituting the GLF (6.9.3) and the result (6.6.17) for $Q^*$ of Section 6.6 into Theorem 6.7.1.

16. More generally, we could have considered $D \geq 0$ but did not do so because (6.9.14) reduces to (6.9.2) when $D = 0$.

17. The naming of the LIOUVILLE-GREEN-LANCHESTER (LGL) approximation was arrived at in the following manner. The LIOUVILLE-GREEN (LG) approximation [34] (also called the WKB approximation [33, pp. 790–791; 34], the JWKB approximation [28; 34], or even the WKBJ approximation [30]) to the solution of a second-order linear differential equation is a very useful approximation that is frequently made in applied mathematics. Since we have applied the theory of the LG approximation to LANCHESTER-type equations of modern warfare, we have called the result the LGL approximation.

18. The LG approximation (see OLVER [34, Chapter 6]) is a widely used approximation to the solution of a second-order linear ordinary differential equation. See the previous footnote for further details.

19. Actually, additional hypotheses are required. For simplicity we have omitted them here (see TAYLOR [48] or Section 8.10 below).

20. An equivalent result is given by MORSE and KIMBALL [31, p. 72]. However, their result is in a considerably less convenient form for determining the qualitative behavior of the model (6.15.1). For example, the behavior shown in Table 6.XIII was not detected by MORSE and KIMBALL, and consequently incorrect battle-outcome-prediction conditions are implied in [31, p. 72].

21. This interpretation only holds for cases of no replacements and withdrawals or, more generally, when the rates of replacement and withdrawal are equal.

22. In TAYLOR [44] we erroneously stated that (under the stated assumptions) (6.17.7) was a condition sufficient to predict an X victory in a fixed-force-ratio-breakpoint battle. Subsequently, we discovered the counterexample mentioned in Footnote 10 above (i.e. see TAYLOR [50]) that shows that such a battle need never end when $b(t) \in L(0,+\infty)$, i.e. limited cumulative firepower is available to the X force. Consequently, for example, Theorems 6.6.2 and 6.13.3 each contain the assumption that $b(t) \notin L(0,+\infty)$, and Theorem 6.17.1.

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## APPENDIX D: TABLES OF LCS FUNCTIONS FOR ANALYZING HOMOGENEOUS-FORCE BATTLES

### 1. Introduction.

This appendix contains the most extensive set of tables of the LANCHESTER-CLIFFORD-SCHLÄFLI (LCS) functions (see Section 6.9) which are currently available for analyzing homogeneous-force "aimed-fire" combat modelled by power attrition-rate coefficients with "no offset"

$$a(t) = k_a(t+C)^{\mu} , \quad \text{and} \quad b(t) = k_b(t+C)^{\nu} , \tag{D.1}$$

or by certain other attrition-rate coefficients that yield force-level equations equivalent to (6.9.7). Some military situations modelled with these coefficients have been discussed above in Section 6.2, e.g. "aimed-fire" force-on-force combat between two opposing weapon-system types with the same maximum effective range. These tabulations of LCS functions allow one to analyze such combat modelled by the power attrition-rate coefficients (D.1) with somewhat the same facility as one can for the constant-coefficient case, and thus they can aid in parametric analyses (see Section 6.9 for further details).

Tabulations of the hyperbolic-like LCS functions $F_{\alpha}(\xi)$, $H_{1-\alpha}(\xi)$, and $T_{\alpha}(\xi)$ are given in this appendix for various values of the argument $\xi$ and for $\alpha$ = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 2/7, 3/7, 4/7, 5/7, 4/9, 5/9, 3/11, 5/11, 6/11, 8/11, 5/13, 8/13, 5/17, 12/17, 5/21, and 16/21. As we have seen in Section 6.9 above,

the LCS functions $F_\alpha(\xi)$ and $H_\alpha(\xi)$ may be represented for $\alpha \neq 0, -1, -2, \ldots$ as the infinite series

$$F_\alpha(\xi) = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k}}{\{k!\Gamma(k+\alpha)\}} , \tag{D.2}$$

and

$$H_\alpha(\xi) = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(\xi/2)^{2(k+\alpha)}}{\{k!\Gamma(k+\alpha+1)\}} , \tag{D.3}$$

while $T_\alpha(\xi)$ is defined by

$$T_\alpha(\xi) = \frac{H_{1-\alpha}(\xi)}{F_\alpha(\xi)} . \tag{D.4}$$

The LCS function $F_\alpha(\xi)$ corresponds to the hyperbolic cosine, $H_{1-\alpha}(\xi)$ to the hyperbolic sine, while $T_\alpha(\xi)$ corresponds to the hyperbolic tangent. A key result that is used to develop force-annihilation-prediction conditions is that (TAYLOR and BROWN [5]; see also Section 6.9 above)

$$\lim_{\xi \to +\infty} T_\alpha(\xi) = \frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} . \tag{D.5}$$

2. Use of LCS Functions for Analyzing Homogeneous-Force Combat.

The LANCHESTER-CLIFFORD-SCHLÄFLI (LCS) functions $F_\alpha(\xi)$ and $H_\alpha(\xi)$ are very useful for analyzing "aimed-fire" combat modelled by

the power attrition-rate coefficients with "no offset" (D.1). In other words, the LCS functions arise in solving the differential-equation force-on-force combat model (6.5.1) with attrition-rate coefficients (D.1). In order that both $a(t)$ and $b(t) \in L(t_0,T)$, we must have $\mu$ and $\nu > -1$, and we will assume that this latter condition is satisfied. For such combat, these LCS functions may be used to

(T1) compute the force levels as functions of time,

(T2) predict force annihilation,

and (T3) compute the time of force annihilation.

Although we have given results for accomplishing these tasks in Section 6.9, for the reader's convenience we will review the salient points and collect the main results here.

According to (6.5.6) and (6.9.3), the $X$ force level $x(t)$ may be written as

$$x(t) = x_0\{F_p(\tau_0)F_q(\tau) - H_q(\tau_0)H_p(\tau)\}$$

$$- y_0 \lambda_R\left(\frac{\lambda_I}{\mu+\nu+2}\right)\{F_q(\tau_0)H_p(\tau) - H_p(\tau_0)F_q(\tau)\} , \qquad (D.6)$$

where $p = (\mu+1)/(\mu+\nu+2)$, $q = 1-p$,

$$\tau(t) = \left(\frac{2\lambda_I}{\mu+\nu+2}\right)(t+C)^{(\mu+\nu+2)/2} , \qquad (D.7)$$

$\tau_0$ denotes $\tau(0)$, $\lambda_I = \sqrt{k_a k_b}$, and $\lambda_R = k_a/k_b$. Let us observe that from the condition that both $\mu$ and $\nu > -1$, it follows that both $p$

$q \in (0,1)$. From (D.5) and (D.6) (see TAYLOR [3] for details) we may conclude the following force-annihilation-prediction result. [Alternatively, we may substitute (6.9.3) and (6.9.8) into Theorem 6.6.1 to obtain Theorem D.1.]

THEOREM D.1 (TAYLOR and BROWN [5]): Consider combat between two homogeneous forces modelled by the $F|F$ LANCHESTER-type equations (6.5.1) with power attrition-rate coefficients (D.1). Assume that both $\mu$ and $\nu > -1$. Then the $X$ force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p}\frac{\Gamma(p)}{\Gamma(q)}\left\{\frac{F_q(\tau_0)-\left(\frac{\Gamma(q)}{\Gamma(p)}\right)H_p(\tau_0)}{F_p(\tau_0)-\left(\frac{\Gamma(p)}{\Gamma(q)}\right)H_q(\tau_0)}\right\} \quad \text{(D.8)}$$

When $\tau_0 = 0$ (i.e. $C = 0$), the $X$ force will be annihilated in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p}\frac{\Gamma(p)}{\Gamma(q)}\ . \quad \text{(D.8a)}$$

When (D.8) is satisfied, the time to annihilate the $X$ force, $t_a^X$, is determinedly $x(t_a^X) = 0$. It follows that

$$T_q[\tau(t_a^X)] = \frac{x_0F_p(\tau_0) + y_0\sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p} H_p(\tau_0)}{x_0H_q(\tau_0) + y_0\sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p} F_q(\tau_0)} \tag{D.9}$$

or, more explicitly,

$$t_a^X = \tau^{-1} \quad T_q^{-1}\left\{\left[\frac{x_0F_p(\tau_0) + y_0\sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p} H_p(\tau_0)}{x_0H_q(\tau_0) + y_0\sqrt{\lambda_R}\left(\frac{\lambda_I}{\mu+\nu+2}\right)^{q-p} F_q(\tau_0)}\right]\right\} \tag{D.10}$$

where $\tau^{-1}$ and $T_q^{-1}$ denote inverse functions. Numerical examples using the above analytical results have been given in Section 6.9 above, and these examples show the use of the LCS functions for analyzing homogeneous-force combat.

3. Tables of LCS Functions.

This appendix contains the most extensive set of tables of the LANCHESTER-CLIFFORD-SCHLÄFLI functions currently available. The Annex contains tables of five-decimal-place values of the hyperbolic-like LCS functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for various values of the argument $x$, namely $x = 0.00\ (0.01)\ 2.00\ (0.1)\ 10.0$, and $\alpha = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 2/7, 3/7, 4/7, 5/7, 4/9, 5/9, 3/11, 5/11, 6/11, 8/11, 5/13, 8/13, 5/17, 12/17, 5/21$, and $16/21$. These values of the index $\alpha$ correspond to $\mu, \nu = 0, 1/4, 1/2, 1, 1\frac{1}{2}, 2,$ and $3$ in (D.1)

and allow one to analyze, for example, a fairly wide variety of range capabilities for weapon systems in the constant-speed-attack model of Section 6.2. These tables have been calculated by the recursive methods given in TAYLOR and BROWN [4, Section 8].

A representative tabulation of the hyperbolic-like LCS functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ is given in, for example, Tables D.VIIIA and D.VIIIB of the Annex for $\alpha = 3/5$. The values of the argument $x$ are the same as those used for the tabulation of the hyperbolic functions by ABRAMOWITZ and STEGUN [1]. These particular tables for $\alpha = 3/5$ also appear in Section 6.9 and have been used to compute the numerical examples given there. The reader should note in Table D.VIIIB that from (D.5) the limiting value of $T_\alpha(x)$ as $x \to +\infty$ (here $\alpha = 3/5$) is quickly reached, with three-decimal-place agreement by $x = 4.5$. Also, the reader should recall from Section 6.9 (e.g. see Table 6.II) that $F_{1/2}(\xi) = \cosh \xi$, $H_{1/2}(\xi) = \sinh \xi$, and $T_{1/2}(\xi) = \tanh \xi$, and consequently Tables D.IA and D.IB for $\alpha = 1/2$ are simply tabulations of the hyperbolic functions.

## 4. Outline of Computational Procedure.

The above-mentioned tabluations of these LCS functions make the analysis of several important classes of LANCHESTER-type battles (see Section 6.2) a comparatively easy matter. A couple of numerical examples have been given in Section 6.9 to show how these LCS functions may be used to analyze homogeneous-force "aimed-fire" combat modelled by the power attrition-rate coefficients with "no offset" (D.1). For such analysis of homogeneous-force combat, the author suggests the following

computational procedure (based on the results given above in Section D.2):

(TASK 1) determine from (D.8) whether the $X$ force can be annihilated,

(TASK 2) if annihilation is possible, determine the time of the $X$ force's annihilation as follows:

(SUBTASK 2a) compute $T_q(\tau_a^X)$ by (D.9) [here $\tau_a^X = (t_a^X)$],

(SUBTASK 2b) using interpolation, determine $\tau_a^X$ from the appropriate tabulation of $T_q$,

(SUBTASK 2c) using (D.7), compute $t_a^X = \tau^{-1}(t_a^X)$.

From the above, it should be noted that these two determinations involve only the initial force ratio $u_0 = x_0/y_0$ (and not the individual initial force levels themselves). For the numerical examples given in Section 6.9, when the $X$ force is not annihilated with a given time $t_{max}$, the final $X$ force level has been calculated by (D.6) with the help of our tabulations.

5. Final Remarks.

In Section 6.9 above, we have shown how the LCS functions allow one to conveniently obtain much valuable information about the "aimed-fire" force-on-force attrition model (6.5.1) with power attrition-rate coefficients (D.1) without having to explicitly compute the entire force-level trajectories. Previously one was limited to only being able to compute force-level trajectories (see TAYLOR [2] and TAYLOR and BROWN [4]). With the availability of these tabulations of LCS functions (see the Annex to this appendix), one can now tell which side is going to be annihilated and when this event will occur without explicitly computing the trajectories. Not only did we answer questions about the qualitative behavior of the force-on-force combat model (e.g. force annihilation) for specific values of, for example, initial force levels but also for the entire possible range of values for the initial force ratio (i.e. parametric analysis of model behavior).

The results of this appendix may be used for other parametric analyses, e.g. parametric dependence of battle outcome on weapon-system capabilities. Thus, the contents of this appendix (see also Section 6.9 above) allow one to develop important insights into the dynamics of combat between two homogeneous forces with temporal variations in fire effectiveness. With the availability of these tabulations of the LCS functions, one can now analyze combat modelled by the power attrition-rate coefficients (D.1) with somewhat the same facility as he can for the constant-coefficient case of $F|F$ LANCHESTER-type equations and thus aid in parametric analyses of such homogeneous-force battles. For a further discussion of the significance of such results for military operations research, the reader is directed to TAYLOR and BROWN [5].

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## ANNEX to Appendix D:

Tabulations of the LCS Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha$ = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 2/7, 3/7, 4/7, 5/7, 4/9, 5/9, 3/11, 5/11, 6/11, 8/11, 5/13, 8/13, 5/17, 12/17, 5/21, and 16/21.

| $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ | $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ | $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.12763 | 0.52110 | 0.46212 | 1.00 | 1.54308 | 1.17520 | 0.76159 |
| 0.01 | 1.00005 | 0.01000 | 0.01000 | 0.51 | 1.13289 | 0.53240 | 0.46995 | 1.01 | 1.55491 | 1.19069 | 0.76576 |
| 0.02 | 1.00020 | 0.02000 | 0.02000 | 0.52 | 1.13827 | 0.54375 | 0.47770 | 1.02 | 1.56689 | 1.20630 | 0.76987 |
| 0.03 | 1.00045 | 0.03000 | 0.02999 | 0.53 | 1.14377 | 0.55516 | 0.48538 | 1.03 | 1.57904 | 1.22203 | 0.77391 |
| 0.04 | 1.00080 | 0.04001 | 0.03998 | 0.54 | 1.14938 | 0.56663 | 0.49299 | 1.04 | 1.59134 | 1.23788 | 0.77789 |
| 0.05 | 1.00125 | 0.05002 | 0.04996 | 0.55 | 1.15510 | 0.57815 | 0.50052 | 1.05 | 1.60379 | 1.25386 | 0.78181 |
| 0.06 | 1.00180 | 0.06004 | 0.05993 | 0.56 | 1.16094 | 0.58973 | 0.50798 | 1.06 | 1.61641 | 1.26996 | 0.78566 |
| 0.07 | 1.00245 | 0.07006 | 0.06989 | 0.57 | 1.16690 | 0.60137 | 0.51536 | 1.07 | 1.62919 | 1.28619 | 0.78946 |
| 0.08 | 1.00320 | 0.08009 | 0.07983 | 0.58 | 1.17297 | 0.61307 | 0.52267 | 1.08 | 1.64214 | 1.30254 | 0.79320 |
| 0.09 | 1.00405 | 0.09012 | 0.08976 | 0.59 | 1.17916 | 0.62483 | 0.52990 | 1.09 | 1.65525 | 1.31903 | 0.79688 |
| 0.10 | 1.00500 | 0.10017 | 0.09967 | 0.60 | 1.18547 | 0.63665 | 0.53705 | 1.10 | 1.66852 | 1.33565 | 0.80050 |
| 0.11 | 1.00606 | 0.11022 | 0.10956 | 0.61 | 1.19189 | 0.64854 | 0.54413 | 1.11 | 1.68196 | 1.35240 | 0.80406 |
| 0.12 | 1.00721 | 0.12029 | 0.11943 | 0.62 | 1.19844 | 0.66049 | 0.55113 | 1.12 | 1.69557 | 1.36929 | 0.80757 |
| 0.13 | 1.00846 | 0.13037 | 0.12927 | 0.63 | 1.20510 | 0.67251 | 0.55805 | 1.13 | 1.70934 | 1.38631 | 0.81102 |
| 0.14 | 1.00982 | 0.14046 | 0.13909 | 0.34 | 1.21189 | 0.68459 | 0.56490 | 1.14 | 1.72329 | 1.40347 | 0.81441 |
| 0.15 | 1.01127 | 0.15056 | 0.14889 | 0.65 | 1.21879 | 0.69675 | 0.57167 | 1.15 | 1.73741 | 1.42078 | 0.81775 |
| 0.16 | 1.01283 | 0.16068 | 0.15865 | 0.66 | 1.22582 | 0.70897 | 0.57836 | 1.16 | 1.75171 | 1.43822 | 0.82104 |
| 0.17 | 1.01448 | 0.17082 | 0.16838 | 0.67 | 1.23297 | 0.72126 | 0.58498 | 1.17 | 1.76618 | 1.45581 | 0.82427 |
| 0.18 | 1.01624 | 0.18097 | 0.17808 | 0.68 | 1.24025 | 0.73363 | 0.59152 | 1.18 | 1.78083 | 1.47355 | 0.82745 |
| 0.19 | 1.01810 | 0.19115 | 0.18775 | 0.69 | 1.24765 | 0.74607 | 0.59798 | 1.19 | 1.79565 | 1.49143 | 0.83058 |
| 0.20 | 1.02007 | 0.20134 | 0.19738 | 0.70 | 1.25517 | 0.75858 | 0.60437 | 1.20 | 1.81066 | 1.50946 | 0.83365 |
| 0.21 | 1.02213 | 0.21155 | 0.20697 | 0.71 | 1.26282 | 0.77117 | 0.61068 | 1.21 | 1.82584 | 1.52764 | 0.83668 |
| 0.22 | 1.02430 | 0.22178 | 0.21652 | 0.72 | 1.27059 | 0.78384 | 0.61691 | 1.22 | 1.84121 | 1.54598 | 0.83965 |
| 0.23 | 1.02657 | 0.23203 | 0.22603 | 0.73 | 1.27849 | 0.79659 | 0.62307 | 1.23 | 1.85676 | 1.56447 | 0.84258 |
| 0.24 | 1.02894 | 0.24231 | 0.23550 | 0.74 | 1.28652 | 0.80941 | 0.62915 | 1.24 | 1.87250 | 1.58311 | 0.84546 |
| 0.25 | 1.03141 | 0.25261 | 0.24492 | 0.75 | 1.29468 | 0.82232 | 0.63515 | 1.25 | 1.88842 | 1.60192 | 0.84828 |
| 0.26 | 1.03399 | 0.26294 | 0.25430 | 0.76 | 1.30297 | 0.83530 | 0.64108 | 1.26 | 1.90454 | 1.62088 | 0.85106 |
| 0.27 | 1.03667 | 0.27329 | 0.26362 | 0.77 | 1.31139 | 0.84838 | 0.64693 | 1.27 | 1.92084 | 1.64001 | 0.85380 |
| 0.28 | 1.03946 | 0.28367 | 0.27291 | 0.78 | 1.31994 | 0.86153 | 0.65271 | 1.28 | 1.93734 | 1.65930 | 0.85648 |
| 0.29 | 1.04235 | 0.29408 | 0.28213 | 0.79 | 1.32862 | 0.87478 | 0.65841 | 1.29 | 1.95403 | 1.67876 | 0.85913 |
| 0.30 | 1.04534 | 0.30452 | 0.29131 | 0.80 | 1.33743 | 0.88811 | 0.66404 | 1.30 | 1.97091 | 1.69838 | 0.86172 |
| 0.31 | 1.04844 | 0.31499 | 0.30044 | 0.81 | 1.34638 | 0.90152 | 0.66959 | 1.31 | 1.98800 | 1.71818 | 0.86428 |
| 0.32 | 1.05164 | 0.32549 | 0.30951 | 0.82 | 1.35547 | 0.91503 | 0.67507 | 1.32 | 2.00528 | 1.73814 | 0.86678 |
| 0.33 | 1.05495 | 0.33602 | 0.31852 | 0.83 | 1.36468 | 0.92863 | 0.68048 | 1.33 | 2.02276 | 1.75828 | 0.86925 |
| 0.34 | 1.05836 | 0.34659 | 0.32748 | 0.84 | 1.37404 | 0.94233 | 0.68581 | 1.34 | 2.04044 | 1.77860 | 0.87167 |
| 0.35 | 1.06188 | 0.35719 | 0.33638 | 0.85 | 1.38353 | 0.95612 | 0.69107 | 1.35 | 2.05833 | 1.79909 | 0.87405 |
| 0.36 | 1.06550 | 0.36783 | 0.34521 | 0.86 | 1.39316 | 0.97000 | 0.69626 | 1.36 | 2.07643 | 1.81977 | 0.87639 |
| 0.37 | 1.06923 | 0.37850 | 0.35399 | 0.87 | 1.40293 | 0.98398 | 0.70137 | 1.37 | 2.09473 | 1.84062 | 0.87869 |
| 0.38 | 1.07307 | 0.38921 | 0.36271 | 0.88 | 1.41284 | 0.99806 | 0.70642 | 1.36 | 2.11324 | 1.86166 | 0.88095 |
| 0.39 | 1.07702 | 0.39996 | 0.37136 | 0.89 | 1.42289 | 1.01224 | 0.71139 | 1.39 | 2.13196 | 1.88289 | 0.88317 |
| 0.40 | 1.08107 | 0.41075 | 0.37995 | 0.90 | 1.43309 | 1.02652 | 0.71630 | 1.40 | 2.15090 | 1.90430 | 0.88535 |
| 0.41 | 1.08523 | 0.42158 | 0.38847 | 0.91 | 1.44342 | 1.04090 | 0.72113 | 1.41 | 2.17005 | 1.92591 | 0.88749 |
| 0.42 | 1.08950 | 0.43246 | 0.39693 | 0.92 | 1.45390 | 1.05539 | 0.72590 | 1.42 | 2.18942 | 1.94770 | 0.88960 |
| 0.43 | 1.09388 | 0.44337 | 0.40532 | 0.93 | 1.46453 | 1.06998 | 0.73059 | 1.43 | 2.20900 | 1.96970 | 0.89167 |
| 0.44 | 1.09837 | 0.45434 | 0.41364 | 0.54 | 1.47530 | 1.08468 | 0.73522 | 1.44 | 2.22881 | 1.99188 | 0.89370 |
| 0.45 | 1.10297 | 0.46534 | 0.42190 | 0.95 | 1.48623 | 1.09948 | 0.73978 | 1.45 | 2.24884 | 2.01427 | 0.89569 |
| 0.46 | 1.10768 | 0.47640 | 0.43008 | 0.96 | 1.49729 | 1.11440 | 0.74428 | 1.46 | 2.26910 | 2.03686 | 0.89765 |
| 0.47 | 1.11250 | 0.48750 | 0.43820 | 0.97 | 1.50851 | 1.12943 | 0.74870 | 1.47 | 2.28958 | 2.05965 | 0.89958 |
| 0.48 | 1.11743 | 0.49865 | 0.44624 | 0.98 | 1.51988 | 1.14457 | 0.75307 | 1.48 | 2.31029 | 2.08265 | 0.90147 |
| 0.49 | 1.12247 | 0.50984 | 0.45422 | 0.99 | 1.53141 | 1.15983 | 0.75736 | 1.49 | 2.33123 | 2.10586 | 0.90332 |
| 0.50 | 1.12763 | 0.52110 | 0.46212 | 1.00 | 1.54308 | 1.17520 | 0.76159 | 1.50 | 2.35241 | 2.12928 | 0.90515 |

TABLE D.IA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/2$ and x from 0.00 to 1.50.

$\alpha = 1/2$

| $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ |
|---|---|---|---|
| 1.50 | 2.35241 | 2.12928 | 0.90515 |
| 1.51 | 2.37382 | 2.15291 | 0.90694 |
| 1.52 | 2.39547 | 2.17676 | 0.90870 |
| 1.53 | 2.41736 | 2.20082 | 0.91042 |
| 1.54 | 2.43949 | 2.22510 | 0.91212 |
| 1.55 | 2.46186 | 2.24961 | 0.91379 |
| 1.56 | 2.48448 | 2.27434 | 0.91542 |
| 1.57 | 2.50735 | 2.29930 | 0.91703 |
| 1.58 | 2.53047 | 2.32449 | 0.91860 |
| 1.59 | 2.55384 | 2.34991 | 0.92015 |
| 1.60 | 2.57746 | 2.37557 | 0.92167 |
| 1.61 | 2.60135 | 2.40146 | 0.92316 |
| 1.62 | 2.62549 | 2.42760 | 0.92462 |
| 1.63 | 2.64990 | 2.45397 | 0.92604 |
| 1.64 | 2.67457 | 2.48059 | 0.92747 |
| 1.65 | 2.69951 | 2.50746 | 0.92886 |
| 1.66 | 2.72472 | 2.53459 | 0.93022 |
| 1.67 | 2.75021 | 2.56196 | 0.93155 |
| 1.68 | 2.77596 | 2.58959 | 0.93286 |
| 1.69 | 2.80200 | 2.61748 | 0.93415 |
| 1.70 | 2.82832 | 2.64563 | 0.93541 |
| 1.71 | 2.85491 | 2.67405 | 0.93665 |
| 1.72 | 2.88180 | 2.70273 | 0.93786 |
| 1.73 | 2.90897 | 2.73168 | 0.93906 |
| 1.74 | 2.93643 | 2.76091 | 0.94023 |
| 1.75 | 2.96419 | 2.79041 | 0.94138 |
| 1.76 | 2.99224 | 2.82020 | 0.94250 |
| 1.77 | 3.02059 | 2.85026 | 0.94361 |
| 1.78 | 3.04925 | 2.88061 | 0.94470 |
| 1.79 | 3.07821 | 2.91125 | 0.94576 |
| 1.80 | 3.10747 | 2.94217 | 0.94681 |
| 1.81 | 3.13705 | 2.97340 | 0.94783 |
| 1.82 | 3.16694 | 3.00492 | 0.94884 |
| 1.83 | 3.19715 | 3.03674 | 0.94983 |
| 1.84 | 3.22768 | 3.06886 | 0.95080 |
| 1.85 | 3.25853 | 3.10129 | 0.95175 |
| 1.86 | 3.28970 | 3.13403 | 0.95268 |
| 1.87 | 3.32121 | 3.16709 | 0.95359 |
| 1.88 | 3.35305 | 3.20046 | 0.95449 |
| 1.89 | 3.38522 | 3.23415 | 0.95537 |
| 1.90 | 3.41773 | 3.26816 | 0.95624 |
| 1.91 | 3.45058 | 3.30250 | 0.95709 |
| 1.92 | 3.48378 | 3.33718 | 0.95792 |
| 1.93 | 3.51733 | 3.37218 | 0.95873 |
| 1.94 | 3.55123 | 3.40752 | 0.95953 |
| 1.95 | 3.58548 | 3.44321 | 0.96032 |
| 1.96 | 3.62009 | 3.47923 | 0.96109 |
| 1.97 | 3.65507 | 3.51561 | 0.96185 |
| 1.98 | 3.69041 | 3.55234 | 0.96259 |
| 1.99 | 3.72611 | 3.58942 | 0.96331 |
| 2.00 | 3.76220 | 3.62686 | 0.96403 |

| $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ |
|---|---|---|---|
| 2.0 | 3.76220 | 3.62686 | 0.96403 |
| 2.1 | 4.14431 | 4.02186 | 0.97045 |
| 2.2 | 4.56791 | 4.45711 | 0.97574 |
| 2.3 | 5.03722 | 4.93696 | 0.98010 |
| 2.4 | 5.55695 | 5.46623 | 0.98367 |
| 2.5 | 6.13229 | 6.05020 | 0.98661 |
| 2.6 | 6.76901 | 6.69473 | 0.98903 |
| 2.7 | 7.47347 | 7.40626 | 0.99101 |
| 2.8 | 8.25273 | 8.19192 | 0.99263 |
| 2.9 | 9.11458 | 9.05956 | 0.99396 |
| 3.0 | 10.06766 | 10.01787 | 0.99505 |
| 3.1 | 11.12150 | 11.07645 | 0.99595 |
| 3.2 | 12.27665 | 12.24588 | 0.99668 |
| 3.3 | 13.57476 | 13.53788 | 0.99728 |
| 3.4 | 14.99874 | 14.96536 | 0.99777 |
| 3.5 | 16.57282 | 16.54263 | 0.99818 |
| 3.6 | 18.31278 | 18.28546 | 0.99851 |
| 3.7 | 20.23601 | 20.21129 | 0.99878 |
| 3.8 | 22.36178 | 22.33941 | 0.99900 |
| 3.9 | 24.71135 | 24.69110 | 0.99918 |
| 4.0 | 27.30823 | 27.28992 | 0.99933 |
| 4.1 | 30.17843 | 30.16186 | 0.99945 |
| 4.2 | 33.35066 | 33.33567 | 0.99955 |
| 4.3 | 36.85668 | 36.84311 | 0.99963 |
| 4.4 | 40.73157 | 40.71930 | 0.99970 |
| 4.5 | 45.01412 | 45.00301 | 0.99975 |
| 4.6 | 49.74718 | 49.73713 | 0.99980 |
| 4.7 | 54.97813 | 54.96904 | 0.99983 |
| 4.8 | 60.75932 | 60.75109 | 0.99986 |
| 4.9 | 67.14861 | 67.14117 | 0.99989 |
| 5.0 | 74.20935 | 74.20321 | 0.99991 |
| 5.1 | 82.01400 | 82.00791 | 0.99993 |
| 5.2 | 90.63888 | 90.63336 | 0.99994 |
| 5.3 | 100.17090 | 100.16591 | 0.99995 |
| 5.4 | 110.70547 | 110.70095 | 0.99996 |
| 5.5 | 122.34801 | 122.34392 | 0.99997 |
| 5.6 | 135.21505 | 135.21135 | 0.99997 |
| 5.7 | 149.43537 | 149.43203 | 0.99998 |
| 5.8 | 165.15129 | 165.14827 | 0.99998 |
| 5.9 | 182.52010 | 182.51736 | 0.99998 |
| 6.0 | 201.71564 | 201.71316 | 0.99999 |

| $x$ | $F_{1/2}(x)$ | $H_{1/2}(x)$ | $T_{1/2}(x)$ |
|---|---|---|---|
| 6.0 | 201.71564 | 201.71316 | 0.99999 |
| 6.1 | 222.93001 | 222.92776 | 0.99999 |
| 6.2 | 246.37554 | 246.37351 | 0.99999 |
| 6.3 | 272.28687 | 272.28504 | 0.99999 |
| 6.4 | 300.92335 | 300.92169 | 0.99999 |
| 6.5 | 332.57157 | 332.57006 | 1.00000 |
| 6.6 | 367.54827 | 367.54691 | 1.00000 |
| 6.7 | 406.20353 | 406.20230 | 1.00000 |
| 6.8 | 448.92420 | 448.92309 | 1.00000 |
| 6.9 | 496.13786 | 496.13685 | 1.03000 |
| 7.0 | 548.31704 | 548.31612 | 1.00000 |
| 7.1 | 605.98395 | 605.98312 | 1.00000 |
| 7.2 | 669.71576 | 669.71501 | 1.00000 |
| 7.3 | 740.15030 | 740.14963 | 1.00000 |
| 7.4 | 617.99252 | 817.99191 | 1.00000 |
| 7.5 | 904.02148 | 904.02093 | 1.00000 |
| 7.6 | 999.09820 | 999.09770 | 1.00000 |
| 7.7 | 1104.17422 | 1104.17377 | 1.00000 |
| 7.8 | 1220.30119 | 1220.30078 | 1.00000 |
| 7.9 | 1348.64135 | 1348.64098 | 1.00000 |
| 8.0 | 1490.47916 | 1490.47883 | 1.00000 |
| 8.1 | 1647.23419 | 1647.23389 | 1.00000 |
| 8.2 | 1820.47529 | 1820.47502 | 1.00000 |
| 8.3 | 2011.93632 | 2011.93607 | 1.00000 |
| 8.4 | 2223.53349 | 2223.53326 | 1.00000 |
| 8.5 | 2457.38452 | 2457.38432 | 1.00000 |
| 8.6 | 2715.82989 | 2715.82970 | 1.00000 |
| 8.7 | 3001.45619 | 3001.45603 | 1.00000 |
| 8.8 | 3317.12208 | 3317.12193 | 1.00000 |
| 8.9 | 3665.98684 | 3665.98670 | 1.00000 |
| 9.0 | 4051.54203 | 4051.54190 | 1.00000 |
| 9.1 | 4477.64641 | 4477.64630 | 1.00000 |
| 9.2 | 4948.56458 | 4948.56448 | 1.00000 |
| 9.3 | 5469.00965 | 5469.00956 | 1.00000 |
| 9.4 | 6044.19041 | 6044.19032 | 1.00000 |
| 9.5 | 6679.85345 | 6679.86338 | 1.00000 |
| 9.6 | 7382.39082 | 7382.39075 | 1.00000 |
| 9.7 | 8158.80363 | 8158.80357 | 1.00000 |
| 9.8 | 9016.87249 | 9016.87244 | 1.00000 |
| 9.9 | 9965.18524 | 9965.18519 | 1.00000 |
| 10.0 | 11013.23292 | 11013.23287 | 1.00000 |

TABLE D.IB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/2$ and $x$ from 1.50 to 10.0.

$\alpha = 1/3$

| $x$ | $F_{1/3}(x)$ | $H_{2/3}(x)$ | $T_{1/3}(x)$ | $x$ | $F_{1/3}(x)$ | $H_{2/3}(x)$ | $T_{1/3}(x)$ | $x$ | $F_{1/3}(x)$ | $H_{2/3}(x)$ | $T_{1/3}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.19193 | 0.24520 | 0.20571 | 1.00 | 1.82287 | 0.68885 | 0.37789 |
| 0.01 | 1.00008 | 0.00128 | 0.00128 | 0.51 | 1.19988 | 0.25214 | 0.21013 | 1.01 | 1.84096 | 0.70005 | 0.38026 |
| 0.02 | 1.00030 | 0.00323 | 0.00323 | 0.52 | 1.20799 | 0.25914 | 0.21452 | 1.02 | 1.85929 | 0.71135 | 0.38259 |
| 0.03 | 1.00068 | 0.00555 | 0.00555 | 0.53 | 1.21628 | 0.26622 | 0.21888 | 1.03 | 1.87787 | 0.72276 | 0.38488 |
| 0.04 | 1.00120 | 0.00815 | 0.00814 | 0.54 | 1.22474 | 0.27337 | 0.22321 | 1.04 | 1.89670 | 0.73428 | 0.38714 |
| 0.05 | 1.00188 | 0.01097 | 0.01095 | 0.55 | 1.23338 | 0.28059 | 0.22750 | 1.05 | 1.91578 | 0.74592 | 0.38935 |
| 0.06 | 1.00270 | 0.01399 | 0.01395 | 0.56 | 1.24219 | 0.28789 | 0.23176 | 1.06 | 1.93511 | 0.75766 | 0.39153 |
| 0.07 | 1.00368 | 0.01719 | 0.01712 | 0.57 | 1.25118 | 0.29525 | 0.23598 | 1.07 | 1.95469 | 0.76952 | 0.39368 |
| 0.08 | 1.00480 | 0.02054 | 0.02044 | 0.58 | 1.26035 | 0.30269 | 0.24016 | 1.08 | 1.97453 | 0.78149 | 0.39578 |
| 0.09 | 1.00608 | 0.02404 | 0.02389 | 0.59 | 1.26970 | 0.31020 | 0.24431 | 1.09 | 1.99463 | 0.79357 | 0.39785 |
| 0.10 | 1.00751 | 0.02767 | 0.02747 | 0.60 | 1.27923 | 0.31779 | 0.24842 | 1.10 | 2.01500 | 0.80578 | 0.39989 |
| 0.11 | 1.00909 | 0.03143 | 0.03115 | 0.61 | 1.28894 | 0.32545 | 0.25249 | 1.11 | 2.03562 | 0.81810 | 0.40189 |
| 0.12 | 1.01081 | 0.03531 | 0.03493 | 0.62 | 1.29883 | 0.33318 | 0.25652 | 1.12 | 2.05651 | 0.83054 | 0.40386 |
| 0.13 | 1.01270 | 0.03930 | 0.03881 | 0.63 | 1.30891 | 0.34099 | 0.26052 | 1.13 | 2.07767 | 0.84310 | 0.40579 |
| 0.14 | 1.01473 | 0.04340 | 0.04277 | 0.64 | 1.31917 | 0.34888 | 0.26447 | 1.14 | 2.09910 | 0.85578 | 0.40769 |
| 0.15 | 1.01691 | 0.04760 | 0.04681 | 0.65 | 1.32962 | 0.35684 | 0.26838 | 1.15 | 2.12081 | 0.86858 | 0.40955 |
| 0.16 | 1.01925 | 0.05191 | 0.05093 | 0.66 | 1.34025 | 0.36488 | 0.27225 | 1.16 | 2.14279 | 0.88151 | 0.41138 |
| 0.17 | 1.02173 | 0.05630 | 0.05511 | 0.67 | 1.35107 | 0.37299 | 0.27607 | 1.17 | 2.16504 | 0.89456 | 0.41318 |
| 0.18 | 1.02437 | 0.06079 | 0.05935 | 0.68 | 1.36208 | 0.38119 | 0.27985 | 1.18 | 2.18758 | 0.90774 | 0.41495 |
| 0.19 | 1.02717 | 0.06537 | 0.06365 | 0.69 | 1.37329 | 0.38946 | 0.28360 | 1.19 | 2.21040 | 0.92105 | 0.41669 |
| 0.20 | 1.03011 | 0.07004 | 0.06799 | 0.70 | 1.38468 | 0.39781 | 0.28729 | 1.20 | 2.23350 | 0.93449 | 0.41840 |
| 0.21 | 1.03321 | 0.07480 | 0.07239 | 0.71 | 1.39627 | 0.40624 | 0.29095 | 1.21 | 2.25690 | 0.94806 | 0.42007 |
| 0.22 | 1.03646 | 0.07963 | 0.07683 | 0.72 | 1.40805 | 0.41475 | 0.29456 | 1.22 | 2.28058 | 0.96176 | 0.42172 |
| 0.23 | 1.03987 | 0.08455 | 0.08131 | 0.73 | 1.42003 | 0.42334 | 0.29812 | 1.23 | 2.30456 | 0.97560 | 0.42333 |
| 0.24 | 1.04343 | 0.08955 | 0.08583 | 0.74 | 1.43220 | 0.43202 | 0.30164 | 1.24 | 2.32883 | 0.98957 | 0.42492 |
| 0.25 | 1.04715 | 0.09463 | 0.09037 | 0.75 | 1.44457 | 0.44077 | 0.30512 | 1.25 | 2.35340 | 1.00367 | 0.42648 |
| 0.26 | 1.05102 | 0.09979 | 0.09494 | 0.76 | 1.45715 | 0.44961 | 0.30856 | 1.26 | 2.37827 | 1.01792 | 0.42801 |
| 0.27 | 1.05505 | 0.10502 | 0.09954 | 0.77 | 1.46992 | 0.45853 | 0.31195 | 1.27 | 2.40345 | 1.03230 | 0.42951 |
| 0.28 | 1.05923 | 0.11033 | 0.10416 | 0.78 | 1.48290 | 0.46754 | 0.31529 | 1.28 | 2.42893 | 1.04683 | 0.43098 |
| 0.29 | 1.06357 | 0.11571 | 0.10880 | 0.79 | 1.49608 | 0.47664 | 0.31859 | 1.29 | 2.45473 | 1.06150 | 0.43243 |
| 0.30 | 1.06807 | 0.12117 | 0.11345 | 0.80 | 1.50947 | 0.48582 | 0.32185 | 1.30 | 2.48083 | 1.07631 | 0.43385 |
| 0.31 | 1.07273 | 0.12670 | 0.11811 | 0.81 | 1.52306 | 0.49508 | 0.32506 | 1.31 | 2.50725 | 1.09127 | 0.43525 |
| 0.32 | 1.07754 | 0.13230 | 0.12278 | 0.82 | 1.53686 | 0.50444 | 0.32822 | 1.32 | 2.53399 | 1.10638 | 0.43661 |
| 0.33 | 1.08251 | 0.13798 | 0.12746 | 0.83 | 1.55088 | 0.51388 | 0.33135 | 1.33 | 2.56105 | 1.12163 | 0.43796 |
| 0.34 | 1.08764 | 0.14372 | 0.13214 | 0.84 | 1.56510 | 0.52341 | 0.33443 | 1.34 | 2.58844 | 1.13704 | 0.43928 |
| 0.35 | 1.09293 | 0.14954 | 0.13683 | 0.85 | 1.57954 | 0.53304 | 0.33746 | 1.35 | 2.61615 | 1.15260 | 0.44057 |
| 0.36 | 1.09839 | 0.15543 | 0.14151 | 0.86 | 1.59419 | 0.54275 | 0.34045 | 1.36 | 2.64419 | 1.16831 | 0.44184 |
| 0.37 | 1.10400 | 0.16139 | 0.14618 | 0.87 | 1.60906 | 0.55256 | 0.34340 | 1.37 | 2.67257 | 1.18418 | 0.44309 |
| 0.38 | 1.10977 | 0.16742 | 0.15086 | 0.88 | 1.62415 | 0.56246 | 0.34631 | 1.38 | 2.70128 | 1.20020 | 0.44431 |
| 0.39 | 1.11571 | 0.17352 | 0.15552 | 0.89 | 1.63946 | 0.57245 | 0.34917 | 1.39 | 2.73034 | 1.21639 | 0.44551 |
| 0.40 | 1.12181 | 0.17968 | 0.16017 | 0.90 | 1.65499 | 0.58254 | 0.35199 | 1.40 | 2.75973 | 1.23273 | 0.44668 |
| 0.41 | 1.12807 | 0.18592 | 0.16481 | 0.91 | 1.67074 | 0.59272 | 0.35477 | 1.41 | 2.78947 | 1.24924 | 0.44784 |
| 0.42 | 1.13450 | 0.19223 | 0.16944 | 0.92 | 1.68672 | 0.60300 | 0.35750 | 1.42 | 2.81957 | 1.26591 | 0.44897 |
| 0.43 | 1.14109 | 0.19861 | 0.17405 | 0.93 | 1.70292 | 0.61338 | 0.36019 | 1.43 | 2.85001 | 1.28274 | 0.45008 |
| 0.44 | 1.14785 | 0.20505 | 0.17864 | 0.94 | 1.71936 | 0.62386 | 0.36284 | 1.44 | 2.88081 | 1.29975 | 0.45117 |
| 0.45 | 1.15478 | 0.21157 | 0.18321 | 0.95 | 1.73602 | 0.63443 | 0.36545 | 1.45 | 2.91197 | 1.31692 | 0.45224 |
| 0.46 | 1.16187 | 0.21815 | 0.18776 | 0.96 | 1.75292 | 0.64511 | 0.36802 | 1.46 | 2.94350 | 1.33427 | 0.45329 |
| 0.47 | 1.16913 | 0.22481 | 0.19229 | 0.97 | 1.77005 | 0.65589 | 0.37055 | 1.47 | 2.97539 | 1.35179 | 0.45432 |
| 0.48 | 1.17656 | 0.23154 | 0.19679 | 0.98 | 1.78742 | 0.66677 | 0.37304 | 1.48 | 3.00765 | 1.36948 | 0.45533 |
| 0.49 | 1.18416 | 0.23833 | 0.20127 | 0.99 | 1.80503 | 0.67776 | 0.37548 | 1.49 | 3.04029 | 1.38735 | 0.45632 |
| 0.50 | 1.19193 | 0.24520 | 0.20571 | 1.00 | 1.82287 | 0.68885 | 0.37789 | 1.50 | 3.07330 | 1.40540 | 0.45729 |

TABLE D.IIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 1/3$ and $x$ from 0.00 to 1.50.

$\alpha = 1/3$

| $x$ | $F_{1/3}(x)$ | $H_{2/3}(x)$ | $T_{1/3}(x)$ |
|---|---|---|---|
| 1.50 | 3.07330 | 1.40540 | 0.45729 |
| 1.51 | 3.10669 | 1.42363 | 0.45825 |
| 1.52 | 3.14047 | 1.44204 | 0.45918 |
| 1.53 | 3.17464 | 1.46064 | 0.46010 |
| 1.54 | 3.20920 | 1.47943 | 0.46100 |
| 1.55 | 3.24416 | 1.49840 | 0.46188 |
| 1.56 | 3.27951 | 1.51757 | 0.46274 |
| 1.57 | 3.31527 | 1.53692 | 0.46359 |
| 1.58 | 3.35144 | 1.55648 | 0.46442 |
| 1.59 | 3.38802 | 1.57622 | 0.46524 |
| 1.60 | 3.42501 | 1.59617 | 0.46603 |
| 1.61 | 3.46243 | 1.61632 | 0.46682 |
| 1.62 | 3.50027 | 1.63668 | 0.46759 |
| 1.63 | 3.53854 | 1.65724 | 0.46834 |
| 1.64 | 3.57724 | 1.67800 | 0.46908 |
| 1.65 | 3.61637 | 1.69898 | 0.46980 |
| 1.66 | 3.65595 | 1.72017 | 0.47051 |
| 1.67 | 3.69598 | 1.74158 | 0.47121 |
| 1.68 | 3.73645 | 1.76320 | 0.47189 |
| 1.69 | 3.77738 | 1.78504 | 0.47256 |
| 1.70 | 3.81877 | 1.80710 | 0.47322 |
| 1.71 | 3.86062 | 1.82939 | 0.47386 |
| 1.72 | 3.90294 | 1.85191 | 0.47449 |
| 1.73 | 3.94573 | 1.87465 | 0.47511 |
| 1.74 | 3.98900 | 1.89762 | 0.47571 |
| 1.75 | 4.03275 | 1.92083 | 0.47631 |
| 1.76 | 4.07699 | 1.94428 | 0.47689 |
| 1.77 | 4.12172 | 1.96796 | 0.47746 |
| 1.78 | 4.16695 | 1.99189 | 0.47802 |
| 1.79 | 4.21267 | 2.01606 | 0.47857 |
| 1.80 | 4.25891 | 2.04048 | 0.47911 |
| 1.81 | 4.30566 | 2.06514 | 0.47963 |
| 1.82 | 4.35292 | 2.09006 | 0.48015 |
| 1.83 | 4.40071 | 2.11523 | 0.48066 |
| 1.84 | 4.44903 | 2.14067 | 0.48115 |
| 1.85 | 4.49787 | 2.16636 | 0.48164 |
| 1.86 | 4.54726 | 2.19231 | 0.48212 |
| 1.87 | 4.59719 | 2.21853 | 0.48258 |
| 1.88 | 4.64767 | 2.24502 | 0.48304 |
| 1.89 | 4.69870 | 2.27178 | 0.48349 |
| 1.90 | 4.75030 | 2.29882 | 0.48393 |
| 1.91 | 4.80246 | 2.32613 | 0.48436 |
| 1.92 | 4.85519 | 2.35373 | 0.48479 |
| 1.93 | 4.90850 | 2.38160 | 0.48520 |
| 1.94 | 4.96239 | 2.40977 | 0.48561 |
| 1.95 | 5.01688 | 2.43822 | 0.48600 |
| 1.96 | 5.07195 | 2.46697 | 0.48639 |
| 1.97 | 5.12763 | 2.49601 | 0.48678 |
| 1.98 | 5.18392 | 2.52535 | 0.48715 |
| 1.99 | 5.24082 | 2.55499 | 0.48752 |
| 2.00 | 5.29834 | 2.58494 | 0.48788 |
| 2.0 | 5.29834 | 2.58494 | 0.48788 |
| 2.1 | 5.90906 | 2.90196 | 0.49110 |
| 2.2 | 6.58916 | 3.25333 | 0.49374 |
| 2.3 | 7.34609 | 3.64286 | 0.49589 |
| 2.4 | 8.18812 | 4.07680 | 0.49765 |
| 2.5 | 9.12445 | 4.55385 | 0.49908 |
| 2.6 | 10.16526 | 5.08521 | 0.50025 |
| 2.7 | 11.32190 | 5.67464 | 0.50121 |
| 2.8 | 12.60691 | 6.32854 | 0.50199 |
| 2.9 | 14.03424 | 7.05399 | 0.50263 |
| 3.0 | 15.61932 | 7.85882 | 0.50315 |
| 3.1 | 17.37931 | 8.75173 | 0.50357 |
| 3.2 | 19.33319 | 9.74236 | 0.50392 |
| 3.3 | 21.50203 | 10.84137 | 0.50420 |
| 3.4 | 23.90917 | 12.06058 | 0.50443 |
| 3.5 | 26.58046 | 13.41311 | 0.50462 |
| 3.6 | 29.54465 | 14.91345 | 0.50478 |
| 3.7 | 32.83344 | 16.57770 | 0.50490 |
| 3.8 | 36.48208 | 18.42367 | 0.50501 |
| 3.9 | 40.52959 | 20.47111 | 0.50509 |
| 4.0 | 45.01920 | 22.74186 | 0.50516 |
| 4.1 | 49.99881 | 25.26018 | 0.50522 |
| 4.2 | 55.52151 | 28.05289 | 0.50526 |
| 4.3 | 61.64609 | 31.14972 | 0.50530 |
| 4.4 | 68.43768 | 34.58361 | 0.50533 |
| 4.5 | 75.96846 | 38.39105 | 0.50536 |
| 4.6 | 84.31835 | 42.61245 | 0.50538 |
| 4.7 | 93.57590 | 47.29256 | 0.50539 |
| 4.8 | 103.83919 | 52.48098 | 0.50541 |
| 4.9 | 115.21685 | 58.23262 | 0.50542 |
| 5.0 | 127.82922 | 64.60830 | 0.50543 |
| 5.1 | 141.80955 | 71.67541 | 0.50543 |
| 5.2 | 157.30546 | 79.50854 | 0.50544 |
| 5.3 | 174.48042 | 88.19033 | 0.50545 |
| 5.4 | 193.51547 | 97.81231 | 0.50545 |
| 5.5 | 214.61111 | 108.47582 | 0.50545 |
| 5.6 | 237.98938 | 120.29309 | 0.50546 |
| 5.7 | 263.89617 | 133.38841 | 0.50546 |
| 5.8 | 292.60376 | 147.89943 | 0.50546 |
| 5.9 | 324.41368 | 163.97856 | 0.50546 |
| 6.0 | 359.65982 | 181.79456 | 0.50546 |
| 6.0 | 359.65982 | 181.79456 | 0.50546 |
| 6.1 | 398.71187 | 201.53431 | 0.50546 |
| 6.2 | 441.97922 | 223.40475 | 0.50546 |
| 6.3 | 489.91515 | 247.63498 | 0.50547 |
| 6.4 | 543.02156 | 274.47872 | 0.50547 |
| 6.5 | 601.85414 | 304.21683 | 0.50547 |
| 6.6 | 667.02816 | 337.16033 | 0.50547 |
| 6.7 | 739.22482 | 373.65355 | 0.50547 |
| 6.8 | 819.19834 | 414.07770 | 0.50547 |
| 6.9 | 907.78370 | 458.65486 | 0.50547 |
| 7.0 | 1005.90533 | 508.45229 | 0.50547 |
| 7.1 | 1114.58667 | 563.38731 | 0.50547 |
| 7.2 | 1234.96073 | 624.23262 | 0.50547 |
| 7.3 | 1368.28181 | 691.62223 | 0.50547 |
| 7.4 | 1515.93846 | 766.25801 | 0.50547 |
| 7.5 | 1679.46784 | 848.91694 | 0.50547 |
| 7.6 | 1860.57158 | 940.45914 | 0.50547 |
| 7.7 | 2061.13333 | 1041.83675 | 0.50547 |
| 7.8 | 2283.23826 | 1154.10374 | 0.50547 |
| 7.9 | 2529.19452 | 1278.42680 | 0.50547 |
| 8.0 | 2801.55706 | 1416.09741 | 0.50547 |
| 8.1 | 3103.15401 | 1568.54507 | 0.50547 |
| 8.2 | 3437.11583 | 1737.35213 | 0.50547 |
| 8.3 | 3806.90755 | 1924.27007 | 0.50547 |
| 8.4 | 4216.36450 | 2131.23751 | 0.50547 |
| 8.5 | 4669.73184 | 2360.40025 | 0.50547 |
| 8.6 | 5171.70827 | 2614.13334 | 0.50547 |
| 8.7 | 5727.49440 | 2895.06550 | 0.50547 |
| 8.8 | 6342.84633 | 3206.10628 | 0.50547 |
| 8.9 | 7024.13486 | 3550.47590 | 0.50547 |
| 9.0 | 7778.41109 | 3931.73868 | 0.50547 |
| 9.1 | 8613.47893 | 4353.83863 | 0.50547 |
| 9.2 | 9537.97538 | 4821.14209 | 0.50547 |
| 9.3 | 10561.45936 | 5338.48059 | 0.50547 |
| 9.4 | 11694.51000 | 5911.20154 | 0.50547 |
| 9.5 | 12948.83537 | 6545.22299 | 0.50547 |
| 9.6 | 14337.39281 | 7247.09447 | 0.50547 |
| 9.7 | 15874.52207 | 8024.06427 | 0.50547 |
| 9.8 | 17576.09265 | 8884.15390 | 0.50547 |
| 9.9 | 19459.66675 | 9836.24050 | 0.50547 |
| 10.0 | 21544.67965 | 10690.14799 | 0.50547 |

TABLE D.IIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/3$ and $x$ from 1.50 to 10.0.

$\alpha = 2/3$

| $x$ | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ | $x$ | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ | $x$ | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.09552 | 1.24711 | 1.13837 | 1.00 | 1.40402 | 2.26370 | 1.61230 |
| 0.01 | 1.00004 | 0.08772 | 0.08772 | 0.51 | 1.09946 | 1.26603 | 1.15151 | 1.01 | 1.41275 | 2.28674 | 1.61864 |
| 0.02 | 1.00015 | 0.13926 | 0.13924 | 0.52 | 1.10347 | 1.28495 | 1.16446 | 1.02 | 1.42160 | 2.30994 | 1.62488 |
| 0.03 | 1.00034 | 0.18250 | 0.18244 | 0.53 | 1.10758 | 1.30389 | 1.17724 | 1.03 | 1.43056 | 2.33328 | 1.63103 |
| 0.04 | 1.00060 | 0.22111 | 0.22098 | 0.54 | 1.11176 | 1.32283 | 1.18985 | 1.04 | 1.43964 | 2.35677 | 1.63708 |
| 0.05 | 1.00094 | 0.25662 | 0.25638 | 0.55 | 1.11604 | 1.34178 | 1.20228 | 1.05 | 1.44882 | 2.38046 | 1.64303 |
| 0.06 | 1.00135 | 0.28984 | 0.28945 | 0.56 | 1.12039 | 1.36076 | 1.21454 | 1.06 | 1.45813 | 2.40429 | 1.64889 |
| 0.07 | 1.00184 | 0.32129 | 0.32070 | 0.57 | 1.12484 | 1.37976 | 1.22663 | 1.07 | 1.46755 | 2.42828 | 1.65465 |
| 0.08 | 1.00240 | 0.35130 | 0.35046 | 0.58 | 1.12937 | 1.39879 | 1.23856 | 1.08 | 1.47709 | 2.45245 | 1.66033 |
| 0.09 | 1.00304 | 0.38012 | 0.37897 | 0.59 | 1.13398 | 1.41784 | 1.25032 | 1.09 | 1.48674 | 2.47678 | 1.66591 |
| 0.10 | 1.00375 | 0.40793 | 0.40640 | 0.60 | 1.13869 | 1.43694 | 1.26192 | 1.10 | 1.49652 | 2.50129 | 1.67140 |
| 0.11 | 1.00454 | 0.43496 | 0.43289 | 0.61 | 1.14348 | 1.45606 | 1.27337 | 1.11 | 1.50641 | 2.52597 | 1.67681 |
| 0.12 | 1.00541 | 0.46103 | 0.45855 | 0.62 | 1.14836 | 1.47523 | 1.28465 | 1.12 | 1.51643 | 2.55083 | 1.68213 |
| 0.13 | 1.00635 | 0.48653 | 0.48346 | 0.63 | 1.15332 | 1.49445 | 1.29578 | 1.13 | 1.52656 | 2.57587 | 1.68736 |
| 0.14 | 1.00736 | 0.51142 | 0.50769 | 0.64 | 1.15838 | 1.51371 | 1.30675 | 1.14 | 1.53682 | 2.60109 | 1.69251 |
| 0.15 | 1.00845 | 0.53579 | 0.53130 | 0.65 | 1.16352 | 1.53302 | 1.31757 | 1.15 | 1.54721 | 2.62650 | 1.69758 |
| 0.16 | 1.00962 | 0.55967 | 0.55434 | 0.66 | 1.16876 | 1.55239 | 1.32824 | 1.16 | 1.55772 | 2.65210 | 1.70256 |
| 0.17 | 1.01086 | 0.58311 | 0.57685 | 0.67 | 1.17409 | 1.57181 | 1.33875 | 1.17 | 1.56835 | 2.67789 | 1.70746 |
| 0.18 | 1.01218 | 0.60616 | 0.59886 | 0.68 | 1.17950 | 1.59130 | 1.34913 | 1.18 | 1.57911 | 2.70388 | 1.71228 |
| 0.19 | 1.01357 | 0.62884 | 0.62042 | 0.69 | 1.18501 | 1.61084 | 1.35935 | 1.19 | 1.58999 | 2.73006 | 1.71702 |
| 0.20 | 1.01505 | 0.65119 | 0.64154 | 0.70 | 1.19061 | 1.63046 | 1.36943 | 1.20 | 1.60101 | 2.75644 | 1.72169 |
| 0.21 | 1.01659 | 0.67323 | 0.66225 | 0.71 | 1.19630 | 1.65014 | 1.37937 | 1.21 | 1.61215 | 2.78302 | 1.72627 |
| 0.22 | 1.01822 | 0.69500 | 0.68236 | 0.72 | 1.20208 | 1.66990 | 1.38917 | 1.22 | 1.62343 | 2.80981 | 1.73079 |
| 0.23 | 1.01992 | 0.71650 | 0.70251 | 0.73 | 1.20796 | 1.68973 | 1.39883 | 1.23 | 1.63484 | 2.83680 | 1.73522 |
| 0.24 | 1.02169 | 0.73777 | 0.72211 | 0.74 | 1.21393 | 1.70964 | 1.40835 | 1.24 | 1.64637 | 2.86401 | 1.73959 |
| 0.25 | 1.02355 | 0.75882 | 0.74136 | 0.75 | 1.21999 | 1.72963 | 1.41774 | 1.25 | 1.65805 | 2.89143 | 1.74388 |
| 0.26 | 1.02548 | 0.77966 | 0.76029 | 0.76 | 1.22615 | 1.74970 | 1.42699 | 1.26 | 1.66985 | 2.91906 | 1.74809 |
| 0.27 | 1.02749 | 0.80032 | 0.77891 | 0.77 | 1.23241 | 1.76987 | 1.43610 | 1.27 | 1.68179 | 2.94691 | 1.75224 |
| 0.28 | 1.02957 | 0.82080 | 0.79722 | 0.78 | 1.23876 | 1.79012 | 1.44509 | 1.28 | 1.69387 | 2.97499 | 1.75632 |
| 0.29 | 1.03175 | 0.84112 | 0.81525 | 0.79 | 1.24521 | 1.81046 | 1.45394 | 1.29 | 1.70609 | 3.00328 | 1.76033 |
| 0.30 | 1.03398 | 0.86129 | 0.83299 | 0.80 | 1.25175 | 1.83090 | 1.46267 | 1.30 | 1.71844 | 3.03181 | 1.76427 |
| 0.31 | 1.03630 | 0.88133 | 0.85046 | 0.81 | 1.25840 | 1.85143 | 1.47127 | 1.31 | 1.73096 | 3.06056 | 1.76815 |
| 0.32 | 1.03870 | 0.90124 | 0.86766 | 0.82 | 1.26514 | 1.87207 | 1.47974 | 1.32 | 1.74358 | 3.08955 | 1.77196 |
| 0.33 | 1.04117 | 0.92103 | 0.88461 | 0.83 | 1.27198 | 1.89281 | 1.48809 | 1.33 | 1.75635 | 3.11877 | 1.77571 |
| 0.34 | 1.04373 | 0.94071 | 0.90130 | 0.84 | 1.27892 | 1.91365 | 1.49631 | 1.34 | 1.76928 | 3.14823 | 1.77939 |
| 0.35 | 1.04636 | 0.96030 | 0.91775 | 0.85 | 1.28595 | 1.93461 | 1.50441 | 1.35 | 1.78234 | 3.17794 | 1.78301 |
| 0.36 | 1.04907 | 0.97979 | 0.93396 | 0.86 | 1.29309 | 1.95567 | 1.51240 | 1.36 | 1.79556 | 3.20788 | 1.78657 |
| 0.37 | 1.05187 | 0.99921 | 0.94994 | 0.87 | 1.30034 | 1.97685 | 1.52026 | 1.37 | 1.80892 | 3.23807 | 1.79006 |
| 0.38 | 1.05474 | 1.01855 | 0.96569 | 0.88 | 1.30768 | 1.99815 | 1.52801 | 1.38 | 1.82242 | 3.26852 | 1.79350 |
| 0.39 | 1.05769 | 1.03782 | 0.98121 | 0.89 | 1.31513 | 2.01956 | 1.53564 | 1.39 | 1.83606 | 3.29921 | 1.79688 |
| 0.40 | 1.06072 | 1.05703 | 0.99652 | 0.90 | 1.32268 | 2.04110 | 1.54316 | 1.40 | 1.84989 | 3.33017 | 1.80020 |
| 0.41 | 1.06384 | 1.07619 | 1.01161 | 0.91 | 1.33033 | 2.06275 | 1.55056 | 1.41 | 1.86385 | 3.36139 | 1.80346 |
| 0.42 | 1.06703 | 1.09530 | 1.02649 | 0.92 | 1.33809 | 2.08454 | 1.55785 | 1.42 | 1.87796 | 3.39285 | 1.80667 |
| 0.43 | 1.07030 | 1.11436 | 1.04116 | 0.93 | 1.34595 | 2.10646 | 1.56503 | 1.43 | 1.89223 | 3.42459 | 1.80982 |
| 0.44 | 1.07366 | 1.13339 | 1.05563 | 0.94 | 1.35392 | 2.12850 | 1.57210 | 1.44 | 1.90665 | 3.45660 | 1.81291 |
| 0.45 | 1.07710 | 1.15239 | 1.06990 | 0.95 | 1.36200 | 2.15068 | 1.57906 | 1.45 | 1.92123 | 3.48885 | 1.81596 |
| 0.46 | 1.08062 | 1.17137 | 1.08398 | 0.96 | 1.37019 | 2.17300 | 1.58592 | 1.46 | 1.93597 | 3.52143 | 1.81895 |
| 0.47 | 1.08422 | 1.19032 | 1.09786 | 0.97 | 1.37848 | 2.19546 | 1.59267 | 1.47 | 1.95087 | 3.55426 | 1.82188 |
| 0.48 | 1.08790 | 1.20926 | 1.11155 | 0.98 | 1.38688 | 2.21806 | 1.59931 | 1.48 | 1.96593 | 3.58737 | 1.82477 |
| 0.49 | 1.09167 | 1.22819 | 1.12505 | 0.99 | 1.39540 | 2.24081 | 1.60586 | 1.49 | 1.98115 | 3.62076 | 1.82761 |
| 0.50 | 1.09552 | 1.24711 | 1.13837 | 1.00 | 1.40402 | 2.26370 | 1.61230 | 1.50 | 1.99654 | 3.65444 | 1.83039 |

TABLE D.IIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 2/3$ and $x$ from 0.00 to 1.50.

$\alpha = 2/3$

| x | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ |
|---|---|---|---|
| 1.50 | 1.99654 | 3.65444 | 1.83039 |
| 1.51 | 2.01209 | 3.68842 | 1.83313 |
| 1.52 | 2.02781 | 3.72268 | 1.83582 |
| 1.53 | 2.04369 | 3.75724 | 1.83846 |
| 1.54 | 2.05975 | 3.79211 | 1.84105 |
| 1.55 | 2.07598 | 3.82727 | 1.84360 |
| 1.56 | 2.09237 | 3.86274 | 1.84610 |
| 1.57 | 2.10895 | 3.89853 | 1.84856 |
| 1.58 | 2.12570 | 3.93462 | 1.85098 |
| 1.59 | 2.14262 | 3.97103 | 1.85335 |
| 1.60 | 2.15973 | 4.00777 | 1.85568 |
| 1.61 | 2.17701 | 4.04483 | 1.85797 |
| 1.62 | 2.19448 | 4.08221 | 1.86022 |
| 1.63 | 2.21213 | 4.11993 | 1.86243 |
| 1.64 | 2.22996 | 4.15798 | 1.86459 |
| 1.65 | 2.24798 | 4.19636 | 1.86672 |
| 1.66 | 2.26619 | 4.23509 | 1.86881 |
| 1.67 | 2.28459 | 4.27417 | 1.87087 |
| 1.68 | 2.30318 | 4.31360 | 1.87288 |
| 1.69 | 2.32197 | 4.35337 | 1.87486 |
| 1.70 | 2.34095 | 4.39351 | 1.87681 |
| 1.71 | 2.36012 | 4.43400 | 1.87872 |
| 1.72 | 2.37950 | 4.47486 | 1.88059 |
| 1.73 | 2.39907 | 4.51609 | 1.88243 |
| 1.74 | 2.41885 | 4.55768 | 1.88424 |
| 1.75 | 2.43883 | 4.59966 | 1.88601 |
| 1.76 | 2.45901 | 4.64201 | 1.88775 |
| 1.77 | 2.47940 | 4.68475 | 1.88946 |
| 1.78 | 2.50001 | 4.72787 | 1.89114 |
| 1.79 | 2.52082 | 4.77139 | 1.89279 |
| 1.80 | 2.54185 | 4.81530 | 1.89441 |
| 1.81 | 2.56309 | 4.85961 | 1.89600 |
| 1.82 | 2.58455 | 4.90433 | 1.89756 |
| 1.83 | 2.60623 | 4.94945 | 1.89909 |
| 1.84 | 2.62813 | 4.99499 | 1.90059 |
| 1.85 | 2.65025 | 5.04094 | 1.90207 |
| 1.86 | 2.67259 | 5.08732 | 1.90351 |
| 1.87 | 2.69517 | 5.13412 | 1.90493 |
| 1.88 | 2.71797 | 5.18135 | 1.90633 |
| 1.89 | 2.74100 | 5.22901 | 1.90770 |
| 1.90 | 2.76427 | 5.27731 | 1.90904 |
| 1.91 | 2.78777 | 5.32565 | 1.91036 |
| 1.92 | 2.81152 | 5.37465 | 1.91165 |
| 1.93 | 2.83550 | 5.42309 | 1.91293 |
| 1.94 | 2.85972 | 5.47399 | 1.91417 |
| 1.95 | 2.88418 | 5.52435 | 1.91540 |
| 1.96 | 2.90890 | 5.57518 | 1.91660 |
| 1.97 | 2.93386 | 5.62648 | 1.91778 |
| 1.98 | 2.95907 | 5.67825 | 1.91893 |
| 1.99 | 2.98453 | 5.73051 | 1.92007 |
| 2.00 | 3.01025 | 5.78325 | 1.92118 |

| x | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ |
|---|---|---|---|
| 2.0 | 3.01025 | 5.78325 | 1.92118 |
| 2.1 | 3.28205 | 6.33842 | 1.93124 |
| 2.2 | 3.58216 | 6.94779 | 1.93955 |
| 2.3 | 3.91334 | 7.61702 | 1.94643 |
| 2.4 | 4.27863 | 8.35230 | 1.95209 |
| 2.5 | 4.68142 | 9.16046 | 1.95677 |
| 2.6 | 5.12542 | 10.04900 | 1.96062 |
| 2.7 | 5.61474 | 11.02615 | 1.96379 |
| 2.8 | 6.15390 | 12.10099 | 1.96639 |
| 2.9 | 6.74791 | 13.28351 | 1.96854 |
| 3.0 | 7.40228 | 14.58471 | 1.97030 |
| 3.1 | 8.12310 | 16.01669 | 1.97175 |
| 3.2 | 8.91707 | 17.59281 | 1.97294 |
| 3.3 | 9.79160 | 19.32776 | 1.97391 |
| 3.4 | 10.75486 | 21.23775 | 1.97471 |
| 3.5 | 11.81582 | 23.34063 | 1.97537 |
| 3.6 | 12.98444 | 25.65607 | 1.97591 |
| 3.7 | 14.27164 | 28.20577 | 1.97635 |
| 3.8 | 15.68949 | 31.01364 | 1.97671 |
| 3.9 | 17.25130 | 34.10602 | 1.97701 |
| 4.0 | 18.97175 | 37.51199 | 1.97726 |
| 4.1 | 20.86700 | 41.26558 | 1.97746 |
| 4.2 | 22.95492 | 45.39611 | 1.97762 |
| 4.3 | 25.25517 | 49.94852 | 1.97775 |
| 4.4 | 27.78944 | 54.96375 | 1.97786 |
| 4.5 | 30.58167 | 60.48916 | 1.97795 |
| 4.6 | 33.65824 | 66.57697 | 1.97803 |
| 4.7 | 37.04825 | 73.28475 | 1.97809 |
| 4.8 | 40.78379 | 80.67601 | 1.97814 |
| 4.9 | 44.90025 | 88.82076 | 1.97818 |
| 5.0 | 49.43666 | 97.79625 | 1.97821 |
| 5.1 | 54.43607 | 107.68763 | 1.97824 |
| 5.2 | 59.94596 | 118.58886 | 1.97826 |
| 5.3 | 66.01869 | 130.60352 | 1.97828 |
| 5.4 | 72.71202 | 143.84591 | 1.97830 |
| 5.5 | 80.08966 | 158.44205 | 1.97831 |
| 5.6 | 88.22187 | 174.53096 | 1.97832 |
| 5.7 | 97.18616 | 192.26599 | 1.97833 |
| 5.8 | 107.06803 | 211.81628 | 1.97833 |
| 5.9 | 117.96178 | 233.36842 | 1.97834 |
| 6.0 | 129.97145 | 257.12820 | 1.97834 |

| x | $F_{2/3}(x)$ | $H_{1/3}(x)$ | $T_{2/3}(x)$ |
|---|---|---|---|
| 6.0 | 129.97145 | 257.12620 | 1.97634 |
| 6.1 | 143.21179 | 283.32268 | 1.97835 |
| 6.2 | 157.80940 | 312.20230 | 1.97835 |
| 6.3 | 173.90393 | 344.04335 | 1.97835 |
| 6.4 | 191.64943 | 379.15061 | 1.97836 |
| 6.5 | 211.21588 | 417.36035 | 1.97836 |
| 6.6 | 232.79078 | 460.54351 | 1.97836 |
| 6.7 | 256.58098 | 507.60934 | 1.97836 |
| 6.8 | 282.81472 | 559.50936 | 1.97836 |
| 6.9 | 311.74379 | 616.74170 | 1.97836 |
| 7.0 | 343.64598 | 679.85596 | 1.97836 |
| 7.1 | 378.82779 | 749.45850 | 1.97835 |
| 7.2 | 417.62741 | 826.21836 | 1.97836 |
| 7.3 | 460.41794 | 910.67370 | 1.97836 |
| 7.4 | 507.61109 | 1004.23990 | 1.97836 |
| 7.5 | 559.66109 | 1107.21292 | 1.97836 |
| 7.6 | 617.06916 | 1220.78706 | 1.97836 |
| 7.7 | 680.38836 | 1346.05555 | 1.97836 |
| 7.8 | 750.22894 | 1484.22569 | 1.97836 |
| 7.9 | 827.[illegible]0427 | 1636.62967 | 1.97636 |
| 8.0 | 912.23739 | 1804.73751 | 1.97836 |
| 8.1 | 1005.96825 | 1990.17133 | 1.37836 |
| 8.2 | 1109.36164 | 2194.72114 | 1.97836 |
| 8.3 | 1223.41598 | 2420.36220 | 1.97836 |
| 8.4 | 1349.23309 | 2669.27429 | 1.97836 |
| 8.5 | 1488.02882 | 2943.86283 | 1.97836 |
| 8.6 | 1641.14495 | 3246.78232 | 1.97836 |
| 8.7 | 1810.06220 | 3580.96219 | 1.97836 |
| 8.8 | 1996.41466 | 3949.63526 | 1.97836 |
| 8.9 | 2202.00570 | 4356.36923 | 1.97836 |
| 9.0 | 2428.82550 | 4805.10143 | 1.97836 |
| 9.1 | 2679.07046 | 5300.17714 | 1.97836 |
| 9.2 | 2955.16461 | 5846.39194 | 1.97836 |
| 9.3 | 3259.78315 | 6449.03039 | 1.97836 |
| 9.4 | 3595.87862 | 7113.95767 | 1.97836 |
| 9.5 | 3966.70960 | 7847.59643 | 1.97836 |
| 9.6 | 4375.87251 | 8657.06970 | 1.97836 |
| 9.7 | 4827.33666 | 9550.23026 | 1.97836 |
| 9.8 | 5325.48299 | 10535.74517 | 1.97836 |
| 9.9 | 5875.14679 | 11623.18037 | 1.97836 |
| 10.0 | 6481.66483 | 12823.09399 | 1.97836 |

TABLE D.IIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 2/3$ and $x$ from 1.50 to 10.0.

$\alpha = 1/4$

| $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ | $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ | $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.25631 | 0.17269 | 0.13746 | 1.00 | 2.10378 | 0.54138 | 0.25757 |
| 0.01 | 1.00010 | 0.00047 | 0.00047 | 0.51 | 1.26693 | 0.17815 | 0.14061 | 1.01 | 2.12817 | 0.55153 | 0.25916 |
| 0.02 | 1.00040 | 0.00133 | 0.00133 | 0.52 | 1.27779 | 0.18368 | 0.14375 | 1.02 | 2.15290 | 0.56129 | 0.26071 |
| 0.03 | 1.00090 | 0.00245 | 0.00245 | 0.53 | 1.28887 | 0.18928 | 0.14686 | 1.03 | 2.17796 | 0.57115 | 0.26224 |
| 0.04 | 1.00160 | 0.00377 | 0.00377 | 0.54 | 1.30020 | 0.19496 | 0.14995 | 1.04 | 2.20337 | 0.58111 | 0.26374 |
| 0.05 | 1.00250 | 0.00527 | 0.00526 | 0.55 | 1.31175 | 0.20071 | 0.15301 | 1.05 | 2.22912 | 0.59118 | 0.26521 |
| 0.06 | 1.00360 | 0.00693 | 0.00691 | 0.56 | 1.32355 | 0.20653 | 0.15604 | 1.06 | 2.25522 | 0.60136 | 0.26665 |
| 0.07 | 1.00490 | 0.00874 | 0.00869 | 0.57 | 1.33558 | 0.21242 | 0.15905 | 1.07 | 2.28165 | 0.61165 | 0.26807 |
| 0.08 | 1.00640 | 0.01068 | 0.01061 | 0.58 | 1.34786 | 0.21839 | 0.16203 | 1.08 | 2.30846 | 0.62205 | 0.26946 |
| 0.09 | 1.00811 | 0.01274 | 0.01264 | 0.59 | 1.36037 | 0.22443 | 0.16498 | 1.09 | 2.33561 | 0.63256 | 0.27083 |
| 0.10 | 1.01001 | 0.01493 | 0.01478 | 0.60 | 1.37313 | 0.23054 | 0.16789 | 1.10 | 2.36313 | 0.64318 | 0.27217 |
| 0.11 | 1.01211 | 0.01723 | 0.01702 | 0.61 | 1.38614 | 0.23673 | 0.17078 | 1.11 | 2.39100 | 0.65391 | 0.27349 |
| 0.12 | 1.01442 | 0.01954 | 0.01936 | 0.62 | 1.39939 | 0.24299 | 0.17364 | 1.12 | 2.41924 | 0.66475 | 0.27478 |
| 0.13 | 1.01693 | 0.02215 | 0.02175 | 0.63 | 1.41289 | 0.24933 | 0.17647 | 1.13 | 2.44785 | 0.67572 | 0.27604 |
| 0.14 | 1.01964 | 0.02476 | 0.02429 | 0.64 | 1.42663 | 0.25575 | 0.17927 | 1.14 | 2.47683 | 0.68679 | 0.27729 |
| 0.15 | 1.02255 | 0.02747 | 0.02687 | 0.65 | 1.44063 | 0.26224 | 0.18203 | 1.15 | 2.50619 | 0.69799 | 0.27851 |
| 0.16 | 1.02567 | 0.03028 | 0.02952 | 0.66 | 1.45488 | 0.26880 | 0.18476 | 1.16 | 2.53592 | 0.70930 | 0.27970 |
| 0.17 | 1.02898 | 0.03318 | 0.03224 | 0.67 | 1.46939 | 0.27545 | 0.18746 | 1.17 | 2.56604 | 0.72074 | 0.28087 |
| 0.18 | 1.03251 | 0.03617 | 0.03503 | 0.68 | 1.48415 | 0.28217 | 0.19012 | 1.18 | 2.59655 | 0.73229 | 0.28202 |
| 0.19 | 1.03623 | 0.03924 | 0.03787 | 0.69 | 1.49917 | 0.28897 | 0.19275 | 1.19 | 2.62744 | 0.74397 | 0.28315 |
| 0.20 | 1.04016 | 0.04241 | 0.04077 | 0.70 | 1.51445 | 0.29584 | 0.19535 | 1.20 | 2.65873 | 0.75577 | 0.28426 |
| 0.21 | 1.04429 | 0.04565 | 0.04373 | 0.71 | 1.52999 | 0.30280 | 0.19791 | 1.21 | 2.69042 | 0.76770 | 0.28534 |
| 0.22 | 1.04863 | 0.04898 | 0.04671 | 0.72 | 1.54580 | 0.30984 | 0.20044 | 1.22 | 2.72250 | 0.77975 | 0.28641 |
| 0.23 | 1.05318 | 0.05239 | 0.04975 | 0.73 | 1.56186 | 0.31695 | 0.20293 | 1.23 | 2.75499 | 0.79193 | 0.28745 |
| 0.24 | 1.05793 | 0.05588 | 0.05282 | 0.74 | 1.57820 | 0.32415 | 0.20539 | 1.24 | 2.78789 | 0.80424 | 0.28845 |
| 0.25 | 1.06289 | 0.05945 | 0.05594 | 0.75 | 1.59481 | 0.33143 | 0.20782 | 1.25 | 2.82120 | 0.81668 | 0.28948 |
| 0.26 | 1.06806 | 0.06310 | 0.05908 | 0.76 | 1.61168 | 0.33879 | 0.21021 | 1.26 | 2.85493 | 0.82925 | 0.29046 |
| 0.27 | 1.07343 | 0.06683 | 0.06226 | 0.77 | 1.62883 | 0.34623 | 0.21256 | 1.27 | 2.88909 | 0.84195 | 0.29143 |
| 0.28 | 1.07902 | 0.07063 | 0.06546 | 0.78 | 1.64626 | 0.35376 | 0.21488 | 1.28 | 2.92365 | 0.85479 | 0.29237 |
| 0.29 | 1.08481 | 0.07451 | 0.06868 | 0.79 | 1.66396 | 0.36137 | 0.21717 | 1.29 | 2.95865 | 0.86777 | 0.29330 |
| 0.30 | 1.09081 | 0.07846 | 0.07193 | 0.80 | 1.68194 | 0.36906 | 0.21942 | 1.30 | 2.99408 | 0.88089 | 0.29421 |
| 0.31 | 1.09703 | 0.08249 | 0.07519 | 0.81 | 1.70021 | 0.37684 | 0.22164 | 1.31 | 3.02995 | 0.89413 | 0.29510 |
| 0.32 | 1.10345 | 0.08659 | 0.07847 | 0.82 | 1.71875 | 0.38470 | 0.22383 | 1.32 | 3.06626 | 0.90752 | 0.29597 |
| 0.33 | 1.11009 | 0.09076 | 0.08176 | 0.83 | 1.73759 | 0.39266 | 0.22598 | 1.33 | 3.10301 | 0.92105 | 0.29683 |
| 0.34 | 1.11694 | 0.09501 | 0.08506 | 0.84 | 1.75671 | 0.40070 | 0.22809 | 1.34 | 3.14022 | 0.93473 | 0.29766 |
| 0.35 | 1.12401 | 0.09933 | 0.08837 | 0.85 | 1.77612 | 0.40882 | 0.23018 | 1.35 | 3.17787 | 0.94855 | 0.29849 |
| 0.36 | 1.13129 | 0.10372 | 0.09168 | 0.86 | 1.79582 | 0.41704 | 0.23223 | 1.36 | 3.21599 | 0.96252 | 0.29929 |
| 0.37 | 1.13878 | 0.10818 | 0.09500 | 0.87 | 1.81582 | 0.42535 | 0.23425 | 1.37 | 3.25457 | 0.97663 | 0.30008 |
| 0.38 | 1.14650 | 0.11272 | 0.09832 | 0.88 | 1.83612 | 0.43374 | 0.23623 | 1.38 | 3.29361 | 0.99090 | 0.30085 |
| 0.39 | 1.15443 | 0.11732 | 0.10163 | 0.89 | 1.85671 | 0.44223 | 0.23818 | 1.39 | 3.33313 | 1.00531 | 0.30161 |
| 0.40 | 1.16256 | 0.12200 | 0.10494 | 0.90 | 1.87761 | 0.45081 | 0.24010 | 1.40 | 3.37312 | 1.01988 | 0.30235 |
| 0.41 | 1.17094 | 0.12675 | 0.10825 | 0.91 | 1.89881 | 0.45949 | 0.24199 | 1.41 | 3.41360 | 1.03460 | 0.30308 |
| 0.42 | 1.17953 | 0.13157 | 0.11155 | 0.92 | 1.92032 | 0.46825 | 0.24384 | 1.42 | 3.45456 | 1.04948 | 0.30380 |
| 0.43 | 1.18834 | 0.13646 | 0.11483 | 0.93 | 1.94214 | 0.47712 | 0.24566 | 1.43 | 3.49601 | 1.06452 | 0.30449 |
| 0.44 | 1.19736 | 0.14142 | 0.11811 | 0.94 | 1.96427 | 0.48607 | 0.24746 | 1.44 | 3.53795 | 1.07971 | 0.30518 |
| 0.45 | 1.20663 | 0.14646 | 0.12138 | 0.95 | 1.98672 | 0.49513 | 0.24922 | 1.45 | 3.58040 | 1.09507 | 0.30585 |
| 0.46 | 1.21611 | 0.15156 | 0.12462 | 0.96 | 2.00949 | 0.50428 | 0.25095 | 1.46 | 3.62335 | 1.11059 | 0.30651 |
| 0.47 | 1.22582 | 0.15674 | 0.12786 | 0.97 | 2.03257 | 0.51353 | 0.25265 | 1.47 | 3.66681 | 1.12627 | 0.30715 |
| 0.48 | 1.23575 | 0.16198 | 0.13108 | 0.98 | 2.05598 | 0.52288 | 0.25432 | 1.48 | 3.71079 | 1.14212 | 0.30778 |
| 0.49 | 1.24592 | 0.16730 | 0.13428 | 0.99 | 2.07971 | 0.53233 | 0.25596 | 1.49 | 3.75529 | 1.15814 | 0.30840 |
| 0.50 | 1.25631 | 0.17269 | 0.13746 | 1.00 | 2.10378 | 0.54188 | 0.25757 | 1.50 | 3.80031 | 1.17433 | 0.30901 |

TABLE D.IVA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/4$ and $x$ from 0.00 to 1.50.

$\alpha = 1/4$

| $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ |
|---|---|---|---|
| 1.50 | 3.80031 | 1.17433 | 0.30901 |
| 1.51 | 3.84587 | 1.19069 | 0.30960 |
| 1.52 | 3.89196 | 1.20722 | 0.31018 |
| 1.53 | 3.93859 | 1.22393 | 0.31075 |
| 1.54 | 3.98577 | 1.24081 | 0.31131 |
| 1.55 | 4.03350 | 1.25788 | 0.31186 |
| 1.56 | 4.08179 | 1.27512 | 0.31239 |
| 1.57 | 4.13065 | 1.29255 | 0.31292 |
| 1.58 | 4.18007 | 1.31017 | 0.31343 |
| 1.59 | 4.23007 | 1.32797 | 0.31394 |
| 1.60 | 4.28065 | 1.34596 | 0.31443 |
| 1.61 | 4.33182 | 1.36414 | 0.31491 |
| 1.62 | 4.38358 | 1.38251 | 0.31538 |
| 1.63 | 4.43594 | 1.40108 | 0.31585 |
| 1.64 | 4.48890 | 1.41985 | 0.31630 |
| 1.65 | 4.54248 | 1.43882 | 0.31675 |
| 1.66 | 4.59667 | 1.45798 | 0.31718 |
| 1.67 | 4.65149 | 1.47736 | 0.31761 |
| 1.68 | 4.70694 | 1.49694 | 0.31803 |
| 1.69 | 4.76302 | 1.51672 | 0.31844 |
| 1.70 | 4.81975 | 1.53672 | 0.31884 |
| 1.71 | 4.87713 | 1.55693 | 0.31923 |
| 1.72 | 4.93516 | 1.57735 | 0.31962 |
| 1.73 | 4.99386 | 1.59800 | 0.31999 |
| 1.74 | 5.05322 | 1.61886 | 0.32036 |
| 1.75 | 5.11327 | 1.63994 | 0.32072 |
| 1.76 | 5.17400 | 1.66125 | 0.32108 |
| 1.77 | 5.23541 | 1.68279 | 0.32142 |
| 1.78 | 5.29753 | 1.70455 | 0.32176 |
| 1.79 | 5.36035 | 1.72655 | 0.32210 |
| 1.80 | 5.42389 | 1.74878 | 0.32242 |
| 1.81 | 5.48814 | 1.77125 | 0.32274 |
| 1.82 | 5.55312 | 1.79396 | 0.32305 |
| 1.83 | 5.61884 | 1.81591 | 0.32336 |
| 1.84 | 5.68530 | 1.84010 | 0.32366 |
| 1.85 | 5.75251 | 1.86354 | 0.32395 |
| 1.86 | 5.82047 | 1.88724 | 0.32424 |
| 1.87 | 5.88921 | 1.91118 | 0.32452 |
| 1.88 | 5.95871 | 1.93538 | 0.32480 |
| 1.89 | 6.02900 | 1.95984 | 0.32507 |
| 1.90 | 6.10008 | 1.98456 | 0.32533 |
| 1.91 | 6.17196 | 2.00954 | 0.32559 |
| 1.92 | 6.24464 | 2.03479 | 0.32585 |
| 1.93 | 6.31814 | 2.06031 | 0.32610 |
| 1.94 | 6.39246 | 2.08611 | 0.32634 |
| 1.95 | 6.46761 | 2.11217 | 0.32658 |
| 1.96 | 6.54360 | 2.13852 | 0.32681 |
| 1.97 | 6.62044 | 2.16515 | 0.32704 |
| 1.98 | 6.69814 | 2.19206 | 0.32726 |
| 1.99 | 6.77671 | 2.21926 | 0.32748 |
| 2.00 | 6.85616 | 2.24675 | 0.32770 |

| $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ |
|---|---|---|---|
| 2.0 | 6.85616 | 2.24675 | 0.32770 |
| 2.1 | 7.70085 | 2.53835 | 0.32962 |
| 2.2 | 8.64388 | 2.86267 | 0.33118 |
| 2.3 | 9.69609 | 3.22343 | 0.33245 |
| 2.4 | 10.86951 | 3.62473 | 0.33348 |
| 2.5 | 12.17756 | 4.07113 | 0.33431 |
| 2.6 | 13.63512 | 4.56771 | 0.33500 |
| 2.7 | 15.25875 | 5.12007 | 0.33555 |
| 2.8 | 17.06686 | 5.73448 | 0.33600 |
| 2.9 | 19.07988 | 6.41786 | 0.33637 |
| 3.0 | 21.32051 | 7.17792 | 0.33667 |
| 3.1 | 23.81395 | 8.02318 | 0.33691 |
| 3.2 | 26.58819 | 8.96313 | 0.33711 |
| 3.3 | 29.67429 | 10.00828 | 0.33727 |
| 3.4 | 33.10675 | 11.17032 | 0.33740 |
| 3.5 | 36.92382 | 12.46219 | 0.33751 |
| 3.6 | 41.16800 | 13.89826 | 0.33760 |
| 3.7 | 45.88641 | 15.49447 | 0.33767 |
| 3.8 | 51.13136 | 17.26852 | 0.33773 |
| 3.9 | 56.96088 | 19.24004 | 0.33778 |
| 4.0 | 63.43934 | 21.43078 | 0.33782 |
| 4.1 | 70.63817 | 23.86490 | 0.33785 |
| 4.2 | 78.63660 | 26.56919 | 0.33787 |
| 4.3 | 87.52252 | 29.57336 | 0.33789 |
| 4.4 | 97.39340 | 32.91036 | 0.33791 |
| 4.5 | 108.35735 | 36.61674 | 0.33793 |
| 4.6 | 120.53430 | 40.73304 | 0.33794 |
| 4.7 | 134.05721 | 45.30420 | 0.33795 |
| 4.8 | 149.07356 | 50.38008 | 0.33795 |
| 4.9 | 165.74688 | 56.01595 | 0.33796 |
| 5.0 | 184.25850 | 62.27311 | 0.33797 |
| 5.1 | 204.80949 | 69.21950 | 0.33797 |
| 5.2 | 227.62277 | 76.93050 | 0.33797 |
| 5.3 | 252.94553 | 85.48964 | 0.33798 |
| 5.4 | 281.05178 | 94.98954 | 0.33798 |
| 5.5 | 312.24535 | 105.53289 | 0.33798 |
| 5.6 | 346.86301 | 117.23353 | 0.33798 |
| 5.7 | 385.27813 | 130.21764 | 0.33798 |
| 5.8 | 427.90458 | 144.62511 | 0.33798 |
| 5.9 | 475.20112 | 160.61101 | 0.33799 |
| 6.0 | 527.67626 | 178.34720 | 0.33799 |

| $x$ | $F_{1/4}(x)$ | $H_{3/4}(x)$ | $T_{1/4}(x)$ |
|---|---|---|---|
| 6.0 | 527.67626 | 178.34720 | 0.33799 |
| 6.1 | 585.89365 | 198.02419 | 0.33799 |
| 6.2 | 650.47799 | 219.85313 | 0.33799 |
| 6.3 | 722.12168 | 244.06805 | 0.33799 |
| 6.4 | 801.59208 | 270.92829 | 0.33799 |
| 6.5 | 889.73967 | 300.72132 | 0.33799 |
| 6.6 | 987.50692 | 333.76568 | 0.33799 |
| 6.7 | 1095.93831 | 370.41439 | 0.33799 |
| 6.8 | 1216.19128 | 411.05867 | 0.33799 |
| 6.9 | 1349.54844 | 456.13200 | 0.33799 |
| 7.0 | 1497.43105 | 506.11478 | 0.33799 |
| 7.1 | 1661.41396 | 561.53928 | 0.33799 |
| 7.2 | 1843.24219 | 622.99529 | 0.33799 |
| 7.3 | 2044.84923 | 691.13633 | 0.33799 |
| 7.4 | 2268.37739 | 766.68645 | 0.33799 |
| 7.5 | 2516.20022 | 850.44792 | 0.33799 |
| 7.6 | 2790.94750 | 943.30954 | 0.33799 |
| 7.7 | 3095.53275 | 1046.25608 | 0.33799 |
| 7.8 | 3433.18381 | 1160.37849 | 0.33799 |
| 7.9 | 3807.47665 | 1286.88542 | 0.33799 |
| 8.0 | 4222.37283 | 1427.11584 | 0.33799 |
| 8.1 | 4682.26100 | 1582.55306 | 0.33799 |
| 8.2 | 5192.00279 | 1754.84026 | 0.33799 |
| 8.3 | 5756.98374 | 1945.79769 | 0.33799 |
| 8.4 | 6383.16955 | 2157.44170 | 0.33799 |
| 8.5 | 7077.16850 | 2392.00581 | 0.33799 |
| 8.6 | 7846.30050 | 2651.96407 | 0.33799 |
| 8.7 | 8698.67355 | 2940.05690 | 0.33799 |
| 8.8 | 9643.26842 | 3259.31970 | 0.33799 |
| 8.9 | 10690.03241 | 3613.11455 | 0.33799 |
| 9.0 | 11849.98315 | 4005.16529 | 0.33799 |
| 9.1 | 13135.32354 | 4439.59636 | 0.33799 |
| 9.2 | 14559.56902 | 4920.97585 | 0.33799 |
| 9.3 | 16137.68846 | 5454.36306 | 0.33799 |
| 9.4 | 17886.26020 | 6045.36129 | 0.33799 |
| 9.5 | 19823.64472 | 6700.17618 | 0.33799 |
| 9.6 | 21970.17593 | 7425.68038 | 0.33799 |
| 9.7 | 24348.37283 | 8229.48506 | 0.33799 |
| 9.8 | 26983.17397 | 9120.01918 | 0.33799 |
| 9.9 | 29902.19693 | 10106.61719 | 0.33799 |
| 10.0 | 33136.02562 | 11199.61611 | 0.33799 |

TABLE D.IVB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $R_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/4$ and $x$ from 1.50 to 10.0.

$\alpha = 3/4$

| $x$ | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00003 | 0.28285 | 0.28284 |
| 0.02 | 1.00013 | 0.40003 | 0.39998 |
| 0.03 | 1.00030 | 0.48999 | 0.48984 |
| 0.04 | 1.00053 | 0.56587 | 0.56556 |
| 0.05 | 1.00083 | 0.63277 | 0.63224 |
| 0.06 | 1.00120 | 0.69332 | 0.69249 |
| 0.07 | 1.00163 | 0.74907 | 0.74784 |
| 0.08 | 1.00213 | 0.80102 | 0.79932 |
| 0.09 | 1.00270 | 0.84990 | 0.84761 |
| 0.10 | 1.00334 | 0.89622 | 0.89324 |
| 0.11 | 1.00404 | 0.94035 | 0.93657 |
| 0.12 | 1.00480 | 0.98262 | 0.97792 |
| 0.13 | 1.00564 | 1.02325 | 1.01752 |
| 0.14 | 1.00654 | 1.06245 | 1.05555 |
| 0.15 | 1.00751 | 1.10038 | 1.09218 |
| 0.16 | 1.00855 | 1.13717 | 1.12753 |
| 0.17 | 1.00965 | 1.17294 | 1.16173 |
| 0.18 | 1.01083 | 1.20779 | 1.19486 |
| 0.19 | 1.01206 | 1.24180 | 1.22700 |
| 0.20 | 1.01337 | 1.27505 | 1.25823 |
| 0.21 | 1.01475 | 1.30761 | 1.28861 |
| 0.22 | 1.01619 | 1.33953 | 1.31819 |
| 0.23 | 1.01770 | 1.37086 | 1.34702 |
| 0.24 | 1.01928 | 1.40165 | 1.37514 |
| 0.25 | 1.02093 | 1.43195 | 1.40260 |
| 0.26 | 1.02264 | 1.46179 | 1.42943 |
| 0.27 | 1.02443 | 1.49121 | 1.45565 |
| 0.28 | 1.02628 | 1.52023 | 1.48130 |
| 0.29 | 1.02820 | 1.54889 | 1.50641 |
| 0.30 | 1.03019 | 1.57722 | 1.53099 |
| 0.31 | 1.03225 | 1.60523 | 1.55507 |
| 0.32 | 1.03438 | 1.63295 | 1.57867 |
| 0.33 | 1.03658 | 1.66041 | 1.60181 |
| 0.34 | 1.03885 | 1.68762 | 1.62450 |
| 0.35 | 1.04119 | 1.71460 | 1.64676 |
| 0.36 | 1.04360 | 1.74136 | 1.66861 |
| 0.37 | 1.04608 | 1.76793 | 1.69005 |
| 0.38 | 1.04863 | 1.79432 | 1.71110 |
| 0.39 | 1.05125 | 1.82054 | 1.73178 |
| 0.40 | 1.05395 | 1.84661 | 1.75209 |
| 0.41 | 1.05671 | 1.87254 | 1.77204 |
| 0.42 | 1.05954 | 1.89834 | 1.79165 |
| 0.43 | 1.06245 | 1.92402 | 1.81092 |
| 0.44 | 1.06543 | 1.94960 | 1.82987 |
| 0.45 | 1.06848 | 1.97508 | 1.84849 |
| 0.46 | 1.07161 | 2.00048 | 1.86680 |
| 0.47 | 1.07480 | 2.02580 | 1.88481 |
| 0.48 | 1.07807 | 2.05105 | 1.90252 |
| 0.49 | 1.08142 | 2.07625 | 1.91994 |
| 0.50 | 1.08483 | 2.10140 | 1.93707 |

| $x$ | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 0.50 | 1.08483 | 2.10140 | 1.93707 |
| 0.51 | 1.08832 | 2.12650 | 1.95393 |
| 0.52 | 1.09189 | 2.15158 | 1.97051 |
| 0.53 | 1.09553 | 2.17663 | 1.98683 |
| 0.54 | 1.09924 | 2.20166 | 2.00288 |
| 0.55 | 1.10303 | 2.22667 | 2.01868 |
| 0.56 | 1.10690 | 2.25169 | 2.03423 |
| 0.57 | 1.11084 | 2.27670 | 2.04953 |
| 0.58 | 1.11486 | 2.30172 | 2.06459 |
| 0.59 | 1.11895 | 2.32676 | 2.07942 |
| 0.60 | 1.12312 | 2.35182 | 2.09401 |
| 0.61 | 1.12737 | 2.37690 | 2.10837 |
| 0.62 | 1.13169 | 2.40202 | 2.12250 |
| 0.63 | 1.13610 | 2.42717 | 2.13641 |
| 0.64 | 1.14058 | 2.45237 | 2.15011 |
| 0.65 | 1.14514 | 2.47761 | 2.16359 |
| 0.66 | 1.14978 | 2.50291 | 2.17686 |
| 0.67 | 1.15450 | 2.52827 | 2.18993 |
| 0.68 | 1.15930 | 2.55369 | 2.20279 |
| 0.69 | 1.16418 | 2.57917 | 2.21545 |
| 0.70 | 1.16914 | 2.60474 | 2.22792 |
| 0.71 | 1.17418 | 2.63037 | 2.24019 |
| 0.72 | 1.17930 | 2.65609 | 2.25226 |
| 0.73 | 1.18450 | 2.68190 | 2.26415 |
| 0.74 | 1.18979 | 2.70760 | 2.27586 |
| 0.75 | 1.19516 | 2.73379 | 2.28738 |
| 0.76 | 1.20062 | 2.75989 | 2.29872 |
| 0.77 | 1.20615 | 2.78609 | 2.30989 |
| 0.78 | 1.21178 | 2.81239 | 2.32086 |
| 0.79 | 1.21748 | 2.83881 | 2.33170 |
| 0.80 | 1.22328 | 2.86534 | 2.34235 |
| 0.81 | 1.22916 | 2.89200 | 2.35283 |
| 0.82 | 1.23512 | 2.91878 | 2.36315 |
| 0.83 | 1.24117 | 2.94568 | 2.37331 |
| 0.84 | 1.24731 | 2.97272 | 2.38331 |
| 0.85 | 1.25354 | 2.99990 | 2.39315 |
| 0.86 | 1.25985 | 3.02721 | 2.40283 |
| 0.87 | 1.26626 | 3.05467 | 2.41236 |
| 0.88 | 1.27275 | 3.08228 | 2.42174 |
| 0.89 | 1.27934 | 3.11003 | 2.43098 |
| 0.90 | 1.28601 | 3.13795 | 2.44006 |
| 0.91 | 1.29278 | 3.16601 | 2.44900 |
| 0.92 | 1.29963 | 3.19425 | 2.45780 |
| 0.93 | 1.30658 | 3.22264 | 2.46646 |
| 0.94 | 1.31363 | 3.25121 | 2.47499 |
| 0.95 | 1.32076 | 3.27995 | 2.48337 |
| 0.96 | 1.32800 | 3.30886 | 2.49162 |
| 0.97 | 1.33532 | 3.33796 | 2.49974 |
| 0.98 | 1.34274 | 3.36724 | 2.50773 |
| 0.99 | 1.35026 | 3.39670 | 2.51559 |
| 1.00 | 1.35788 | 3.42636 | 2.52332 |

| $x$ | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 1.00 | 1.35788 | 3.42636 | 2.52332 |
| 1.01 | 1.36559 | 3.45621 | 2.53093 |
| 1.02 | 1.37340 | 3.48625 | 2.53841 |
| 1.03 | 1.38131 | 3.51650 | 2.54578 |
| 1.04 | 1.38932 | 3.54695 | 2.55302 |
| 1.05 | 1.39743 | 3.57761 | 2.56015 |
| 1.06 | 1.40564 | 3.60848 | 2.56716 |
| 1.07 | 1.41395 | 3.63957 | 2.57405 |
| 1.08 | 1.42236 | 3.67087 | 2.58083 |
| 1.09 | 1.43088 | 3.70240 | 2.58750 |
| 1.10 | 1.43950 | 3.73415 | 2.59407 |
| 1.11 | 1.44822 | 3.76613 | 2.60052 |
| 1.12 | 1.45705 | 3.79834 | 2.60687 |
| 1.13 | 1.46599 | 3.83079 | 2.61311 |
| 1.14 | 1.47503 | 3.86347 | 2.61925 |
| 1.15 | 1.48418 | 3.89640 | 2.62528 |
| 1.16 | 1.49344 | 3.92958 | 2.63122 |
| 1.17 | 1.50281 | 3.96300 | 2.63706 |
| 1.18 | 1.51229 | 3.99668 | 2.64280 |
| 1.19 | 1.52188 | 4.03061 | 2.64844 |
| 1.20 | 1.53158 | 4.06480 | 2.65400 |
| 1.21 | 1.54139 | 4.09926 | 2.65945 |
| 1.22 | 1.55132 | 4.13398 | 2.66482 |
| 1.23 | 1.56136 | 4.16897 | 2.67009 |
| 1.24 | 1.57152 | 4.20424 | 2.67528 |
| 1.25 | 1.58179 | 4.23979 | 2.68038 |
| 1.26 | 1.59218 | 4.27562 | 2.68539 |
| 1.27 | 1.60268 | 4.31173 | 2.69032 |
| 1.28 | 1.61331 | 4.34813 | 2.69516 |
| 1.29 | 1.62405 | 4.38482 | 2.69992 |
| 1.30 | 1.63492 | 4.42181 | 2.70460 |
| 1.31 | 1.64590 | 4.45909 | 2.70921 |
| 1.32 | 1.65701 | 4.49669 | 2.71373 |
| 1.33 | 1.66825 | 4.53458 | 2.71817 |
| 1.34 | 1.67960 | 4.57279 | 2.72254 |
| 1.35 | 1.69109 | 4.61131 | 2.72683 |
| 1.36 | 1.70270 | 4.65015 | 2.73105 |
| 1.37 | 1.71443 | 4.68931 | 2.73520 |
| 1.38 | 1.72630 | 4.72880 | 2.73927 |
| 1.39 | 1.73829 | 4.76861 | 2.74328 |
| 1.40 | 1.75041 | 4.80876 | 2.74721 |
| 1.41 | 1.76267 | 4.84925 | 2.75108 |
| 1.42 | 1.77506 | 4.89007 | 2.75488 |
| 1.43 | 1.78758 | 4.93125 | 2.75862 |
| 1.44 | 1.80024 | 4.97277 | 2.76228 |
| 1.45 | 1.81303 | 5.01464 | 2.76589 |
| 1.46 | 1.82596 | 5.05687 | 2.76943 |
| 1.47 | 1.83903 | 5.09945 | 2.77291 |
| 1.48 | 1.85223 | 5.14241 | 2.77633 |
| 1.49 | 1.86558 | 5.18573 | 2.77969 |
| 1.50 | 1.87907 | 5.22942 | 2.78299 |

TABLE D.VA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 3/4$ and $x$ from 0.00 to 1.50.

$\alpha = 3/4$

| x | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 1.50 | 1.87907 | 5.22942 | 2.78299 |
| 1.51 | 1.89270 | 5.27350 | 2.78623 |
| 1.52 | 1.90648 | 5.31795 | 2.78941 |
| 1.53 | 1.92040 | 5.36279 | 2.79254 |
| 1.54 | 1.93446 | 5.40801 | 2.79561 |
| 1.55 | 1.94868 | 5.45363 | 2.79863 |
| 1.56 | 1.96304 | 5.49965 | 2.80160 |
| 1.57 | 1.97755 | 5.54607 | 2.80451 |
| 1.58 | 1.99222 | 5.59289 | 2.80737 |
| 1.59 | 2.00704 | 5.64013 | 2.81018 |
| 1.60 | 2.02201 | 5.68778 | 2.81294 |
| 1.61 | 2.03713 | 5.73585 | 2.81565 |
| 1.62 | 2.05242 | 5.78434 | 2.81831 |
| 1.63 | 2.06786 | 5.83326 | 2.82092 |
| 1.64 | 2.08346 | 5.88261 | 2.82349 |
| 1.65 | 2.09922 | 5.93241 | 2.62601 |
| 1.66 | 2.11514 | 5.98264 | 2.82849 |
| 1.67 | 2.13122 | 6.03332 | 2.83092 |
| 1.68 | 2.14747 | 6.08445 | 2.83331 |
| 1.69 | 2.16389 | 6.13603 | 2.83565 |
| 1.70 | 2.18047 | 6.18808 | 2.83795 |
| 1.71 | 2.19723 | 6.24059 | 2.84021 |
| 1.72 | 2.21415 | 6.29357 | 2.84243 |
| 1.73 | 2.23124 | 6.34703 | 2.84461 |
| 1.74 | 2.24851 | 6.40096 | 2.84675 |
| 1.75 | 2.26596 | 6.45538 | 2.84885 |
| 1.76 | 2.28358 | 6.51029 | 2.85092 |
| 1.77 | 2.30138 | 6.56569 | 2.85294 |
| 1.78 | 2.31935 | 6.62159 | 2.85493 |
| 1.79 | 2.33751 | 6.67800 | 2.85688 |
| 1.80 | 2.35585 | 6.73492 | 2.85880 |
| 1.81 | 2.37438 | 6.79235 | 2.86068 |
| 1.82 | 2.39309 | 6.85030 | 2.86253 |
| 1.83 | 2.41199 | 6.90877 | 2.86434 |
| 1.84 | 2.43108 | 6.96778 | 2.86612 |
| 1.85 | 2.45036 | 7.02732 | 2.85787 |
| 1.86 | 2.46983 | 7.08741 | 2.86959 |
| 1.87 | 2.48950 | 7.14804 | 2.87127 |
| 1.88 | 2.50937 | 7.20922 | 2.87292 |
| 1.89 | 2.52943 | 7.27096 | 2.87455 |
| 1.90 | 2.54969 | 7.33326 | 2.87614 |
| 1.91 | 2.57015 | 7.39613 | 2.87770 |
| 1.92 | 2.59081 | 7.45957 | 2.87924 |
| 1.93 | 2.61168 | 7.52360 | 2.88075 |
| 1.94 | 2.63276 | 7.58821 | 2.88222 |
| 1.95 | 2.65405 | 7.65341 | 2.88368 |
| 1.96 | 2.67554 | 7.71921 | 2.88510 |
| 1.97 | 2.69725 | 7.78562 | 2.88650 |
| 1.98 | 2.71918 | 7.85263 | 2.88787 |
| 1.99 | 2.74132 | 7.92025 | 2.88922 |
| 2.00 | 2.76367 | 7.98850 | 2.89054 |

| x | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 2.0 | 2.76367 | 7.98850 | 2.89054 |
| 2.1 | 2.99970 | 8.70654 | 2.90247 |
| 2.2 | 3.25984 | 9.49380 | 2.91235 |
| 2.3 | 3.54639 | 10.35732 | 2.92052 |
| 2.4 | 3.86190 | 11.30482 | 2.92727 |
| 2.5 | 4.20915 | 12.34477 | 2.93284 |
| 2.6 | 4.59124 | 13.48645 | 2.93743 |
| 2.7 | 5.01157 | 14.74009 | 2.94121 |
| 2.8 | 5.47389 | 16.11690 | 2.94433 |
| 2.9 | 5.98232 | 17.62923 | 2.94689 |
| 3.0 | 6.54143 | 19.29064 | 2.94900 |
| 3.1 | 7.15622 | 21.11607 | 2.95073 |
| 3.2 | 7.83223 | 23.12194 | 2.95215 |
| 3.3 | 8.57555 | 25.32635 | 2.95332 |
| 3.4 | 9.39287 | 27.74918 | 2.95428 |
| 3.5 | 10.29158 | 30.41235 | 2.95507 |
| 3.6 | 11.27981 | 33.33997 | 2.95572 |
| 3.7 | 12.36654 | 36.55859 | 2.95625 |
| 3.8 | 13.56162 | 40.09746 | 2.95669 |
| 3.9 | 14.87592 | 43.98876 | 2.95705 |
| 4.0 | 16.32142 | 48.26796 | 2.95734 |
| 4.1 | 17.91130 | 52.97409 | 2.95758 |
| 4.2 | 19.66009 | 58.15016 | 2.95778 |
| 4.3 | 21.58379 | 63.84353 | 2.95794 |
| 4.4 | 23.70002 | 70.10637 | 2.95807 |
| 4.5 | 26.02821 | 75.99614 | 2.95818 |
| 4.6 | 28.58973 | 84.57615 | 2.95827 |
| 4.7 | 31.40816 | 92.91611 | 2.95834 |
| 4.8 | 34.50945 | 102.09286 | 2.95840 |
| 4.9 | 37.92220 | 112.19102 | 2.95845 |
| 5.0 | 41.67792 | 123.30382 | 2.95849 |
| 5.1 | 45.81133 | 135.53399 | 2.95853 |
| 5.2 | 50.36069 | 148.99473 | 2.95855 |
| 5.3 | 55.36813 | 163.81073 | 2.95857 |
| 5.4 | 60.88010 | 180.11942 | 2.95859 |
| 5.5 | 66.94778 | 198.07221 | 2.95861 |
| 5.6 | 73.62756 | 217.83597 | 2.95862 |
| 5.7 | 80.98160 | 239.59457 | 2.95863 |
| 5.8 | 89.07837 | 263.55064 | 2.95864 |
| 5.9 | 97.99335 | 289.92750 | 2.95864 |
| 6.0 | 107.80972 | 318.97126 | 2.95865 |

| x | $F_{3/4}(x)$ | $H_{1/4}(x)$ | $T_{3/4}(x)$ |
|---|---|---|---|
| 6.0 | 107.80972 | 318.97126 | 2.95865 |
| 6.1 | 118.61917 | 350.95316 | 2.95865 |
| 6.2 | 130.52272 | 386.17215 | 2.95866 |
| 6.3 | 143.63175 | 424.95772 | 2.95866 |
| 6.4 | 158.06900 | 467.67305 | 2.95866 |
| 6.5 | 173.96977 | 514.71843 | 2.95867 |
| 6.6 | 191.48319 | 566.53511 | 2.95867 |
| 6.7 | 210.77365 | 623.60946 | 2.95867 |
| 6.8 | 232.02236 | 686.47762 | 2.95867 |
| 6.9 | 255.42908 | 755.73062 | 2.95867 |
| 7.0 | 281.21402 | 832.01997 | 2.95867 |
| 7.1 | 309.61993 | 916.06393 | 2.95867 |
| 7.2 | 340.91444 | 1008.65430 | 2.95867 |
| 7.3 | 375.39259 | 1110.66402 | 2.95867 |
| 7.4 | 413.37966 | 1223.05550 | 2.95867 |
| 7.5 | 455.23428 | 1346.88978 | 2.95867 |
| 7.6 | 501.35184 | 1483.33670 | 2.95867 |
| 7.7 | 552.16828 | 1633.68608 | 2.95867 |
| 7.8 | 608.16425 | 1799.36002 | 2.95867 |
| 7.9 | 669.86974 | 1981.92656 | 2.95867 |
| 8.0 | 737.86911 | 2183.11464 | 2.95867 |
| 8.1 | 812.80670 | 2404.83066 | 2.95867 |
| 8.2 | 895.39301 | 2649.17676 | 2.95867 |
| 8.3 | 986.41151 | 2918.47093 | 2.95867 |
| 8.4 | 1086.72611 | 3215.26927 | 2.95867 |
| 8.5 | 1197.28949 | 3542.39043 | 2.95867 |
| 8.6 | 1319.15223 | 3902.94271 | 2.95867 |
| 8.7 | 1453.47287 | 4300.35785 | 2.95868 |
| 8.8 | 1601.52902 | 4738.40392 | 2.95868 |
| 8.9 | 1764.72967 | 5221.26163 | 2.95868 |
| 9.0 | 1944.62868 | 5753.52437 | 2.95868 |
| 9.1 | 2142.93973 | 6340.26235 | 2.95868 |
| 9.2 | 2361.55278 | 6987.06735 | 2.95868 |
| 9.3 | 2602.55224 | 7700.10640 | 2.95868 |
| 9.4 | 2868.23703 | 8486.18146 | 2.95868 |
| 9.5 | 3161.14265 | 9352.79405 | 2.95868 |
| 9.6 | 3484.06564 | 10308.21826 | 2.95868 |
| 9.7 | 3840.09044 | 11361.57997 | 2.95868 |
| 9.8 | 4232.61910 | 12522.94477 | 2.95868 |
| 9.9 | 4665.40409 | 13803.41494 | 2.95868 |
| 10.0 | 5142.58436 | 15215.23636 | 2.95868 |

TABLE D.VB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/4$ and x from 1.50 to 10.0.

$\alpha = 1/5$

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00013 | 0.00026 | 0.00026 |
| 0.02 | 1.00050 | 0.00079 | 0.00079 |
| 0.03 | 1.00113 | 0.00151 | 0.00151 |
| 0.04 | 1.00200 | 0.00239 | 0.00239 |
| 0.05 | 1.00313 | 0.00342 | 0.00341 |
| 0.06 | 1.00450 | 0.00458 | 0.00456 |
| 0.07 | 1.00613 | 0.00586 | 0.00582 |
| 0.08 | 1.00801 | 0.00725 | 0.00720 |
| 0.09 | 1.01013 | 0.00876 | 0.00867 |
| 0.10 | 1.01251 | 0.01037 | 0.01024 |
| 0.11 | 1.01514 | 0.01208 | 0.01190 |
| 0.12 | 1.01803 | 0.01389 | 0.01365 |
| 0.13 | 1.02116 | 0.01580 | 0.01547 |
| 0.14 | 1.02455 | 0.01779 | 0.01737 |
| 0.15 | 1.02819 | 0.01988 | 0.01933 |
| 0.16 | 1.03209 | 0.02205 | 0.02136 |
| 0.17 | 1.03623 | 0.02431 | 0.02346 |
| 0.18 | 1.04064 | 0.02665 | 0.02561 |
| 0.19 | 1.04529 | 0.02907 | 0.02781 |
| 0.20 | 1.05021 | 0.03157 | 0.03006 |
| 0.21 | 1.05538 | 0.03416 | 0.03236 |
| 0.22 | 1.06081 | 0.03682 | 0.03471 |
| 0.23 | 1.06649 | 0.03956 | 0.03709 |
| 0.24 | 1.07243 | 0.04237 | 0.03951 |
| 0.25 | 1.07863 | 0.04526 | 0.04196 |
| 0.26 | 1.08510 | 0.04823 | 0.04444 |
| 0.27 | 1.09182 | 0.05127 | 0.04696 |
| 0.28 | 1.09880 | 0.05438 | 0.04949 |
| 0.29 | 1.10605 | 0.05757 | 0.05205 |
| 0.30 | 1.11356 | 0.06082 | 0.05462 |
| 0.31 | 1.12133 | 0.06415 | 0.05721 |
| 0.32 | 1.12937 | 0.06756 | 0.05982 |
| 0.33 | 1.13768 | 0.07103 | 0.06243 |
| 0.34 | 1.14625 | 0.07457 | 0.06506 |
| 0.35 | 1.15509 | 0.07819 | 0.06769 |
| 0.36 | 1.16420 | 0.08187 | 0.07033 |
| 0.37 | 1.17358 | 0.08563 | 0.07296 |
| 0.38 | 1.18323 | 0.08945 | 0.07560 |
| 0.39 | 1.19315 | 0.09335 | 0.07824 |
| 0.40 | 1.20335 | 0.09731 | 0.08087 |
| 0.41 | 1.21383 | 0.10135 | 0.08349 |
| 0.42 | 1.22458 | 0.10545 | 0.08611 |
| 0.43 | 1.23561 | 0.10963 | 0.08872 |
| 0.44 | 1.24692 | 0.11387 | 0.09132 |
| 0.45 | 1.25851 | 0.11818 | 0.09391 |
| 0.46 | 1.27038 | 0.12257 | 0.09648 |
| 0.47 | 1.28253 | 0.12702 | 0.09904 |
| 0.48 | 1.29497 | 0.13154 | 0.10158 |
| 0.49 | 1.30770 | 0.13614 | 0.10410 |
| 0.50 | 1.32072 | 0.14080 | 0.10661 |

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 0.50 | 1.32072 | 0.14080 | 0.10661 |
| 0.51 | 1.33402 | 0.14553 | 0.10909 |
| 0.52 | 1.34762 | 0.15034 | 0.11156 |
| 0.53 | 1.36151 | 0.15521 | 0.11400 |
| 0.54 | 1.37569 | 0.16016 | 0.11642 |
| 0.55 | 1.39016 | 0.16518 | 0.11882 |
| 0.56 | 1.40496 | 0.17027 | 0.12119 |
| 0.57 | 1.42004 | 0.17543 | 0.12354 |
| 0.58 | 1.43542 | 0.18066 | 0.12586 |
| 0.59 | 1.45111 | 0.18597 | 0.12816 |
| 0.60 | 1.46711 | 0.19135 | 0.13043 |
| 0.61 | 1.48341 | 0.19680 | 0.13267 |
| 0.62 | 1.50002 | 0.20233 | 0.13488 |
| 0.63 | 1.51695 | 0.20793 | 0.13707 |
| 0.64 | 1.53419 | 0.21360 | 0.13923 |
| 0.65 | 1.55174 | 0.21935 | 0.14136 |
| 0.66 | 1.56962 | 0.22518 | 0.14346 |
| 0.67 | 1.58761 | 0.23108 | 0.14553 |
| 0.68 | 1.60633 | 0.23706 | 0.14758 |
| 0.69 | 1.62518 | 0.24311 | 0.14959 |
| 0.70 | 1.64435 | 0.24924 | 0.15158 |
| 0.71 | 1.66385 | 0.25545 | 0.15353 |
| 0.72 | 1.68369 | 0.26174 | 0.15546 |
| 0.73 | 1.70386 | 0.26811 | 0.15735 |
| 0.74 | 1.72436 | 0.27455 | 0.15922 |
| 0.75 | 1.74521 | 0.28108 | 0.16106 |
| 0.76 | 1.76640 | 0.28769 | 0.16287 |
| 0.77 | 1.78794 | 0.29437 | 0.16464 |
| 0.78 | 1.80982 | 0.30114 | 0.16639 |
| 0.79 | 1.83205 | 0.30800 | 0.16811 |
| 0.80 | 1.85464 | 0.31493 | 0.16981 |
| 0.81 | 1.87759 | 0.32195 | 0.17147 |
| 0.82 | 1.90089 | 0.32905 | 0.17311 |
| 0.83 | 1.92455 | 0.33624 | 0.17471 |
| 0.84 | 1.94858 | 0.34352 | 0.17629 |
| 0.85 | 1.97298 | 0.35088 | 0.17784 |
| 0.86 | 1.99775 | 0.35833 | 0.17937 |
| 0.87 | 2.02289 | 0.36587 | 0.18086 |
| 0.88 | 2.04841 | 0.37349 | 0.18233 |
| 0.89 | 2.07431 | 0.38121 | 0.18378 |
| 0.90 | 2.10059 | 0.38901 | 0.18519 |
| 0.91 | 2.12726 | 0.39691 | 0.18658 |
| 0.92 | 2.15432 | 0.40490 | 0.18795 |
| 0.93 | 2.18177 | 0.41298 | 0.18929 |
| 0.94 | 2.20962 | 0.42116 | 0.19060 |
| 0.95 | 2.23787 | 0.42943 | 0.19189 |
| 0.96 | 2.26652 | 0.43779 | 0.19316 |
| 0.97 | 2.29558 | 0.44626 | 0.19440 |
| 0.98 | 2.32505 | 0.45482 | 0.19562 |
| 0.99 | 2.35494 | 0.46347 | 0.19681 |
| 1.00 | 2.38524 | 0.47223 | 0.19798 |

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 1.00 | 2.38524 | 0.47223 | 0.19798 |
| 1.01 | 2.41596 | 0.48109 | 0.19913 |
| 1.02 | 2.44711 | 0.49005 | 0.20025 |
| 1.03 | 2.47869 | 0.49911 | 0.20136 |
| 1.04 | 2.51070 | 0.50827 | 0.20244 |
| 1.05 | 2.54315 | 0.51753 | 0.20350 |
| 1.06 | 2.57604 | 0.52691 | 0.20454 |
| 1.07 | 2.60937 | 0.53638 | 0.20556 |
| 1.08 | 2.64315 | 0.54597 | 0.20656 |
| 1.09 | 2.67739 | 0.55566 | 0.20754 |
| 1.10 | 2.71209 | 0.56547 | 0.20850 |
| 1.11 | 2.74724 | 0.57538 | 0.20944 |
| 1.12 | 2.78286 | 0.58540 | 0.21036 |
| 1.13 | 2.81895 | 0.59554 | 0.21126 |
| 1.14 | 2.85552 | 0.60579 | 0.21215 |
| 1.15 | 2.89257 | 0.61616 | 0.21301 |
| 1.16 | 2.93010 | 0.62664 | 0.21386 |
| 1.17 | 2.96812 | 0.63723 | 0.21469 |
| 1.18 | 3.00663 | 0.64795 | 0.21551 |
| 1.19 | 3.04564 | 0.65879 | 0.21631 |
| 1.20 | 3.08515 | 0.66975 | 0.21709 |
| 1.21 | 3.12517 | 0.68082 | 0.21785 |
| 1.22 | 3.16570 | 0.69203 | 0.21860 |
| 1.23 | 3.20675 | 0.70335 | 0.21934 |
| 1.24 | 3.24832 | 0.71481 | 0.2200[illegible] |
| 1.25 | 3.29042 | 0.72639 | 0.22076 |
| 1.26 | 3.33305 | 0.73810 | 0.22145 |
| 1.27 | 3.37621 | 0.74994 | 0.22212 |
| 1.28 | 3.41992 | 0.76191 | 0.22278 |
| 1.29 | 3.46418 | 0.77401 | 0.22343 |
| 1.30 | 3.50900 | 0.78624 | 0.22407 |
| 1.31 | 3.55437 | 0.79862 | 0.22469 |
| 1.32 | 3.60030 | 0.81113 | 0.22529 |
| 1.33 | 3.64681 | 0.82377 | 0.22589 |
| 1.34 | 3.69389 | 0.83656 | 0.22647 |
| 1.35 | 3.74156 | 0.84949 | 0.22704 |
| 1.36 | 3.78981 | 0.86256 | 0.22760 |
| 1.37 | 3.83865 | 0.87577 | 0.22815 |
| 1.38 | 3.88809 | 0.88913 | 0.22868 |
| 1.39 | 3.93814 | 0.90264 | 0.22921 |
| 1.40 | 3.98880 | 0.91630 | 0.22972 |
| 1.41 | 4.04008 | 0.93011 | 0.23022 |
| 1.42 | 4.09198 | 0.94407 | 0.23071 |
| 1.43 | 4.14451 | 0.95818 | 0.23119 |
| 1.44 | 4.19767 | 0.97245 | 0.23166 |
| 1.45 | 4.25148 | 0.98688 | 0.23212 |
| 1.46 | 4.30594 | 1.00146 | 0.23258 |
| 1.47 | 4.36105 | 1.01620 | 0.23302 |
| 1.48 | 4.41683 | 1.03111 | 0.23345 |
| 1.49 | 4.47327 | 1.04618 | 0.23387 |
| 1.50 | 4.53040 | 1.06142 | 0.23429 |

TABLE D.VIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 1/5$ and $x$ from 0.00 to 1.50.

$\alpha = 1/5$

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 1.50 | 4.53040 | 1.06142 | 0.23429 |
| 1.51 | 4.58820 | 1.07682 | 0.23469 |
| 1.52 | 4.64669 | 1.09240 | 0.23509 |
| 1.53 | 4.70588 | 1.10814 | 0.23548 |
| 1.54 | 4.76578 | 1.12406 | 0.23586 |
| 1.55 | 4.82639 | 1.14015 | 0.23623 |
| 1.56 | 4.88771 | 1.15642 | 0.23660 |
| 1.57 | 4.94976 | 1.17286 | 0.23695 |
| 1.58 | 5.01255 | 1.18949 | 0.23730 |
| 1.59 | 5.07607 | 1.20630 | 0.23764 |
| 1.60 | 5.14035 | 1.22330 | 0.23798 |
| 1.61 | 5.20538 | 1.24048 | 0.23831 |
| 1.62 | 5.27117 | 1.25785 | 0.23863 |
| 1.63 | 5.33774 | 1.27541 | 0.23894 |
| 1.64 | 5.40509 | 1.29316 | 0.23925 |
| 1.65 | 5.47322 | 1.31111 | 0.23955 |
| 1.66 | 5.54216 | 1.32925 | 0.23984 |
| 1.67 | 5.61189 | 1.34759 | 0.24013 |
| 1.68 | 5.68244 | 1.36614 | 0.24041 |
| 1.69 | 5.75382 | 1.38488 | 0.24069 |
| 1.70 | 5.82602 | 1.40383 | 0.24096 |
| 1.71 | 5.89906 | 1.42299 | 0.24122 |
| 1.72 | 5.97295 | 1.44236 | 0.24148 |
| 1.73 | 6.04770 | 1.46195 | 0.24174 |
| 1.74 | 6.12331 | 1.48174 | 0.24198 |
| 1.75 | 6.19979 | 1.50175 | 0.24223 |
| 1.76 | 6.27716 | 1.52199 | 0.24246 |
| 1.77 | 6.35542 | 1.54244 | 0.24270 |
| 1.78 | 6.43459 | 1.56312 | 0.24292 |
| 1.79 | 6.51467 | 1.58402 | 0.24315 |
| 1.80 | 6.59566 | 1.60516 | 0.24337 |
| 1.81 | 6.67759 | 1.62652 | 0.24358 |
| 1.82 | 6.76046 | 1.64812 | 0.24379 |
| 1.83 | 6.84428 | 1.66995 | 0.24399 |
| 1.84 | 6.92907 | 1.69203 | 0.24419 |
| 1.85 | 7.01482 | 1.71434 | 0.24439 |
| 1.86 | 7.10155 | 1.73690 | 0.24458 |
| 1.87 | 7.18928 | 1.75971 | 0.24477 |
| 1.88 | 7.27801 | 1.78276 | 0.24495 |
| 1.89 | 7.36775 | 1.80607 | 0.24513 |
| 1.90 | 7.45851 | 1.82963 | 0.24531 |
| 1.91 | 7.55031 | 1.85346 | 0.24548 |
| 1.92 | 7.64315 | 1.87754 | 0.24565 |
| 1.93 | 7.73705 | 1.90188 | 0.24581 |
| 1.94 | 7.83201 | 1.92649 | 0.24598 |
| 1.55 | 7.92806 | 1.95137 | 0.24614 |
| 1.96 | 8.02519 | 1.97653 | 0.24629 |
| 1.97 | 8.12343 | 2.00195 | 0.24644 |
| 1.98 | 8.22278 | 2.02766 | 0.24659 |
| 1.99 | 8.32325 | 2.05365 | 0.24674 |
| 2.00 | 8.42486 | 2.07992 | 0.24688 |

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 2.0 | 8.42486 | 2.07992 | 0.24688 |
| 2.1 | 9.50623 | 2.35893 | 0.24815 |
| 2.2 | 10.71544 | 2.67002 | 0.24917 |
| 2.3 | 12.06679 | 3.01677 | 0.25001 |
| 2.4 | 13.57620 | 3.40328 | 0.25068 |
| 2.5 | 15.26139 | 3.83405 | 0.25123 |
| 2.6 | 17.14209 | 4.31413 | 0.25167 |
| 2.7 | 19.24023 | 4.84900 | 0.25203 |
| 2.8 | 21.58022 | 5.44513 | 0.25232 |
| 2.9 | 24.18919 | 6.10916 | 0.25256 |
| 3.0 | 27.09729 | 6.84885 | 0.25275 |
| 3.1 | 30.33805 | 7.67272 | 0.25291 |
| 3.2 | 33.94870 | 8.59022 | 0.25304 |
| 3.3 | 37.97065 | 9.61186 | 0.25314 |
| 3.4 | 42.44987 | 10.74932 | 0.25322 |
| 3.5 | 47.43742 | 12.01556 | 0.25329 |
| 3.6 | 52.99005 | 13.42498 | 0.25335 |
| 3.7 | 59.17075 | 14.99356 | 0.25339 |
| 3.8 | 66.04948 | 16.73907 | 0.25343 |
| 3.9 | 73.70395 | 18.68120 | 0.25346 |
| 4.0 | 82.22040 | 20.84185 | 0.25349 |
| 4.1 | 91.69462 | 23.24531 | 0.25351 |
| 4.2 | 102.23292 | 25.91854 | 0.25352 |
| 4.3 | 113.95332 | 28.89149 | 0.25354 |
| 4.4 | 126.98683 | 32.19738 | 0.25355 |
| 4.5 | 141.47886 | 35.87310 | 0.25356 |
| 4.6 | 157.59081 | 39.95957 | 0.25357 |
| 4.7 | 175.50179 | 44.50223 | 0.25357 |
| 4.8 | 195.41055 | 49.55148 | 0.25358 |
| 4.9 | 217.53767 | 55.16326 | 0.25358 |
| 5.0 | 242.12789 | 61.39965 | 0.25358 |
| 5.1 | 269.45279 | 68.32952 | 0.25359 |
| 5.2 | 299.81367 | 76.02927 | 0.25359 |
| 5.3 | 333.54482 | 84.58370 | 0.25359 |
| 5.4 | 371.01708 | 94.08684 | 0.25359 |
| 5.5 | 412.64189 | 104.64303 | 0.25359 |
| 5.6 | 458.87562 | 116.36803 | 0.25359 |
| 5.7 | 510.22452 | 129.39020 | 0.25359 |
| 5.8 | 567.25012 | 143.85195 | 0.25360 |
| 5.9 | 630.57526 | 159.91124 | 0.25360 |
| 6.0 | 700.89071 | 177.74325 | 0.25360 |

| $x$ | $F_{1/5}(x)$ | $H_{4/5}(x)$ | $T_{1/5}(x)$ |
|---|---|---|---|
| 6.0 | 700.89071 | 177.74325 | 0.25360 |
| 6.1 | 778.96261 | 197.54227 | 0.25360 |
| 6.2 | 865.64062 | 219.52377 | 0.25360 |
| 6.3 | 961.86701 | 243.97671 | 0.25360 |
| 6.4 | 1068.68667 | 271.01608 | 0.25360 |
| 6.5 | 1187.25832 | 301.08572 | 0.25360 |
| 6.6 | 1318.86678 | 334.46147 | 0.25360 |
| 6.7 | 1464.93670 | 371.50461 | 0.25360 |
| 6.8 | 1627.04773 | 412.61574 | 0.25360 |
| 6.9 | 1806.95125 | 458.23901 | 0.25360 |
| 7.0 | 2006.58906 | 508.86685 | 0.25360 |
| 7.1 | 2228.11398 | 565.04522 | 0.25360 |
| 7.2 | 2473.91267 | 627.37937 | 0.25360 |
| 7.3 | 2746.63101 | 696.54329 | 0.25360 |
| 7.4 | 3049.20213 | 773.27182 | 0.25360 |
| 7.5 | 3384.87750 | 858.39852 | 0.25360 |
| 7.6 | 3757.26135 | 952.83413 | 0.25360 |
| 7.7 | 4170.34888 | 1057.59271 | 0.25360 |
| 7.8 | 4628.56851 | 1173.79641 | 0.25360 |
| 7.9 | 5136.82869 | 1302.69030 | 0.25360 |
| 8.0 | 5700.56986 | 1445.65407 | 0.25360 |
| 8.1 | 6325.82188 | 1604.21688 | 0.25360 |
| 8.2 | 7019.26772 | 1780.07351 | 0.25360 |
| 8.3 | 7788.31405 | 1975.10229 | 0.25360 |
| 8.4 | 8641.16934 | 2191.38488 | 0.25360 |
| 8.5 | 9586.93047 | 2431.22821 | 0.25360 |
| 8.6 | 10635.67862 | 2697.18887 | 0.25360 |
| 8.7 | 11798.58552 | 2992.09998 | 0.25360 |
| 8.8 | 13088.03108 | 3319.10107 | 0.25360 |
| 8.9 | 14517.73379 | 3681.67111 | 0.25360 |
| 9.0 | 16102.89504 | 4083.66516 | 0.25360 |
| 9.1 | 17860.35910 | 4525.35490 | 0.25360 |
| 9.2 | 19808.79019 | 5023.47352 | 0.25360 |
| 9.3 | 21968.86871 | 5571.26556 | 0.25360 |
| 9.4 | 24363.50854 | 6178.54192 | 0.25360 |
| 9.5 | 27018.09769 | 6851.74096 | 0.25360 |
| 9.6 | 29950.76485 | 7597.99607 | 0.25360 |
| 9.7 | 33222.67468 | 8425.21055 | 0.25360 |
| 9.8 | 36838.35470 | 9342.14050 | 0.25360 |
| 9.9 | 40846.05746 | 10358.48672 | 0.25360 |
| 10.0 | 45288.16157 | 1148[illegible].99634 | 0.25360 |

TABLE D.VIB. LANCHESTER–CLIFFORD–SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 1/5$ and $x$ from 1.50 to 10.0.

$\alpha = 2/5$

| $x$ | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ | $x$ | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ | $x$ | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.15977 | 0.32826 | 0.28304 | 1.00 | 1.68278 | 0.84439 | 0.50178 |
| 0.01 | 1.00006 | 0.00289 | 0.00289 | 0.51 | 1.16637 | 0.33668 | 0.28865 | 1.01 | 1.69773 | 0.85707 | 0.50483 |
| 0.02 | 1.00025 | 0.00664 | 0.00663 | 0.52 | 1.17312 | 0.34516 | 0.29422 | 1.02 | 1.71288 | 0.86986 | 0.50784 |
| 0.03 | 1.00056 | 0.01079 | 0.01079 | 0.53 | 1.18001 | 0.35371 | 0.29975 | 1.03 | 1.72823 | 0.88276 | 0.51079 |
| 0.04 | 1.00100 | 0.01525 | 0.01523 | 0.54 | 1.18704 | 0.36233 | 0.30524 | 1.04 | 1.74378 | 0.89578 | 0.51370 |
| 0.05 | 1.00156 | 0.01993 | 0.01990 | 0.55 | 1.19422 | 0.37102 | 0.31068 | 1.05 | 1.75954 | 0.90891 | 0.51656 |
| 0.06 | 1.00225 | 0.02481 | 0.02475 | 0.56 | 1.20155 | 0.37977 | 0.31607 | 1.06 | 1.77551 | 0.92216 | 0.51938 |
| 0.07 | 1.00306 | 0.02986 | 0.02977 | 0.57 | 1.20902 | 0.38860 | 0.32141 | 1.07 | 1.79168 | 0.93553 | 0.52215 |
| 0.08 | 1.00400 | 0.03506 | 0.03492 | 0.58 | 1.21664 | 0.39749 | 0.32671 | 1.08 | 1.80806 | 0.94901 | 0.52488 |
| 0.09 | 1.00507 | 0.04039 | 0.04018 | 0.59 | 1.22441 | 0.40646 | 0.33196 | 1.09 | 1.82466 | 0.96261 | 0.52755 |
| 0.10 | 1.00626 | 0.04584 | 0.04556 | 0.60 | 1.23232 | 0.41549 | 0.33716 | 1.10 | 1.84146 | 0.97634 | 0.53020 |
| 0.11 | 1.00757 | 0.05142 | 0.05103 | 0.61 | 1.24039 | 0.42460 | 0.34231 | 1.11 | 1.85848 | 0.99019 | 0.53279 |
| 0.12 | 1.00901 | 0.05710 | 0.05659 | 0.62 | 1.24861 | 0.43378 | 0.34741 | 1.12 | 1.87572 | 1.00416 | 0.53535 |
| 0.13 | 1.01058 | 0.06288 | 0.06222 | 0.63 | 1.25698 | 0.44304 | 0.35246 | 1.13 | 1.89318 | 1.01826 | 0.53786 |
| 0.14 | 1.01227 | 0.06875 | 0.06792 | 0.64 | 1.26550 | 0.45237 | 0.35746 | 1.14 | 1.91086 | 1.03248 | 0.54032 |
| 0.15 | 1.01409 | 0.07472 | 0.07368 | 0.65 | 1.27417 | 0.46177 | 0.36241 | 1.15 | 1.92876 | 1.04683 | 0.54275 |
| 0.16 | 1.01604 | 0.08078 | 0.07950 | 0.66 | 1.28300 | 0.47125 | 0.36730 | 1.16 | 1.94688 | 1.06131 | 0.54514 |
| 0.17 | 1.01811 | 0.08692 | 0.08537 | 0.67 | 1.29198 | 0.48080 | 0.37214 | 1.17 | 1.96523 | 1.07593 | 0.54748 |
| 0.18 | 1.02031 | 0.09314 | 0.09129 | 0.68 | 1.30112 | 0.49043 | 0.37693 | 1.18 | 1.98381 | 1.09067 | 0.54979 |
| 0.19 | 1.02264 | 0.09944 | 0.09724 | 0.69 | 1.31042 | 0.50014 | 0.38166 | 1.19 | 2.00262 | 1.10555 | 0.55206 |
| 0.20 | 1.02509 | 0.10582 | 0.10323 | 0.70 | 1.31988 | 0.50992 | 0.38634 | 1.20 | 2.02166 | 1.12057 | 0.55428 |
| 0.21 | 1.02767 | 0.11227 | 0.10925 | 0.71 | 1.32949 | 0.51979 | 0.39097 | 1.21 | 2.04093 | 1.13572 | 0.55647 |
| 0.22 | 1.03038 | 0.11880 | 0.11529 | 0.72 | 1.33927 | 0.52973 | 0.39554 | 1.22 | 2.06044 | 1.15102 | 0.55863 |
| 0.23 | 1.03322 | 0.12539 | 0.12136 | 0.73 | 1.34921 | 0.53976 | 0.40006 | 1.23 | 2.08019 | 1.16645 | 0.56074 |
| 0.24 | 1.03619 | 0.13206 | 0.12745 | 0.74 | 1.35930 | 0.54987 | 0.40452 | 1.24 | 2.10018 | 1.18202 | 0.56282 |
| 0.25 | 1.03928 | 0.13880 | 0.13355 | 0.75 | 1.36957 | 0.56006 | 0.40893 | 1.25 | 2.12041 | 1.19774 | 0.56486 |
| 0.26 | 1.04251 | 0.14560 | 0.13966 | 0.76 | 1.37999 | 0.57033 | 0.41328 | 1.26 | 2.14089 | 1.21360 | 0.56687 |
| 0.27 | 1.04586 | 0.15247 | 0.14578 | 0.77 | 1.39059 | 0.58069 | 0.41758 | 1.27 | 2.16161 | 1.22961 | 0.56884 |
| 0.28 | 1.04934 | 0.15941 | 0.15191 | 0.78 | 1.40135 | 0.59113 | 0.42183 | 1.28 | 2.18258 | 1.24577 | 0.57078 |
| 0.29 | 1.05296 | 0.16641 | 0.15804 | 0.79 | 1.41227 | 0.60166 | 0.42602 | 1.29 | 2.20380 | 1.26208 | 0.57268 |
| 0.30 | 1.05670 | 0.17348 | 0.16417 | 0.80 | 1.42337 | 0.61227 | 0.43016 | 1.30 | 2.22528 | 1.27854 | 0.57455 |
| 0.31 | 1.06058 | 0.18061 | 0.17030 | 0.81 | 1.43464 | 0.62298 | 0.43424 | 1.31 | 2.24701 | 1.29515 | 0.57639 |
| 0.32 | 1.06459 | 0.18781 | 0.17642 | 0.82 | 1.44608 | 0.63377 | 0.43827 | 1.32 | 2.26900 | 1.31192 | 0.57819 |
| 0.33 | 1.06873 | 0.19507 | 0.18253 | 0.83 | 1.45769 | 0.64465 | 0.44224 | 1.33 | 2.29125 | 1.32884 | 0.57996 |
| 0.34 | 1.07300 | 0.20240 | 0.18863 | 0.84 | 1.46947 | 0.65562 | 0.44616 | 1.34 | 2.31377 | 1.34592 | 0.58170 |
| 0.35 | 1.07740 | 0.20979 | 0.19471 | 0.85 | 1.48143 | 0.66669 | 0.45003 | 1.35 | 2.33654 | 1.36316 | 0.58341 |
| 0.36 | 1.08194 | 0.21724 | 0.20079 | 0.86 | 1.49357 | 0.67784 | 0.45384 | 1.36 | 2.35959 | 1.38057 | 0.58509 |
| 0.37 | 1.08661 | 0.22475 | 0.20684 | 0.87 | 1.50588 | 0.68909 | 0.45760 | 1.37 | 2.38291 | 1.39814 | 0.58674 |
| 0.38 | 1.09142 | 0.23233 | 0.21287 | 0.88 | 1.51838 | 0.70044 | 0.46131 | 1.38 | 2.40650 | 1.41587 | 0.58835 |
| 0.39 | 1.09636 | 0.23997 | 0.21888 | 0.89 | 1.53105 | 0.71188 | 0.46496 | 1.39 | 2.43036 | 1.43377 | 0.58994 |
| 0.40 | 1.10144 | 0.24768 | 0.22487 | 0.90 | 1.54391 | 0.72342 | 0.46856 | 1.40 | 2.45450 | 1.45184 | 0.59150 |
| 0.41 | 1.10665 | 0.25545 | 0.23083 | 0.91 | 1.55695 | 0.73506 | 0.47211 | 1.41 | 2.47892 | 1.47009 | 0.59303 |
| 0.42 | 1.11200 | 0.26328 | 0.23676 | 0.92 | 1.57017 | 0.74679 | 0.47561 | 1.42 | 2.50363 | 1.48850 | 0.59454 |
| 0.43 | 1.11748 | 0.27118 | 0.24267 | 0.93 | 1.58358 | 0.75863 | 0.47906 | 1.43 | 2.52862 | 1.50709 | 0.59601 |
| 0.44 | 1.12311 | 0.27914 | 0.24854 | 0.94 | 1.59718 | 0.77057 | 0.48246 | 1.44 | 2.55390 | 1.52586 | 0.59746 |
| 0.45 | 1.12887 | 0.28716 | 0.25438 | 0.95 | 1.61096 | 0.78261 | 0.48580 | 1.45 | 2.57946 | 1.54480 | 0.59889 |
| 0.46 | 1.13477 | 0.29525 | 0.26019 | 0.96 | 1.62494 | 0.79475 | 0.48910 | 1.46 | 2.60533 | 1.56393 | 0.60028 |
| 0.47 | 1.14081 | 0.30340 | 0.26596 | 0.97 | 1.63911 | 0.80700 | 0.49234 | 1.47 | 2.63148 | 1.58324 | 0.60165 |
| 0.48 | 1.14699 | 0.31162 | 0.27169 | 0.98 | 1.65347 | 0.81936 | 0.49554 | 1.48 | 2.65794 | 1.60273 | 0.60300 |
| 0.49 | 1.15331 | 0.31991 | 0.27738 | 0.99 | 1.66802 | 0.83182 | 0.49868 | 1.49 | 2.68470 | 1.62241 | 0.60432 |
| 0.50 | 1.15977 | 0.32826 | 0.28304 | 1.00 | 1.68278 | 0.84439 | 0.50178 | 1.50 | 2.71176 | 1.64228 | 0.60561 |

TABLE D.VIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 2/5$ and $x$ from 0.00 to 1.50.

$\alpha = 2/5$

| x | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ |
|---|---|---|---|
| 1.50 | 2.71176 | 1.64228 | 0.60561 |
| 1.51 | 2.73913 | 1.66234 | 0.60689 |
| 1.52 | 2.76681 | 1.68259 | 0.60813 |
| 1.53 | 2.79481 | 1.70304 | 0.60936 |
| 1.54 | 2.82312 | 1.72369 | 0.61056 |
| 1.55 | 2.85175 | 1.74453 | 0.61174 |
| 1.56 | 2.88070 | 1.76558 | 0.61290 |
| 1.57 | 2.90991 | 1.78683 | 0.61403 |
| 1.58 | 2.93959 | 1.80828 | 0.61515 |
| 1.59 | 2.96952 | 1.82995 | 0.61624 |
| 1.60 | 2.99980 | 1.85182 | 0.61731 |
| 1.61 | 3.03041 | 1.87391 | 0.61837 |
| 1.62 | 3.06136 | 1.89621 | 0.61940 |
| 1.63 | 3.09266 | 1.91873 | 0.62041 |
| 1.64 | 3.12430 | 1.94147 | 0.62141 |
| 1.65 | 3.15630 | 1.96443 | 0.62238 |
| 1.66 | 3.18865 | 1.98761 | 0.62334 |
| 1.67 | 3.22136 | 2.01102 | 0.62428 |
| 1.68 | 3.25443 | 2.03466 | 0.62520 |
| 1.69 | 3.28786 | 2.05854 | 0.62610 |
| 1.70 | 3.32166 | 2.08264 | 0.62699 |
| 1.71 | 3.35584 | 2.10699 | 0.62786 |
| 1.72 | 3.39039 | 2.13157 | 0.62871 |
| 1.73 | 3.42532 | 2.15640 | 0.62955 |
| 1.74 | 3.46064 | 2.18146 | 0.63037 |
| 1.75 | 3.49634 | 2.20678 | 0.63117 |
| 1.76 | 3.53243 | 2.23235 | 0.63196 |
| 1.77 | 3.56892 | 2.25817 | 0.63273 |
| 1.78 | 3.60580 | 2.28424 | 0.63349 |
| 1.79 | 3.64309 | 2.31057 | 0.63423 |
| 1.80 | 3.68078 | 2.33716 | 0.63496 |
| 1.81 | 3.71888 | 2.36402 | 0.63568 |
| 1.82 | 3.75740 | 2.39114 | 0.63638 |
| 1.83 | 3.79634 | 2.41853 | 0.63707 |
| 1.84 | 3.83569 | 2.44620 | 0.63775 |
| 1.85 | 3.87548 | 2.47413 | 0.63841 |
| 1.86 | 3.91569 | 2.50235 | 0.63906 |
| 1.87 | 3.95634 | 2.53085 | 0.63969 |
| 1.88 | 3.99743 | 2.55963 | 0.64032 |
| 1.89 | 4.03897 | 2.58869 | 0.64093 |
| 1.90 | 4.08095 | 2.61805 | 0.64153 |
| 1.91 | 4.12338 | 2.64770 | 0.64212 |
| 1.92 | 4.16627 | 2.67764 | 0.64270 |
| 1.93 | 4.20963 | 2.70789 | 0.64326 |
| 1.94 | 4.25345 | 2.73844 | 0.64382 |
| 1.95 | 4.29774 | 2.76929 | 0.64436 |
| 1.96 | 4.34250 | 2.80045 | 0.64489 |
| 1.97 | 4.38775 | 2.83192 | 0.64542 |
| 1.98 | 4.43348 | 2.86371 | 0.64593 |
| 1.99 | 4.47970 | 2.89582 | 0.64643 |
| 2.0C | 4.52641 | 2.92825 | 0.64692 |

| x | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ |
|---|---|---|---|
| 2.0 | 4.52641 | 2.92825 | 0.64692 |
| 2.1 | 5.02188 | 3.27101 | 0.65135 |
| 2.2 | 5.57258 | 3.64995 | 0.65498 |
| 2.3 | 6.18434 | 4.06905 | 0.65796 |
| 2.4 | 6.86361 | 4.53271 | 0.66040 |
| 2.5 | 7.61753 | 5.04580 | 0.66239 |
| 2.6 | 8.45406 | 5.61371 | 0.66403 |
| 2.7 | 9.38197 | 6.24240 | 0.66536 |
| 2.8 | 10.41103 | 6.93848 | 0.66645 |
| 2.9 | 11.55204 | 7.70923 | 0.66735 |
| 3.0 | 12.81695 | 8.56274 | 0.66808 |
| 3.1 | 14.21903 | 9.50795 | 0.66868 |
| 3.2 | 15.77295 | 10.55473 | 0.66917 |
| 3.3 | 17.49496 | 11.71404 | 0.66957 |
| 3.4 | 19.40306 | 12.99799 | 0.66989 |
| 3.5 | 21.51718 | 14.41998 | 0.67016 |
| 3.6 | 23.85938 | 15.99486 | 0.67038 |
| 3.7 | 26.45408 | 17.73903 | 0.67056 |
| 3.8 | 29.32832 | 19.67067 | 0.67071 |
| 3.9 | 32.51202 | 21.80988 | 0.67083 |
| 4.0 | 36.03831 | 24.17894 | 0.67092 |
| 4.1 | 39.94386 | 26.80246 | 0.67100 |
| 4.2 | 44.26924 | 29.70771 | 0.67107 |
| 4.3 | 49.05939 | 32.92486 | 0.67112 |
| 4.4 | 54.36401 | 36.48730 | 0.67117 |
| 4.5 | 60.23811 | 40.43196 | 0.67120 |
| 4.6 | 66.74260 | 44.79975 | 0.67123 |
| 4.7 | 73.94486 | 49.63591 | 0.67126 |
| 4.8 | 81.91947 | 54.99052 | 0.67128 |
| 4.9 | 90.74895 | 60.91C99 | 0.67129 |
| 5.0 | 100.52464 | 67.48265 | 0.67130 |
| 5.1 | 111.34759 | 74.74935 | 0.67132 |
| 5.2 | 123.32965 | 82.79418 | 0.67132 |
| 5.3 | 136.59457 | 91.70023 | 0.67133 |
| 5.4 | 151.27930 | 101.55944 | 0.67134 |
| 5.5 | 167.53537 | 112.47355 | 0.67134 |
| 5.6 | 185.53044 | 124.55513 | 0.67135 |
| 5.7 | 205.45005 | 137.92875 | 0.67135 |
| 5.8 | 227.49948 | 152.73222 | 0.67135 |
| 5.9 | 251.90588 | 169.11805 | 0.67135 |
| 6.0 | 278.92057 | 187.25497 | 0.67136 |

| x | $F_{2/5}(x)$ | $H_{3/5}(x)$ | $T_{2/5}(x)$ |
|---|---|---|---|
| 6.0 | 278.92057 | 187.25497 | 0.67136 |
| 6.1 | 308.82162 | 207.32967 | 0.67136 |
| 6.2 | 341.91671 | 229.54871 | 0.67136 |
| 6.3 | 378.54625 | 254.14063 | 0.67136 |
| 6.4 | 419.08687 | 281.35829 | 0.67136 |
| 6.5 | 463.95527 | 311.48146 | 0.67136 |
| 6.6 | 513.61248 | 344.81964 | 0.67136 |
| 6.7 | 568.56856 | 381.71529 | 0.67136 |
| 6.8 | 629.38782 | 422.54725 | 0.67136 |
| 6.9 | 696.69459 | 467.73468 | 0.67136 |
| 7.0 | 771.17956 | 517.74129 | 0.67136 |
| 7.1 | 853.60686 | 573.08008 | 0.67136 |
| 7.2 | 946.82186 | 634.31861 | 0.67136 |
| 7.3 | 1045.75977 | 702.08474 | 0.67136 |
| 7.4 | 1157.45522 | 777.07309 | 0.67136 |
| 7.5 | 1281.05275 | 860.05207 | 0.67136 |
| 7.6 | 1417.81857 | 951.87175 | 0.67136 |
| 7.7 | 1569.15340 | 1053.47253 | 0.67136 |
| 7.8 | 1736.60676 | 1165.89472 | 0.67136 |
| 7.9 | 1921.89281 | 1290.28911 | 0.67136 |
| 8.0 | 2126.90775 | 1427.92878 | 0.67136 |
| 8.1 | 2353.74921 | 1580.22198 | 0.67136 |
| 8.2 | 2604.73758 | 1748.72653 | 0.67136 |
| 8.3 | 2882.43966 | 1935.16571 | 0.67136 |
| 8.4 | 3189.69479 | 2141.44573 | 0.67136 |
| 8.5 | 3529.64379 | 2369.67524 | 0.67136 |
| 8.6 | 3905.76091 | 2622.18672 | 0.67136 |
| 8.7 | 4321.88925 | 2901.56028 | 0.67136 |
| 8.8 | 4782.27981 | 3210.64990 | 0.67136 |
| 8.9 | 5291.63481 | 3552.61247 | 0.67136 |
| 9.0 | 5855.15553 | 3930.93995 | 0.67136 |
| 9.1 | 6478.59521 | 4349.49486 | 0.67136 |
| 9.2 | 7168.31763 | 4812.54961 | 0.67136 |
| 9.3 | 7931.36179 | 5324.82992 | 0.67136 |
| 9.4 | 8775.51354 | 5891.56294 | 0.67136 |
| 9.5 | 9709.38476 | 6518.53038 | 0.67136 |
| 9.6 | 10742.50093 | 7212.12729 | 0.67136 |
| 9.7 | 11885.39794 | 7979.42709 | 0.67136 |
| 9.8 | 13149.72926 | 8828.25351 | 0.67136 |
| 9.9 | 14548.38436 | 9767.26006 | 0.67136 |
| 10.0 | 16095.61973 | 10806.01804 | 0.67136 |

TABLE D.VIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 2/5$ and x from 1.50 to 10.0.

$\alpha = 3/5$

| x | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ | x | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ | x | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.10622 | 0.86199 | 0.77922 | 1.00 | 1.45028 | 1.70597 | 1.17630 |
| 0.01 | 1.00004 | 0.03607 | 0.03607 | 0.51 | 1.11060 | 0.87731 | 0.78994 | 1.01 | 1.46004 | 1.72536 | 1.18172 |
| 0.02 | 1.00017 | 0.06280 | 0.06279 | 0.52 | 1.11507 | 0.89265 | 0.80054 | 1.02 | 1.46993 | 1.74489 | 1.18706 |
| 0.03 | 1.00038 | 0.08687 | 0.08684 | 0.53 | 1.11963 | 0.90803 | 0.81100 | 1.03 | 1.47995 | 1.76456 | 1.19231 |
| 0.04 | 1.00067 | 0.10937 | 0.10929 | 0.54 | 1.12429 | 0.92343 | 0.82134 | 1.04 | 1.49009 | 1.78436 | 1.19748 |
| 0.05 | 1.00104 | 0.13076 | 0.13063 | 0.55 | 1.12905 | 0.93887 | 0.83156 | 1.05 | 1.50037 | 1.80431 | 1.20258 |
| 0.06 | 1.00150 | 0.15133 | 0.15110 | 0.56 | 1.13390 | 0.95435 | 0.84166 | 1.06 | 1.51077 | 1.82439 | 1.20759 |
| 0.07 | 1.00204 | 0.17123 | 0.17088 | 0.57 | 1.13885 | 0.96987 | 0.85163 | 1.07 | 1.52131 | 1.84463 | 1.21253 |
| 0.08 | 1.00267 | 0.19058 | 0.19008 | 0.58 | 1.14389 | 0.98544 | 0.86148 | 1.08 | 1.53198 | 1.86500 | 1.21738 |
| 0.09 | 1.00338 | 0.20948 | 0.20877 | 0.59 | 1.14903 | 1.00104 | 0.87121 | 1.09 | 1.54278 | 1.88553 | 1.22217 |
| 0.10 | 1.00417 | 0.22798 | 0.22703 | 0.60 | 1.15427 | 1.01670 | 0.88081 | 1.10 | 1.55371 | 1.90621 | 1.22687 |
| 0.11 | 1.00505 | 0.24613 | 0.24490 | 0.61 | 1.15960 | 1.03240 | 0.89030 | 1.11 | 1.56479 | 1.92704 | 1.23150 |
| 0.12 | 1.00601 | 0.26398 | 0.26241 | 0.62 | 1.16504 | 1.04815 | 0.89968 | 1.12 | 1.57599 | 1.94802 | 1.23606 |
| 0.13 | 1.00705 | 0.28157 | 0.27959 | 0.63 | 1.17057 | 1.06396 | 0.90893 | 1.13 | 1.58734 | 1.96917 | 1.24055 |
| 0.14 | 1.00818 | 0.29891 | 0.29648 | 0.64 | 1.17620 | 1.07983 | 0.91807 | 1.14 | 1.59882 | 1.99047 | 1.24496 |
| 0.15 | 1.00939 | 0.31603 | 0.31309 | 0.65 | 1.18193 | 1.09575 | 0.92709 | 1.15 | 1.61044 | 2.01193 | 1.24930 |
| 0.16 | 1.01069 | 0.33296 | 0.32944 | 0.66 | 1.18776 | 1.11174 | 0.93599 | 1.16 | 1.62221 | 2.03356 | 1.25357 |
| 0.17 | 1.01207 | 0.34971 | 0.34554 | 0.67 | 1.19370 | 1.12778 | 0.94478 | 1.17 | 1.63411 | 2.05535 | 1.25778 |
| 0.18 | 1.01353 | 0.36631 | 0.36141 | 0.68 | 1.19973 | 1.14389 | 0.95346 | 1.18 | 1.64616 | 2.07731 | 1.26191 |
| 0.19 | 1.01508 | 0.38275 | 0.37736 | 0.69 | 1.20587 | 1.16007 | 0.96202 | 1.19 | 1.65836 | 2.09944 | 1.26598 |
| 0.20 | 1.01672 | 0.39906 | 0.39250 | 0.70 | 1.21211 | 1.17632 | 0.97048 | 1.20 | 1.67069 | 2.12175 | 1.26998 |
| 0.21 | 1.01844 | 0.41525 | 0.40773 | 0.71 | 1.21845 | 1.19264 | 0.97882 | 1.21 | 1.68318 | 2.14423 | 1.27392 |
| 0.22 | 1.02024 | 0.43132 | 0.42276 | 0.72 | 1.22489 | 1.20903 | 0.98705 | 1.22 | 1.69581 | 2.16688 | 1.27779 |
| 0.23 | 1.02213 | 0.44729 | 0.43761 | 0.73 | 1.23145 | 1.22550 | 0.99517 | 1.23 | 1.70859 | 2.18972 | 1.28159 |
| 0.24 | 1.02411 | 0.46317 | 0.45227 | 0.74 | 1.23810 | 1.24204 | 1.00318 | 1.24 | 1.72152 | 2.21274 | 1.28534 |
| 0.25 | 1.02617 | 0.47897 | 0.46675 | 0.75 | 1.24486 | 1.25866 | 1.01109 | 1.25 | 1.73460 | 2.23594 | 1.28902 |
| 0.26 | 1.02832 | 0.49468 | 0.48106 | 0.76 | 1.25173 | 1.27537 | 1.01889 | 1.26 | 1.74784 | 2.25932 | 1.29264 |
| 0.27 | 1.03055 | 0.51032 | 0.49520 | 0.77 | 1.25870 | 1.29216 | 1.02658 | 1.27 | 1.76123 | 2.28290 | 1.29620 |
| 0.28 | 1.03287 | 0.52590 | 0.50917 | 0.78 | 1.26579 | 1.30903 | 1.03416 | 1.28 | 1.77477 | 2.30663 | 1.29970 |
| 0.29 | 1.03527 | 0.54142 | 0.52298 | 0.79 | 1.27298 | 1.32599 | 1.04165 | 1.29 | 1.78847 | 2.33063 | 1.30314 |
| 0.30 | 1.03776 | 0.55689 | 0.53662 | 0.80 | 1.28028 | 1.34304 | 1.04903 | 1.30 | 1.80233 | 2.35479 | 1.30653 |
| 0.31 | 1.04034 | 0.57231 | 0.55012 | 0.81 | 1.28769 | 1.36019 | 1.05630 | 1.31 | 1.81634 | 2.37914 | 1.30985 |
| 0.32 | 1.04301 | 0.58769 | 0.56345 | 0.82 | 1.29521 | 1.37742 | 1.06348 | 1.32 | 1.83052 | 2.40369 | 1.31312 |
| 0.33 | 1.04576 | 0.60303 | 0.57664 | 0.83 | 1.30284 | 1.39475 | 1.07055 | 1.33 | 1.84485 | 2.42845 | 1.31634 |
| 0.34 | 1.04860 | 0.61834 | 0.58968 | 0.84 | 1.31058 | 1.41218 | 1.07753 | 1.34 | 1.85935 | 2.45342 | 1.31950 |
| 0.35 | 1.05153 | 0.63362 | 0.60257 | 0.85 | 1.31843 | 1.42971 | 1.08440 | 1.35 | 1.87402 | 2.47859 | 1.32261 |
| 0.36 | 1.05455 | 0.64887 | 0.61531 | 0.86 | 1.32640 | 1.44734 | 1.09118 | 1.36 | 1.88885 | 2.50397 | 1.32566 |
| 0.37 | 1.05765 | 0.66411 | 0.62791 | 0.87 | 1.33448 | 1.46508 | 1.09786 | 1.37 | 1.90384 | 2.52956 | 1.32866 |
| 0.38 | 1.06085 | 0.67933 | 0.64037 | 0.88 | 1.34268 | 1.48292 | 1.10445 | 1.38 | 1.91901 | 2.55537 | 1.33161 |
| 0.39 | 1.06413 | 0.69454 | 0.65263 | 0.89 | 1.35099 | 1.50086 | 1.11094 | 1.39 | 1.93434 | 2.58140 | 1.33451 |
| 0.40 | 1.06750 | 0.70974 | 0.66486 | 0.90 | 1.35942 | 1.51892 | 1.11733 | 1.40 | 1.94985 | 2.60765 | 1.33736 |
| 0.41 | 1.07097 | 0.72494 | 0.67690 | 0.91 | 1.36797 | 1.53709 | 1.12363 | 1.41 | 1.96553 | 2.63412 | 1.34016 |
| 0.42 | 1.07452 | 0.74013 | 0.68880 | 0.92 | 1.37663 | 1.55537 | 1.12984 | 1.42 | 1.98138 | 2.66082 | 1.34291 |
| 0.43 | 1.07816 | 0.75532 | 0.70057 | 0.93 | 1.38541 | 1.57377 | 1.13596 | 1.43 | 1.99741 | 2.68775 | 1.34562 |
| 0.44 | 1.08189 | 0.77052 | 0.71220 | 0.94 | 1.39431 | 1.59229 | 1.14199 | 1.44 | 2.01361 | 2.71490 | 1.34828 |
| 0.45 | 1.08572 | 0.78573 | 0.72370 | 0.95 | 1.40333 | 1.61092 | 1.14792 | 1.45 | 2.03000 | 2.74229 | 1.35089 |
| 0.46 | 1.08963 | 0.80095 | 0.73506 | 0.96 | 1.41248 | 1.62968 | 1.15377 | 1.46 | 2.04656 | 2.76992 | 1.35345 |
| 0.47 | 1.09364 | 0.81618 | 0.74630 | 0.97 | 1.42174 | 1.64856 | 1.15953 | 1.47 | 2.06331 | 2.79779 | 1.35597 |
| 0.48 | 1.09774 | 0.83143 | 0.75740 | 0.98 | 1.43113 | 1.66757 | 1.16521 | 1.48 | 2.08024 | 2.82589 | 1.35845 |
| 0.49 | 1.10193 | 0.84670 | 0.76838 | 0.99 | 1.44064 | 1.68670 | 1.17080 | 1.49 | 2.09735 | 2.85425 | 1.36088 |
| 0.50 | 1.10622 | 0.86199 | 0.77922 | 1.00 | 1.45028 | 1.70597 | 1.17630 | 1.50 | 2.11465 | 2.88285 | 1.36327 |

TABLE D.VIIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and x from 0.00 to 1.50.

$\alpha = 3/5$

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 1.50 | 2.11465 | 2.88285 | 1.36327 |
| 1.51 | 2.13214 | 2.91169 | 1.36562 |
| 1.52 | 2.14982 | 2.94080 | 1.36792 |
| 1.53 | 2.16770 | 2.97015 | 1.37019 |
| 1.54 | 2.18576 | 2.99977 | 1.37241 |
| 1.55 | 2.20402 | 3.02965 | 1.37460 |
| 1.56 | 2.22248 | 3.05979 | 1.37675 |
| 1.57 | 2.24113 | 3.09020 | 1.37886 |
| 1.58 | 2.25999 | 3.12088 | 1.38093 |
| 1.59 | 2.27904 | 3.15183 | 1.38296 |
| 1.60 | 2.29830 | 3.18306 | 1.38496 |
| 1.61 | 2.31777 | 3.21456 | 1.38692 |
| 1.62 | 2.33744 | 3.24635 | 1.38885 |
| 1.63 | 2.35733 | 3.27843 | 1.39074 |
| 1.64 | 2.37742 | 3.31079 | 1.39260 |
| 1.65 | 2.39773 | 3.34344 | 1.39442 |
| 1.66 | 2.41825 | 3.37639 | 1.39621 |
| 1.67 | 2.43899 | 3.40964 | 1.39797 |
| 1.68 | 2.45995 | 3.44319 | 1.39970 |
| 1.69 | 2.48113 | 3.47704 | 1.40139 |
| 1.70 | 2.50253 | 3.51120 | 1.40306 |
| 1.71 | 2.52416 | 3.54567 | 1.40469 |
| 1.72 | 2.54601 | 3.58045 | 1.40630 |
| 1.73 | 2.56810 | 3.61555 | 1.40787 |
| 1.74 | 2.59041 | 3.65098 | 1.40942 |
| 1.75 | 2.61296 | 3.68672 | 1.41094 |
| 1.76 | 2.63574 | 3.72280 | 1.41243 |
| 1.77 | 2.65876 | 3.75920 | 1.41389 |
| 1.78 | 2.68202 | 3.79594 | 1.41533 |
| 1.79 | 2.70553 | 3.83302 | 1.41674 |
| 1.80 | 2.72927 | 3.87044 | 1.41812 |
| 1.81 | 2.75327 | 3.90820 | 1.41948 |
| 1.82 | 2.77751 | 3.94632 | 1.42081 |
| 1.83 | 2.80200 | 3.98478 | 1.42212 |
| 1.84 | 2.82675 | 4.02361 | 1.42341 |
| 1.85 | 2.85175 | 4.06279 | 1.42467 |
| 1.86 | 2.87701 | 4.10234 | 1.42590 |
| 1.87 | 2.90253 | 4.14225 | 1.42712 |
| 1.88 | 2.92831 | 4.18253 | 1.42831 |
| 1.89 | 2.95436 | 4.22320 | 1.42948 |
| 1.90 | 2.98067 | 4.26423 | 1.43063 |
| 1.91 | 3.00726 | 4.30566 | 1.43175 |
| 1.92 | 3.03412 | 4.34747 | 1.43286 |
| 1.93 | 3.06125 | 4.38967 | 1.43394 |
| 1.94 | 3.08866 | 4.43226 | 1.43501 |
| 1.95 | 3.11636 | 4.47526 | 1.43605 |
| 1.96 | 3.14433 | 4.51865 | 1.43708 |
| 1.97 | 3.17259 | 4.56246 | 1.43809 |
| 1.98 | 3.20114 | 4.60668 | 1.43907 |
| 1.99 | 3.22998 | 4.65131 | 1.44004 |
| 2.00 | 3.25912 | 4.69636 | 1.44099 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 2.0 | 3.25912 | 4.69636 | 1.44099 |
| 2.1 | 3.56725 | 5.17096 | 1.44957 |
| 2.2 | 3.90799 | 5.69257 | 1.45665 |
| 2.3 | 4.28459 | 6.26617 | 1.46249 |
| 2.4 | 4.70063 | 6.89726 | 1.46731 |
| 2.5 | 5.16006 | 7.59185 | 1.47127 |
| 2.6 | 5.66727 | 8.35659 | 1.47453 |
| 2.7 | 6.22710 | 9.19878 | 1.47722 |
| 2.8 | 6.84409 | 10.12649 | 1.47942 |
| 2.9 | 7.52655 | 11.14858 | 1.48123 |
| 3.0 | 8.27860 | 12.27485 | 1.48272 |
| 3.1 | 9.10823 | 13.51608 | 1.48394 |
| 3.2 | 10.02339 | 14.88417 | 1.48494 |
| 3.3 | 11.03285 | 16.39224 | 1.48577 |
| 3.4 | 12.14631 | 18.05476 | 1.48644 |
| 3.5 | 13.37444 | 19.88770 | 1.48699 |
| 3.6 | 14.72906 | 21.90868 | 1.48745 |
| 3.7 | 16.22317 | 24.13713 | 1.48782 |
| 3.8 | 17.87117 | 26.59449 | 1.48812 |
| 3.9 | 19.68890 | 29.30442 | 1.48837 |
| 4.0 | 21.69389 | 32.29303 | 1.48858 |
| 4.1 | 23.90546 | 35.58913 | 1.48874 |
| 4.2 | 26.34495 | 39.22452 | 1.48888 |
| 4.3 | 29.03589 | 43.23429 | 1.48899 |
| 4.4 | 32.00427 | 47.65715 | 1.48909 |
| 4.5 | 35.27878 | 52.53584 | 1.48916 |
| 4.6 | 38.89107 | 57.91754 | 1.48922 |
| 4.7 | 42.87609 | 63.85430 | 1.48928 |
| 4.8 | 47.27239 | 70.40357 | 1.48932 |
| 4.9 | 52.12255 | 77.62878 | 1.48935 |
| 5.0 | 57.47356 | 85.59991 | 1.48938 |
| 5.1 | 63.37728 | 94.39423 | 1.48940 |
| 5.2 | 69.89096 | 104.09703 | 1.48942 |
| 5.3 | 77.07779 | 114.80242 | 1.48944 |
| 5.4 | 85.00753 | 126.61432 | 1.48945 |
| 5.5 | 93.7[illegible] | 139.64762 | 1.48946 |
| 5.6 | 103.4[illegible]70 | 154.02832 | 1.48947 |
| 5.7 | 114.06496 | 169.89679 | 1.48947 |
| 5.8 | 125.82052 | 187.40709 | 1.48948 |
| 5.9 | 138.79270 | 206.72954 | 1.48948 |
| 6.0 | 153.10773 | 228.05212 | 1.48949 |

| $x$ | $F_{3/5}(x)$ | $H_{2/5}(x)$ | $T_{3/5}(x)$ |
|---|---|---|---|
| 6.0 | 153.10773 | 228.05212 | 1.48949 |
| 6.1 | 168.90493 | 251.58239 | 1.48949 |
| 6.2 | 186.33813 | 277.54946 | 1.48949 |
| 6.3 | 205.37712 | 306.23623 | 1.48950 |
| 6.4 | 226.80931 | 337.85198 | 1.48950 |
| 6.5 | 250.24183 | 372.73492 | 1.48950 |
| 6.6 | 276.10298 | 411.25538 | 1.48950 |
| 6.7 | 304.64504 | 453.76907 | 1.48950 |
| 6.8 | 336.14669 | 500.69077 | 1.48950 |
| 6.9 | 370.91477 | 552.47842 | 1.48950 |
| 7.0 | 409.28931 | 609.63759 | 1.48950 |
| 7.1 | 451.64484 | 672.72646 | 1.48950 |
| 7.2 | 498.39511 | 742.36131 | 1.48950 |
| 7.3 | 549.99691 | 819.22253 | 1.48950 |
| 7.4 | 606.95457 | 904.06134 | 1.48950 |
| 7.5 | 669.82494 | 997.70714 | 1.48950 |
| 7.6 | 739.22287 | 1101.07560 | 1.48950 |
| 7.7 | 815.82701 | 1215.17807 | 1.48950 |
| 7.8 | 900.38700 | 1341.13066 | 1.48950 |
| 7.9 | 993.73032 | 1480.15606 | 1.48950 |
| 8.0 | 1096.77069 | 1633.64526 | 1.48950 |
| 8.1 | 1210.51656 | 1803.07095 | 1.48950 |
| 8.2 | 1336.08310 | 1990.10239 | 1.48950 |
| 8.3 | 1474.69908 | 2196.57164 | 1.48950 |
| 8.4 | 1627.72303 | 2424.50162 | 1.48951 |
| 8.5 | 1796.65455 | 2676.12602 | 1.48951 |
| 8.6 | 1983.14952 | 2953.91126 | 1.48951 |
| 8.7 | 2189.03632 | 3260.58073 | 1.48951 |
| 8.8 | 2416.33389 | 3599.14166 | 1.48951 |
| 8.9 | 2667.27158 | 3972.71446 | 1.48951 |
| 9.0 | 2944.31108 | 4385.56663 | 1.48951 |
| 9.1 | 3250.17066 | 4841.14586 | 1.48951 |
| 9.2 | 3587.85195 | 5344.12390 | 1.48951 |
| 9.3 | 3960.66550 | 5899.43758 | 1.48951 |
| 9.4 | 4372.28341 | 6512.53063 | 1.48951 |
| 9.5 | 4826.73538 | 7189.44720 | 1.48951 |
| 9.6 | 5328.43855 | 7936.81115 | 1.48951 |
| 9.7 | 5882.47146 | 8761.97155 | 1.48951 |
| 9.8 | 6494.12659 | 9673.03503 | 1.48951 |
| 9.9 | 7169.46404 | 10678.95366 | 1.48951 |
| 10.0 | 7915.12075 | 11789.61318 | 1.48951 |

TABLE D.VIIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and $x$ from 1.50 to 10.0.

$\alpha = 4/5$

| x | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ | x | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ | x | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.07949 | 3.02345 | 2.80081 | 1.00 | 1.33486 | 4.62477 | 3.46461 |
| 0.01 | 1.00003 | 0.60057 | 0.60056 | 0.51 | 1.08276 | 3.05377 | 2.82035 | 1.01 | 1.34206 | 4.66104 | 3.47305 |
| 0.02 | 1.00013 | 0.79251 | 0.79241 | 0.52 | 1.08610 | 3.08403 | 2.83955 | 1.02 | 1.34936 | 4.69757 | 3.48134 |
| 0.03 | 1.00028 | 0.93215 | 0.93189 | 0.53 | 1.08951 | 3.11425 | 2.85840 | 1.03 | 1.35674 | 4.73435 | 3.48950 |
| 0.04 | 1.00050 | 1.04599 | 1.04547 | 0.54 | 1.09299 | 3.14444 | 2.87693 | 1.04 | 1.36422 | 4.77139 | 3.49752 |
| 0.05 | 1.00078 | 1.14386 | 1.14297 | 0.55 | 1.09654 | 3.17461 | 2.89513 | 1.05 | 1.37179 | 4.80869 | 3.50541 |
| 0.06 | 1.00113 | 1.23068 | 1.22929 | 0.56 | 1.10015 | 3.20477 | 2.91302 | 1.06 | 1.37946 | 4.84626 | 3.51317 |
| 0.07 | 1.00153 | 1.30930 | 1.30730 | 0.57 | 1.10384 | 3.23493 | 2.93060 | 1.07 | 1.38722 | 4.88410 | 3.52080 |
| 0.08 | 1.00290 | 1.38157 | 1.37881 | 0.58 | 1.10761 | 3.26509 | 2.94788 | 1.08 | 1.39507 | 4.92222 | 3.52830 |
| 0.09 | 1.00253 | 1.44873 | 1.44507 | 0.59 | 1.11144 | 3.29526 | 2.96486 | 1.09 | 1.40302 | 4.96061 | 3.53567 |
| 0.10 | 1.00313 | 1.51169 | 1.50698 | 0.60 | 1.11534 | 3.32546 | 2.98156 | 1.10 | 1.41107 | 4.99929 | 3.54292 |
| 0.11 | 1.00378 | 1.57112 | 1.56520 | 0.61 | 1.11932 | 3.35569 | 2.99798 | 1.11 | 1.41921 | 5.03826 | 3.55005 |
| 0.12 | 1.00450 | 1.62754 | 1.62024 | 0.62 | 1.12337 | 3.38596 | 3.01411 | 1.12 | 1.42745 | 5.07752 | 3.55706 |
| 0.13 | 1.00529 | 1.68137 | 1.67253 | 0.63 | 1.12749 | 3.41627 | 3.02998 | 1.13 | 1.43579 | 5.11708 | 3.56395 |
| 0.14 | 1.00613 | 1.73293 | 1.72237 | 0.64 | 1.13169 | 3.44664 | 3.04558 | 1.14 | 1.44423 | 5.15693 | 3.57072 |
| 0.15 | 1.00704 | 1.78249 | 1.77003 | 0.65 | 1.13595 | 3.47706 | 3.06092 | 1.15 | 1.45276 | 5.19710 | 3.57739 |
| 0.16 | 1.00801 | 1.83029 | 1.81574 | 0.66 | 1.14030 | 3.50755 | 3.07600 | 1.16 | 1.46140 | 5.23757 | 3.58393 |
| 0.17 | 1.00905 | 1.87650 | 1.85967 | 0.67 | 1.14471 | 3.53812 | 3.09083 | 1.17 | 1.47014 | 5.27835 | 3.59037 |
| 0.18 | 1.01015 | 1.92129 | 1.90199 | 0.68 | 1.14920 | 3.56876 | 3.10542 | 1.18 | 1.47898 | 5.31946 | 3.59670 |
| 0.19 | 1.01131 | 1.96481 | 1.94284 | 0.69 | 1.15377 | 3.59949 | 3.11977 | 1.19 | 1.48793 | 5.36087 | 3.60292 |
| 0.20 | 1.01253 | 2.00716 | 1.98231 | 0.70 | 1.15841 | 3.63032 | 3.13387 | 1.20 | 1.49697 | 5.40263 | 3.60903 |
| 0.21 | 1.01382 | 2.04846 | 2.02053 | 0.71 | 1.16313 | 3.66124 | 3.14775 | 1.21 | 1.50613 | 5.44472 | 3.61505 |
| 0.22 | 1.01518 | 2.08880 | 2.05757 | 0.72 | 1.16792 | 3.69226 | 3.16139 | 1.22 | 1.51538 | 5.48714 | 3.62095 |
| 0.23 | 1.01659 | 2.12825 | 2.09352 | 0.73 | 1.17279 | 3.72340 | 3.17482 | 1.23 | 1.52475 | 5.52989 | 3.62676 |
| 0.24 | 1.01807 | 2.16691 | 2.12844 | 0.74 | 1.17774 | 3.75465 | 3.18802 | 1.24 | 1.53421 | 5.57299 | 3.63247 |
| 0.25 | 1.01962 | 2.20482 | 2.16240 | 0.75 | 1.18276 | 3.78603 | 3.20100 | 1.25 | 1.54379 | 5.61644 | 3.63808 |
| 0.26 | 1.02122 | 2.24204 | 2.19545 | 0.76 | 1.18787 | 3.81753 | 3.21377 | 1.26 | 1.55348 | 5.66024 | 3.64360 |
| 0.27 | 1.02290 | 2.27865 | 2.22764 | 0.77 | 1.19305 | 3.84916 | 3.22633 | 1.27 | 1.56327 | 5.70440 | 3.64902 |
| 0.28 | 1.02463 | 2.31467 | 2.25902 | 0.78 | 1.19830 | 3.88093 | 3.23868 | 1.28 | 1.57317 | 5.74892 | 3.65435 |
| 0.29 | 1.02644 | 2.35015 | 2.28963 | 0.79 | 1.20364 | 3.91284 | 3.25084 | 1.29 | 1.58319 | 5.79380 | 3.65958 |
| 0.30 | 1.02830 | 2.38515 | 2.31950 | 0.80 | 1.20906 | 3.94491 | 3.26279 | 1.30 | 1.59331 | 5.83905 | 3.66473 |
| 0.31 | 1.03023 | 2.41968 | 2.34868 | 0.81 | 1.21456 | 3.97712 | 3.27455 | 1.31 | 1.60355 | 5.88468 | 3.66978 |
| 0.32 | 1.03223 | 2.45380 | 2.37718 | 0.82 | 1.22014 | 4.00950 | 3.28611 | 1.32 | 1.61390 | 5.93069 | 3.67475 |
| 0.33 | 1.03429 | 2.48752 | 2.40505 | 0.83 | 1.22579 | 4.04204 | 3.29748 | 1.33 | 1.62437 | 5.97708 | 3.67964 |
| 0.34 | 1.03642 | 2.52088 | 2.43231 | 0.84 | 1.23153 | 4.07474 | 3.30867 | 1.34 | 1.63495 | 6.02385 | 3.68444 |
| 0.35 | 1.03861 | 2.55391 | 2.45897 | 0.85 | 1.23736 | 4.10762 | 3.31968 | 1.35 | 1.64564 | 6.07102 | 3.68915 |
| 0.36 | 1.04087 | 2.58663 | 2.48507 | 0.86 | 1.24326 | 4.14068 | 3.33050 | 1.36 | 1.65645 | 6.11859 | 3.69379 |
| 0.37 | 1.04319 | 2.61906 | 2.51063 | 0.87 | 1.24925 | 4.17392 | 3.34115 | 1.37 | 1.66738 | 6.16655 | 3.69834 |
| 0.38 | 1.04558 | 2.65124 | 2.53566 | 0.88 | 1.25532 | 4.20735 | 3.35162 | 1.38 | 1.67843 | 6.21493 | 3.70281 |
| 0.39 | 1.04804 | 2.68317 | 2.56019 | 0.89 | 1.26147 | 4.24097 | 3.36192 | 1.39 | 1.68960 | 6.26371 | 3.70721 |
| 0.40 | 1.05056 | 2.71488 | 2.58422 | 0.90 | 1.26771 | 4.27478 | 3.37205 | 1.40 | 1.70089 | 6.31291 | 3.71153 |
| 0.41 | 1.05315 | 2.74639 | 2.60779 | 0.91 | 1.27403 | 4.30880 | 3.38201 | 1.41 | 1.71230 | 6.36252 | 3.71577 |
| 0.42 | 1.05580 | 2.77771 | 2.63089 | 0.92 | 1.28044 | 4.34303 | 3.39181 | 1.42 | 1.72383 | 6.41256 | 3.71994 |
| 0.43 | 1.05853 | 2.80886 | 2.65356 | 0.93 | 1.28694 | 4.37746 | 3.40145 | 1.43 | 1.73549 | 6.46303 | 3.72404 |
| 0.44 | 1.06132 | 2.83987 | 2.67579 | 0.94 | 1.29352 | 4.41211 | 3.41093 | 1.44 | 1.74727 | 6.51394 | 3.72806 |
| 0.45 | 1.06418 | 2.87073 | 2.69761 | 0.95 | 1.30019 | 4.44698 | 3.42026 | 1.45 | 1.75918 | 6.56528 | 3.73202 |
| 0.46 | 1.06710 | 2.90147 | 2.71902 | 0.96 | 1.30695 | 4.48207 | 3.42943 | 1.46 | 1.77121 | 6.61707 | 3.73590 |
| 0.47 | 1.07010 | 2.93210 | 2.74003 | 0.97 | 1.31379 | 4.51739 | 3.43844 | 1.47 | 1.78337 | 6.66930 | 3.73972 |
| 0.48 | 1.07316 | 2.96263 | 2.76065 | 0.98 | 1.32072 | 4.55295 | 3.44731 | 1.48 | 1.79566 | 6.72198 | 3.74347 |
| 0.49 | 1.07629 | 2.99308 | 2.78092 | 0.99 | 1.32775 | 4.58874 | 3.45604 | 1.49 | 1.80808 | 6.77513 | 3.74715 |
| 0.50 | 1.07949 | 3.02345 | 2.80081 | 1.00 | 1.33486 | 4.62477 | 3.46461 | 1.50 | 1.82062 | 6.82873 | 3.75077 |

TABLE D.IXA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/5$ and x from 0.00 to 1.50.

$\alpha = 4/5$

| $x$ | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ |
|---|---|---|---|
| 1.50 | 1.82062 | 6.82873 | 3.75077 |
| 1.51 | 1.83330 | 6.88281 | 3.75432 |
| 1.52 | 1.84612 | 6.93736 | 3.75781 |
| 1.53 | 1.85906 | 6.99238 | 3.76124 |
| 1.54 | 1.87214 | 7.04789 | 3.76461 |
| 1.55 | 1.88536 | 7.10388 | 3.76791 |
| 1.56 | 1.89872 | 7.16037 | 3.77116 |
| 1.57 | 1.91221 | 7.21735 | 3.77435 |
| 1.58 | 1.92584 | 7.27484 | 3.77749 |
| 1.59 | 1.93961 | 7.33283 | 3.78057 |
| 1.60 | 1.95353 | 7.39134 | 3.78359 |
| 1.61 | 1.96758 | 7.45037 | 3.78656 |
| 1.62 | 1.98179 | 7.50992 | 3.78947 |
| 1.63 | 1.99613 | 7.57000 | 3.79234 |
| 1.64 | 2.01062 | 7.63062 | 3.79515 |
| 1.65 | 2.02527 | 7.69178 | 3.79791 |
| 1.66 | 2.04006 | 7.75348 | 3.80062 |
| 1.67 | 2.05500 | 7.81573 | 3.80328 |
| 1.68 | 2.07009 | 7.87854 | 3.80590 |
| 1.69 | 2.08533 | 7.94192 | 3.80847 |
| 1.70 | 2.10073 | 8.00586 | 3.81099 |
| 1.71 | 2.11628 | 8.07037 | 3.81346 |
| 1.72 | 2.13200 | 8.13547 | 3.81589 |
| 1.73 | 2.14786 | 8.20115 | 3.81828 |
| 1.74 | 2.16389 | 8.26742 | 3.82062 |
| 1.75 | 2.18008 | 8.33429 | 3.82292 |
| 1.76 | 2.19643 | 8.40176 | 3.82518 |
| 1.77 | 2.21295 | 8.46984 | 3.82740 |
| 1.78 | 2.22963 | 8.53854 | 3.82955 |
| 1.79 | 2.24648 | 8.60786 | 3.83171 |
| 1.80 | 2.26349 | 8.67780 | 3.83381 |
| 1.81 | 2.28068 | 8.74838 | 3.83587 |
| 1.82 | 2.29803 | 8.81960 | 3.83790 |
| 1.83 | 2.31556 | 8.89146 | 3.83988 |
| 1.84 | 2.33326 | 8.96398 | 3.84183 |
| 1.85 | 2.35113 | 9.03716 | 3.84374 |
| 1.86 | 2.36919 | 9.11100 | 3.84562 |
| 1.87 | 2.38762 | 9.18551 | 3.84747 |
| 1.88 | 2.40583 | 9.26070 | 3.84928 |
| 1.89 | 2.42442 | 9.33658 | 3.85105 |
| 1.90 | 2.44320 | 9.41315 | 3.85280 |
| 1.91 | 2.46216 | 9.49041 | 3.85451 |
| 1.92 | 2.48131 | 9.56839 | 3.85619 |
| 1.93 | 2.50064 | 9.64707 | 3.85784 |
| 1.94 | 2.52017 | 9.72647 | 3.85946 |
| 1.95 | 2.53988 | 9.80660 | 3.86104 |
| 1.96 | 2.55979 | 9.88746 | 3.86260 |
| 1.97 | 2.57990 | 9.96906 | 3.86413 |
| 1.98 | 2.60020 | 10.05141 | 3.86564 |
| 1.99 | 2.62070 | 10.13452 | 3.86711 |
| 2.00 | 2.64139 | 10.21838 | 3.86856 |

| $x$ | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ |
|---|---|---|---|
| 2.0 | 2.64139 | 10.21838 | 3.86856 |
| 2.1 | 2.85978 | 11.10058 | 3.88162 |
| 2.2 | 3.10023 | 12.06747 | 3.89244 |
| 2.3 | 3.36483 | 13.12750 | 3.90139 |
| 2.4 | 3.65585 | 14.28993 | 3.90878 |
| 2.5 | 3.97563 | 15.56493 | 3.91488 |
| 2.6 | 4.32755 | 16.96362 | 3.91992 |
| 2.7 | 4.71406 | 18.49825 | 3.92406 |
| 2.8 | 5.13873 | 20.18227 | 3.92748 |
| 2.9 | 5.60529 | 22.03044 | 3.93029 |
| 3.0 | 6.11783 | 24.05901 | 3.93261 |
| 3.1 | 6.68084 | 26.28582 | 3.93451 |
| 3.2 | 7.29929 | 28.73051 | 3.93607 |
| 3.3 | 7.97862 | 31.41467 | 3.93736 |
| 3.4 | 8.72485 | 34.36206 | 3.93841 |
| 3.5 | 9.54459 | 37.59880 | 3.93928 |
| 3.6 | 10.44511 | 41.15365 | 3.93999 |
| 3.7 | 11.43442 | 45.05822 | 3.94058 |
| 3.8 | 12.52134 | 49.34733 | 3.94106 |
| 3.9 | 13.71558 | 54.05928 | 3.94145 |
| 4.0 | 15.02781 | 59.23621 | 3.94177 |
| 4.1 | 16.46979 | 64.92454 | 3.94204 |
| 4.2 | 18.05446 | 71.17530 | 3.94226 |
| 4.3 | 19.79607 | 78.04472 | 3.94243 |
| 4.4 | 21.71031 | 85.59465 | 3.94258 |
| 4.5 | 23.81444 | 93.89320 | 3.94270 |
| 4.6 | 26.12747 | 103.01536 | 3.94280 |
| 4.7 | 28.67033 | 113.04365 | 3.94288 |
| 4.8 | 31.46606 | 124.06897 | 3.94295 |
| 4.9 | 34.54004 | 136.19136 | 3.94300 |
| 5.0 | 37.92019 | 149.52099 | 3.94304 |
| 5.1 | 41.63729 | 164.17919 | 3.94308 |
| 5.2 | 45.72521 | 180.29953 | 3.94311 |
| 5.3 | 50.22124 | 198.02911 | 3.94313 |
| 5.4 | 55.16647 | 217.52993 | 3.94315 |
| 5.5 | 60.60614 | 238.98039 | 3.94317 |
| 5.6 | 66.59008 | 262.57698 | 3.94318 |
| 5.7 | 73.17316 | 288.53610 | 3.94320 |
| 5.8 | 80.41583 | 317.09608 | 3.94320 |
| 5.9 | 88.38466 | 348.51946 | 3.94321 |
| 6.0 | 97.15298 | 383.09540 | 3.94322 |

| $x$ | $F_{4/5}(x)$ | $H_{1/5}(x)$ | $T_{4/5}(x)$ |
|---|---|---|---|
| 6.0 | 97.15298 | 383.09540 | 3.94322 |
| 6.1 | 106.80157 | 421.14242 | 3.94322 |
| 6.2 | 117.41941 | 463.01142 | 3.94323 |
| 6.3 | 129.10452 | 509.08890 | 3.94323 |
| 6.4 | 141.96489 | 559.80068 | 3.94323 |
| 6.5 | 156.11948 | 615.61587 | 3.94324 |
| 6.6 | 171.69934 | 677.05126 | 3.94324 |
| 6.7 | 188.84889 | 744.67626 | 3.94324 |
| 6.8 | 207.72720 | 819.11820 | 3.94324 |
| 6.9 | 228.50956 | 901.06828 | 3.94324 |
| 7.0 | 251.38912 | 991.28809 | 3.94324 |
| 7.1 | 276.57868 | 1090.61681 | 3.94324 |
| 7.2 | 304.31274 | 1195.97911 | 3.94324 |
| 7.3 | 334.84970 | 1320.39393 | 3.94324 |
| 7.4 | 368.47431 | 1452.98408 | 3.94324 |
| 7.5 | 405.50035 | 1558.98691 | 3.94324 |
| 7.6 | 446.27361 | 1759.76597 | 3.94324 |
| 7.7 | 491.17519 | 1936.82395 | 3.94324 |
| 7.8 | 540.62505 | 2131.81696 | 3.94324 |
| 7.9 | 595.08607 | 2346.57016 | 3.94324 |
| 8.0 | 655.06836 | 2583.09512 | 3.94325 |
| 8.1 | 721.13418 | 2843.60889 | 3.94325 |
| 8.2 | 793.90321 | 3130.55508 | 3.94325 |
| 8.3 | 874.05848 | 3446.62701 | 3.94325 |
| 8.4 | 962.35283 | 3794.79337 | 3.94325 |
| 8.5 | 1059.61611 | 4178.32639 | 3.94325 |
| 8.6 | 1166.76302 | 4600.83300 | 3.94325 |
| 8.7 | 1284.80185 | 5066.28910 | 3.94325 |
| 8.8 | 1414.84405 | 5579.07745 | 3.94325 |
| 8.9 | 1558.11484 | 6144.02939 | 3.94325 |
| 9.0 | 1715.96487 | 6766.47083 | 3.94325 |
| 9.1 | 1889.88308 | 7452.27307 | 3.94325 |
| 9.2 | 2081.51091 | 8207.90869 | 3.94325 |
| 9.3 | 2292.65795 | 9040.51334 | 3.94325 |
| 9.4 | 2525.31920 | 9957.95379 | 3.94325 |
| 9.5 | 2781.69411 | 10968.90301 | 3.94325 |
| 9.6 | 3064.20757 | 12082.92298 | 3.94325 |
| 9.7 | 3375.53310 | 13310.55601 | 3.94325 |
| 9.8 | 3718.61835 | 14663.42546 | 3.94325 |
| 9.9 | 4096.71333 | 16154.34682 | 3.94325 |
| 10.0 | 4513.40140 | 17797.45024 | 3.94325 |

TABLE D.IXB. LANCKESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/5$ and $x$ from 1.50 to 10.0.

$\alpha = 2/7$

| $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ | $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ | $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.22412 | 0.20034 | 0.16366 | 1.00 | 1.96323 | 0.59952 | 0.30537 |
| 0.01 | 1.00009 | 0.00072 | 0.00072 | 0.51 | 1.23340 | 0.20639 | 0.16733 | 1.01 | 1.98447 | 0.60960 | 0.30728 |
| 0.02 | 1.00035 | 0.00195 | 0.00194 | 0.52 | 1.24288 | 0.21251 | 0.17098 | 1.02 | 2.00599 | 0.62018 | 0.30916 |
| 0.03 | 1.00079 | 0.00347 | 0.00347 | 0.53 | 1.25257 | 0.21870 | 0.17460 | 1.03 | 2.02781 | 0.63067 | 0.31101 |
| 0.04 | 1.00140 | 0.00524 | 0.00523 | 0.54 | 1.26246 | 0.22497 | 0.17820 | 1.04 | 2.04993 | 0.64126 | 0.31282 |
| 0.05 | 1.00219 | 0.00720 | 0.00719 | 0.55 | 1.27256 | 0.23130 | 0.18176 | 1.05 | 2.07233 | 0.65197 | 0.31461 |
| 0.06 | 1.00315 | 0.00935 | 0.00932 | 0.56 | 1.28266 | 0.23771 | 0.18530 | 1.06 | 2.09504 | 0.66278 | 0.31636 |
| 0.07 | 1.00429 | 0.01166 | 0.01161 | 0.57 | 1.29337 | 0.24419 | 0.18800 | 1.07 | 2.11805 | 0.67370 | 0.31808 |
| 0.08 | 1.00560 | 0.01411 | 0.01403 | 0.58 | 1.30410 | 0.25075 | 0.19228 | 1.08 | 2.14137 | 0.68473 | 0.31977 |
| 0.09 | 1.00709 | 0.01670 | 0.01658 | 0.59 | 1.31503 | 0.25738 | 0.19572 | 1.09 | 2.16499 | 0.69588 | 0.32143 |
| 0.10 | 1.00876 | 0.01942 | 0.01925 | 0.60 | 1.32617 | 0.26408 | 0.19913 | 1.10 | 2.18892 | 0.70714 | 0.32306 |
| 0.11 | 1.01060 | 0.02225 | 0.02202 | 0.61 | 1.33753 | 0.27086 | 0.20251 | 1.11 | 2.21317 | 0.71852 | 0.32466 |
| 0.12 | 1.01262 | 0.02521 | 0.02489 | 0.62 | 1.34910 | 0.27772 | 0.20585 | 1.12 | 2.23773 | 0.73001 | 0.32623 |
| 0.13 | 1.01481 | 0.02827 | 0.02784 | 0.63 | 1.36088 | 0.28465 | 0.20916 | 1.13 | 2.26261 | 0.74162 | 0.32777 |
| 0.14 | 1.01718 | 0.03144 | 0.03091 | 0.64 | 1.37289 | 0.29165 | 0.21244 | 1.14 | 2.28781 | 0.75335 | 0.32929 |
| 0.15 | 1.01973 | 0.03471 | 0.03404 | 0.65 | 1.38511 | 0.29873 | 0.21567 | 1.15 | 2.31333 | 0.76519 | 0.33078 |
| 0.16 | 1.02246 | 0.03808 | 0.03725 | 0.66 | 1.39755 | 0.30589 | 0.21888 | 1.16 | 2.33918 | 0.77716 | 0.33224 |
| 0.17 | 1.02536 | 0.04155 | 0.04052 | 0.67 | 1.41021 | 0.31312 | 0.22206 | 1.17 | 2.36536 | 0.78925 | 0.33367 |
| 0.18 | 1.02844 | 0.04511 | 0.04386 | 0.68 | 1.42310 | 0.32044 | 0.22517 | 1.18 | 2.39188 | 0.80146 | 0.33508 |
| 0.19 | 1.03170 | 0.04875 | 0.04726 | 0.69 | 1.43621 | 0.32783 | 0.22826 | 1.19 | 2.41873 | 0.81380 | 0.33646 |
| 0.20 | 1.03514 | 0.05249 | 0.05071 | 0.70 | 1.44956 | 0.33530 | 0.23131 | 1.20 | 2.44592 | 0.82626 | 0.33781 |
| 0.21 | 1.03875 | 0.05631 | 0.05421 | 0.71 | 1.46311 | 0.34285 | 0.23433 | 1.21 | 2.47345 | 0.83886 | 0.33914 |
| 0.22 | 1.04255 | 0.06022 | 0.05776 | 0.72 | 1.47690 | 0.35048 | 0.23731 | 1.22 | 2.50133 | 0.85158 | 0.34045 |
| 0.23 | 1.04653 | 0.06421 | 0.06136 | 0.73 | 1.49092 | 0.35819 | 0.24025 | 1.23 | 2.52955 | 0.86443 | 0.34173 |
| 0.24 | 1.05068 | 0.06828 | 0.06499 | 0.74 | 1.50517 | 0.36598 | 0.24315 | 1.24 | 2.55813 | 0.87741 | 0.34299 |
| 0.25 | 1.05502 | 0.07244 | 0.06866 | 0.75 | 1.51966 | 0.37386 | 0.24601 | 1.25 | 2.58707 | 0.89052 | 0.34422 |
| 0.26 | 1.05954 | 0.07667 | 0.07236 | 0.76 | 1.53439 | 0.38181 | 0.24884 | 1.26 | 2.61636 | 0.90377 | 0.34543 |
| 0.27 | 1.06424 | 0.08098 | 0.07609 | 0.77 | 1.54935 | 0.38985 | 0.25162 | 1.27 | 2.64601 | 0.91715 | 0.34662 |
| 0.28 | 1.06912 | 0.08536 | 0.07984 | 0.78 | 1.56455 | 0.39798 | 0.25437 | 1.28 | 2.67603 | 0.93068 | 0.34778 |
| 0.29 | 1.07419 | 0.08982 | 0.08362 | 0.79 | 1.57999 | 0.40618 | 0.25708 | 1.29 | 2.70642 | 0.94434 | 0.34892 |
| 0.30 | 1.07944 | 0.09436 | 0.08742 | 0.80 | 1.59567 | 0.41448 | 0.25975 | 1.30 | 2.73718 | 0.95814 | 0.35004 |
| 0.31 | 1.08488 | 0.09898 | 0.09123 | 0.81 | 1.61159 | 0.42286 | 0.26239 | 1.31 | 2.76831 | 0.97208 | 0.35114 |
| 0.32 | 1.09050 | 0.10366 | 0.09506 | 0.82 | 1.62777 | 0.43133 | 0.26498 | 1.32 | 2.79983 | 0.98615 | 0.35222 |
| 0.33 | 1.09630 | 0.10842 | 0.09890 | 0.83 | 1.64419 | 0.43988 | 0.26754 | 1.33 | 2.83173 | 1.00039 | 0.35328 |
| 0.34 | 1.10229 | 0.11326 | 0.10275 | 0.84 | 1.66086 | 0.44852 | 0.27005 | 1.34 | 2.86401 | 1.01477 | 0.35432 |
| 0.35 | 1.10847 | 0.11816 | 0.10660 | 0.85 | 1.67778 | 0.45725 | 0.27254 | 1.35 | 2.89669 | 1.02929 | 0.35533 |
| 0.36 | 1.11484 | 0.12314 | 0.11046 | 0.86 | 1.69496 | 0.46608 | 0.27498 | 1.36 | 2.92975 | 1.04396 | 0.35633 |
| 0.37 | 1.12139 | 0.12819 | 0.11432 | 0.87 | 1.71239 | 0.47499 | 0.27738 | 1.37 | 2.96322 | 1.05878 | 0.35731 |
| 0.38 | 1.12813 | 0.13332 | 0.11818 | 0.88 | 1.73008 | 0.48399 | 0.27975 | 1.38 | 2.99709 | 1.07376 | 0.35827 |
| 0.39 | 1.13507 | 0.13851 | 0.12203 | 0.89 | 1.74803 | 0.49309 | 0.28208 | 1.39 | 3.03136 | 1.08889 | 0.35921 |
| 0.40 | 1.14219 | 0.14378 | 0.12588 | 0.90 | 1.76624 | 0.50228 | 0.28438 | 1.40 | 3.06605 | 1.10417 | 0.36013 |
| 0.41 | 1.14951 | 0.14911 | 0.12972 | 0.91 | 1.78471 | 0.51157 | 0.28664 | 1.41 | 3.10114 | 1.11962 | 0.36103 |
| 0.42 | 1.15701 | 0.15452 | 0.13355 | 0.92 | 1.80345 | 0.52095 | 0.28886 | 1.42 | 3.13666 | 1.13521 | 0.36192 |
| 0.43 | 1.16472 | 0.16000 | 0.13737 | 0.93 | 1.82246 | 0.53042 | 0.29105 | 1.43 | 3.17259 | 1.15097 | 0.36279 |
| 0.44 | 1.17261 | 0.16555 | 0.14118 | 0.94 | 1.84174 | 0.53999 | 0.29320 | 1.44 | 3.20895 | 1.16691 | 0.36364 |
| 0.45 | 1.18070 | 0.17117 | 0.14497 | 0.95 | 1.86130 | 0.54966 | 0.29531 | 1.45 | 3.24574 | 1.18300 | 0.36448 |
| 0.46 | 1.18899 | 0.17686 | 0.14875 | 0.96 | 1.88112 | 0.55943 | 0.29739 | 1.46 | 3.28297 | 1.19925 | 0.36530 |
| 0.47 | 1.19747 | 0.18263 | 0.15251 | 0.97 | 1.90123 | 0.56930 | 0.29944 | 1.47 | 3.32063 | 1.21568 | 0.36610 |
| 0.48 | 1.20615 | 0.18845 | 0.15625 | 0.98 | 1.92161 | 0.57927 | 0.30145 | 1.48 | 3.35874 | 1.23227 | 0.36688 |
| 0.49 | 1.21503 | 0.19436 | 0.15997 | 0.99 | 1.94228 | 0.58924 | 0.30343 | 1.49 | 3.39729 | 1.24903 | 0.36766 |
| 0.50 | 1.22412 | 0.20034 | 0.16366 | 1.00 | 1.96323 | 0.59952 | 0.30537 | 1.50 | 3.43629 | 1.26597 | 0.36841 |

TABLE D.XA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 2/7$ and $x$ from 0.00 to 1.50.

$\alpha = 2/7$

| $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ |
|---|---|---|---|
| 1.50 | 3.43629 | 1.26597 | 0.36841 |
| 1.51 | 3.47575 | 1.28309 | 0.36915 |
| 1.52 | 3.51567 | 1.30038 | 0.36988 |
| 1.53 | 3.55606 | 1.31785 | 0.37059 |
| 1.54 | 3.59691 | 1.33550 | 0.37129 |
| 1.55 | 3.63824 | 1.35333 | 0.37197 |
| 1.56 | 3.68005 | 1.37135 | 0.37264 |
| 1.57 | 3.72234 | 1.38956 | 0.37330 |
| 1.58 | 3.76511 | 1.40795 | 0.37395 |
| 1.59 | 3.80839 | 1.42654 | 0.37458 |
| 1.60 | 3.85216 | 1.44531 | 0.37520 |
| 1.61 | 3.89643 | 1.46429 | 0.37580 |
| 1.62 | 3.94121 | 1.48346 | 0.37640 |
| 1.63 | 3.98651 | 1.50283 | 0.37698 |
| 1.64 | 4.03232 | 1.52240 | 0.37755 |
| 1.65 | 4.07865 | 1.54217 | 0.37811 |
| 1.66 | 4.12552 | 1.56216 | 0.37866 |
| 1.67 | 4.17292 | 1.58234 | 0.37919 |
| 1.68 | 4.22086 | 1.60274 | 0.37972 |
| 1.69 | 4.26934 | 1.62336 | 0.38024 |
| 1.70 | 4.31838 | 1.64418 | 0.38074 |
| 1.71 | 4.36797 | 1.66523 | 0.38124 |
| 1.72 | 4.41812 | 1.68649 | 0.38172 |
| 1.73 | 4.46884 | 1.70798 | 0.38220 |
| 1.74 | 4.52014 | 1.72969 | 0.38266 |
| 1.75 | 4.57201 | 1.75162 | 0.38312 |
| 1.76 | 4.62447 | 1.77379 | 0.38357 |
| 1.77 | 4.67752 | 1.79618 | 0.38400 |
| 1.78 | 4.73116 | 1.81882 | 0.38443 |
| 1.79 | 4.78541 | 1.84168 | 0.38485 |
| 1.80 | 4.84027 | 1.86479 | 0.38527 |
| 1.81 | 4.89574 | 1.88814 | 0.38567 |
| 1.82 | 4.95183 | 1.91173 | 0.38607 |
| 1.83 | 5.00856 | 1.93557 | 0.38645 |
| 1.84 | 5.06591 | 1.95966 | 0.38683 |
| 1.85 | 5.12391 | 1.98401 | 0.38721 |
| 1.86 | 5.18256 | 2.00860 | 0.38757 |
| 1.87 | 5.24136 | 2.03346 | 0.38793 |
| 1.88 | 5.30182 | 2.05858 | 0.38828 |
| 1.89 | 5.36245 | 2.08396 | 0.38862 |
| 1.90 | 5.42375 | 2.10960 | 0.38896 |
| 1.91 | 5.48573 | 2.13552 | 0.38929 |
| 1.92 | 5.54841 | 2.16170 | 0.38961 |
| 1.93 | 5.61177 | 2.18817 | 0.38992 |
| 1.94 | 5.67585 | 2.21490 | 0.39023 |
| 1.95 | 5.74063 | 2.24193 | 0.39054 |
| 1.96 | 5.80612 | 2.26923 | 0.39083 |
| 1.97 | 5.87235 | 2.29682 | 0.39112 |
| 1.98 | 5.93930 | 2.32470 | 0.39141 |
| 1.99 | 6.00700 | 2.35288 | 0.39169 |
| 2.00 | 6.07544 | 2.38135 | 0.39196 |

| $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ |
|---|---|---|---|
| 2.0 | 6.07544 | 2.38135 | 0.39196 |
| 2.1 | 6.80269 | 2.68307 | 0.39441 |
| 2.2 | 7.61372 | 3.01814 | 0.39641 |
| 2.3 | 8.51765 | 3.39030 | 0.39803 |
| 2.4 | 9.52461 | 3.80370 | 0.39935 |
| 2.5 | 10.64589 | 4.26296 | 0.40043 |
| 2.6 | 11.89402 | 4.77319 | 0.40131 |
| 2.7 | 13.28291 | 5.34007 | 0.40203 |
| 2.8 | 14.82801 | 5.96989 | 0.40261 |
| 2.9 | 16.54648 | 6.66962 | 0.40308 |
| 3.0 | 18.45736 | 7.44701 | 0.40347 |
| 3.1 | 20.58178 | 8.31065 | 0.40379 |
| 3.2 | 22.94317 | 9.27007 | 0.40404 |
| 3.3 | 25.56756 | 10.33582 | 0.40426 |
| 3.4 | 28.48380 | 11.51961 | 0.40443 |
| 3.5 | 31.72390 | 12.83443 | 0.40457 |
| 3.6 | 35.32336 | 14.29469 | 0.40468 |
| 3.7 | 39.32155 | 15.91635 | 0.40477 |
| 3.8 | 43.76214 | 17.71712 | 0.40485 |
| 3.9 | 48.69354 | 19.71663 | 0.40491 |
| 4.0 | 54.16945 | 21.93665 | 0.40496 |
| 4.1 | 60.24938 | 24.40130 | 0.40501 |
| 4.2 | 66.99933 | 27.13734 | 0.40504 |
| 4.3 | 74.49248 | 30.17442 | 0.40507 |
| 4.4 | 82.80993 | 33.54542 | 0.40509 |
| 4.5 | 92.04161 | 37.28678 | 0.40511 |
| 4.6 | 102.28718 | 41.43889 | 0.40512 |
| 4.7 | 113.65714 | 46.04653 | 0.40514 |
| 4.8 | 126.27394 | 51.15932 | 0.40515 |
| 4.9 | 140.27331 | 56.83226 | 0.40515 |
| 5.0 | 155.80568 | 63.12631 | 0.40516 |
| 5.1 | 173.03779 | 70.10904 | 0.40517 |
| 5.2 | 192.15439 | 77.85530 | 0.40517 |
| 5.3 | 213.36024 | 86.44808 | 0.40517 |
| 5.4 | 236.88226 | 95.97931 | 0.40518 |
| 5.5 | 262.97188 | 106.55083 | 0.40518 |
| 5.6 | 291.90773 | 118.27569 | 0.40518 |
| 5.7 | 323.99854 | 131.27884 | 0.40518 |
| 5.8 | 359.58639 | 145.59894 | 0.40518 |
| 5.9 | 399.05030 | 161.68955 | 0.40519 |
| 6.0 | 442.81022 | 179.42086 | 0.40519 |

| $x$ | $F_{2/7}(x)$ | $H_{5/7}(x)$ | $T_{2/7}(x)$ |
|---|---|---|---|
| 6.0 | 442.81022 | 179.42086 | 0.40519 |
| 6.1 | 491.33142 | 199.08137 | 0.40519 |
| 6.2 | 545.12935 | 220.37995 | 0.40519 |
| 6.3 | 604.77509 | 245.04800 | 0.40519 |
| 6.4 | 670.90131 | 271.84186 | 0.40519 |
| 6.5 | 744.20885 | 301.54552 | 0.40519 |
| 6.6 | 825.47413 | 334.47357 | 0.40519 |
| 6.7 | 915.55761 | 370.97450 | 0.40519 |
| 6.8 | 1015.61081 | 411.43425 | 0.40519 |
| 6.9 | 1126.09024 | 456.28073 | 0.40519 |
| 7.0 | 1248.76448 | 505.98728 | 0.40519 |
| 7.1 | 1384.72833 | 561.07865 | 0.40519 |
| 7.2 | 1535.41592 | 622.13594 | 0.40519 |
| 7.3 | 1702.41573 | 689.80279 | 0.40519 |
| 7.4 | 1887.48712 | 764.79207 | 0.40519 |
| 7.5 | 2092.57865 | 847.89333 | 0.40519 |
| 7.6 | 2319.84844 | 939.98101 | 0.40519 |
| 7.7 | 2571.68659 | 1042.02357 | 0.40519 |
| 7.8 | 2850.74013 | 1155.09356 | 0.40519 |
| 7.9 | 3159.94055 | 1280.37876 | 0.40519 |
| 8.0 | 3502.53425 | 1419.19463 | 0.40519 |
| 8.1 | 3882.11651 | 1572.99791 | 0.40519 |
| 8.2 | 4302.66855 | 1743.40178 | 0.40519 |
| 8.3 | 4768.59922 | 1932.19265 | 0.40519 |
| 8.4 | 5284.79066 | 2141.34873 | 0.40519 |
| 8.5 | 5856.64912 | 2373.06056 | 0.40519 |
| 8.6 | 6490.16098 | 2629.75379 | 0.40519 |
| 8.7 | 7191.95501 | 2914.11433 | 0.40519 |
| 8.8 | 7969.37115 | 3229.11626 | 0.40519 |
| 8.9 | 8830.53666 | 3578.05267 | 0.40519 |
| 9.0 | 9784.45041 | 3964.56982 | 0.40519 |
| 9.1 | 10841.07625 | 4392.70497 | 0.40519 |
| 9.2 | 12011.44627 | 4866.92821 | 0.40519 |
| 9.3 | 13307.77514 | 5392.18883 | 0.40519 |
| 9.4 | 14743.56670 | 5973.96657 | 0.40519 |
| 9.5 | 16333.85403 | 6618.32837 | 0.40519 |
| 9.6 | 18095.15451 | 7331.99124 | 0.40519 |
| 9.7 | 20045.84147 | 8122.39178 | 0.40519 |
| 9.8 | 22206.23412 | 8997.76315 | 0.40519 |
| 9.9 | 24598.82785 | 9967.22027 | 0.40519 |
| 10.0 | 27248.52692 | 11040.85413 | 0.40519 |

TABLE D.XB. LANCHESTER-CLIFFORD-SCHLÄFI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 2/7$ and $x$ from 1.50 to 10.0.

$\alpha = 3/7$

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00006 | 0.00410 | 0.00410 |
| 0.02 | 1.00023 | 0.00906 | 0.00906 |
| 0.03 | 1.00053 | 0.01441 | 0.01440 |
| 0.04 | 1.00093 | 0.02002 | 0.02000 |
| 0.05 | 1.00146 | 0.02584 | 0.02580 |
| 0.06 | 1.00210 | 0.03183 | 0.03176 |
| 0.07 | 1.00286 | 0.03797 | 0.03786 |
| 0.08 | 1.00374 | 0.04424 | 0.04408 |
| 0.09 | 1.00473 | 0.05063 | 0.05039 |
| 0.10 | 1.00584 | 0.05713 | 0.05679 |
| 0.11 | 1.00707 | 0.06372 | 0.06327 |
| 0.12 | 1.00841 | 0.07041 | 0.06982 |
| 0.13 | 1.00987 | 0.07719 | 0.07643 |
| 0.14 | 1.01145 | 0.08404 | 0.08309 |
| 0.15 | 1.01315 | 0.09098 | 0.08980 |
| 0.16 | 1.01497 | 0.09799 | 0.09655 |
| 0.17 | 1.01690 | 0.10508 | 0.10333 |
| 0.18 | 1.01895 | 0.11223 | 0.11015 |
| 0.19 | 1.02112 | 0.11946 | 0.11699 |
| 0.20 | 1.02342 | 0.12675 | 0.12385 |
| 0.21 | 1.02582 | 0.13410 | 0.13073 |
| 0.22 | 1.02835 | 0.14152 | 0.13762 |
| 0.23 | 1.03100 | 0.14900 | 0.14452 |
| 0.24 | 1.03377 | 0.15655 | 0.15143 |
| 0.25 | 1.03666 | 0.16415 | 0.15835 |
| 0.26 | 1.03967 | 0.17182 | 0.16526 |
| 0.27 | 1.04280 | 0.17954 | 0.17217 |
| 0.28 | 1.04605 | 0.18732 | 0.17908 |
| 0.29 | 1.04942 | 0.19516 | 0.18597 |
| 0.30 | 1.05291 | 0.20306 | 0.19286 |
| 0.31 | 1.05653 | 0.21102 | 0.19973 |
| 0.32 | 1.06027 | 0.21904 | 0.20658 |
| 0.33 | 1.06413 | 0.22711 | 0.21342 |
| 0.34 | 1.06812 | 0.23524 | 0.22024 |
| 0.35 | 1.07223 | 0.24343 | 0.22703 |
| 0.36 | 1.07646 | 0.25167 | 0.23380 |
| 0.37 | 1.08082 | 0.25998 | 0.24054 |
| 0.38 | 1.08530 | 0.26834 | 0.24725 |
| 0.39 | 1.08991 | 0.27676 | 0.25393 |
| 0.40 | 1.09465 | 0.28524 | 0.26058 |
| 0.41 | 1.09951 | 0.29378 | 0.26719 |
| 0.42 | 1.10450 | 0.30238 | 0.27377 |
| 0.43 | 1.10961 | 0.31104 | 0.28031 |
| 0.44 | 1.11486 | 0.31976 | 0.28681 |
| 0.45 | 1.12023 | 0.32853 | 0.29327 |
| 0.46 | 1.12574 | 0.33737 | 0.29969 |
| 0.47 | 1.13137 | 0.34627 | 0.30607 |
| 0.48 | 1.13713 | 0.35523 | 0.31240 |
| 0.49 | 1.14303 | 0.36426 | 0.31868 |
| 0.50 | 1.14905 | 0.37335 | 0.32492 |

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 0.50 | 1.14905 | 0.37335 | 0.32492 |
| 0.51 | 1.15521 | 0.38250 | 0.33111 |
| 0.52 | 1.16150 | 0.39171 | 0.33725 |
| 0.53 | 1.16792 | 0.40099 | 0.34333 |
| 0.54 | 1.17449 | 0.41033 | 0.34937 |
| 0.55 | 1.18118 | 0.41974 | 0.35536 |
| 0.56 | 1.18801 | 0.42922 | 0.36129 |
| 0.57 | 1.19497 | 0.43876 | 0.36717 |
| 0.58 | 1.20208 | 0.44837 | 0.37300 |
| 0.59 | 1.20932 | 0.45805 | 0.37877 |
| 0.60 | 1.21670 | 0.46780 | 0.38449 |
| 0.61 | 1.22422 | 0.47762 | 0.39015 |
| 0.62 | 1.23188 | 0.48751 | 0.39575 |
| 0.63 | 1.23968 | 0.49748 | 0.40130 |
| 0.64 | 1.24762 | 0.50751 | 0.40678 |
| 0.65 | 1.25570 | 0.51762 | 0.41221 |
| 0.66 | 1.26393 | 0.52780 | 0.41759 |
| 0.67 | 1.27230 | 0.53806 | 0.42290 |
| 0.68 | 1.28082 | 0.54839 | 0.42815 |
| 0.69 | 1.28949 | 0.55880 | 0.43335 |
| 0.70 | 1.29830 | 0.56928 | 0.43849 |
| 0.71 | 1.30726 | 0.57985 | 0.44356 |
| 0.72 | 1.31636 | 0.59049 | 0.44858 |
| 0.73 | 1.32562 | 0.60122 | 0.45354 |
| 0.74 | 1.33503 | 0.61202 | 0.45843 |
| 0.75 | 1.34459 | 0.62291 | 0.46327 |
| 0.76 | 1.35430 | 0.63388 | 0.46805 |
| 0.77 | 1.36417 | 0.64494 | 0.47277 |
| 0.78 | 1.37419 | 0.65608 | 0.47743 |
| 0.79 | 1.38437 | 0.66730 | 0.48203 |
| 0.80 | 1.39470 | 0.67862 | 0.48657 |
| 0.81 | 1.40520 | 0.69002 | 0.49105 |
| 0.82 | 1.41585 | 0.70151 | 0.49547 |
| 0.83 | 1.42666 | 0.71309 | 0.49984 |
| 0.84 | 1.43764 | 0.72476 | 0.50414 |
| 0.85 | 1.44877 | 0.73653 | 0.50838 |
| 0.86 | 1.46007 | 0.74839 | 0.51257 |
| 0.87 | 1.47154 | 0.76034 | 0.51670 |
| 0.88 | 1.48317 | 0.77239 | 0.52077 |
| 0.89 | 1.49497 | 0.78453 | 0.52478 |
| 0.90 | 1.50693 | 0.79678 | 0.52874 |
| 0.91 | 1.51907 | 0.80912 | 0.53264 |
| 0.92 | 1.53138 | 0.82156 | 0.53648 |
| 0.93 | 1.54386 | 0.83411 | 0.54027 |
| 0.94 | 1.55651 | 0.84675 | 0.54401 |
| 0.95 | 1.56934 | 0.85950 | 0.54768 |
| 0.96 | 1.58235 | 0.87236 | 0.55131 |
| 0.97 | 1.59553 | 0.88532 | 0.55488 |
| 0.98 | 1.60889 | 0.89839 | 0.55839 |
| 0.99 | 1.62243 | 0.91157 | 0.56186 |
| 1.00 | 1.63616 | 0.92486 | 0.56526 |

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 1.00 | 1.63616 | 0.92486 | 0.56526 |
| 1.01 | 1.65007 | 0.93826 | 0.56862 |
| 1.02 | 1.66416 | 0.95178 | 0.57193 |
| 1.03 | 1.67844 | 0.96541 | 0.57518 |
| 1.04 | 1.69291 | 0.97915 | 0.57838 |
| 1.05 | 1.70756 | 0.99301 | 0.58154 |
| 1.06 | 1.72241 | 1.00699 | 0.58464 |
| 1.07 | 1.73745 | 1.02109 | 0.58770 |
| 1.08 | 1.75268 | 1.03531 | 0.59070 |
| 1.09 | 1.76811 | 1.04965 | 0.59366 |
| 1.10 | 1.78374 | 1.06412 | 0.59657 |
| 1.11 | 1.79956 | 1.07871 | 0.59943 |
| 1.12 | 1.81559 | 1.09343 | 0.60224 |
| 1.13 | 1.83181 | 1.10827 | 0.60501 |
| 1.14 | 1.84824 | 1.12325 | 0.60774 |
| 1.15 | 1.86488 | 1.13836 | 0.61042 |
| 1.16 | 1.88172 | 1.15360 | 0.61305 |
| 1.17 | 1.89878 | 1.16897 | 0.61564 |
| 1.18 | 1.91604 | 1.18448 | 0.61819 |
| 1.19 | 1.93352 | 1.20013 | 0.62070 |
| 1.20 | 1.95121 | 1.21591 | 0.62316 |
| 1.21 | 1.96912 | 1.23184 | 0.62558 |
| 1.22 | 1.98724 | 1.24791 | 0.62796 |
| 1.23 | 2.00559 | 1.26412 | 0.63030 |
| 1.24 | 2.02416 | 1.28048 | 0.63260 |
| 1.25 | 2.04295 | 1.29698 | 0.63486 |
| 1.26 | 2.06196 | 1.31363 | 0.63708 |
| 1.27 | 2.08121 | 1.33044 | 0.63926 |
| 1.28 | 2.10068 | 1.34739 | 0.64141 |
| 1.29 | 2.12039 | 1.36450 | 0.64351 |
| 1.30 | 2.14033 | 1.38177 | 0.64558 |
| 1.31 | 2.16051 | 1.39919 | 0.64762 |
| 1.32 | 2.18093 | 1.41677 | 0.64962 |
| 1.33 | 2.20158 | 1.43451 | 0.65158 |
| 1.34 | 2.22248 | 1.45241 | 0.65351 |
| 1.35 | 2.24362 | 1.47048 | 0.65541 |
| 1.36 | 2.26501 | 1.48872 | 0.65727 |
| 1.37 | 2.28665 | 1.50712 | 0.65910 |
| 1.38 | 2.30854 | 1.52569 | 0.66089 |
| 1.39 | 2.33068 | 1.54444 | 0.66265 |
| 1.40 | 2.35308 | 1.56336 | 0.66439 |
| 1.41 | 2.37574 | 1.58245 | 0.66609 |
| 1.42 | 2.39866 | 1.60172 | 0.66776 |
| 1.43 | 2.42184 | 1.62118 | 0.66940 |
| 1.44 | 2.44529 | 1.64081 | 0.67101 |
| 1.45 | 2.46901 | 1.66063 | 0.67259 |
| 1.46 | 2.49299 | 1.68063 | 0.67414 |
| 1.47 | 2.51725 | 1.70082 | 0.67567 |
| 1.48 | 2.54178 | 1.72120 | 0.67716 |
| 1.49 | 2.56660 | 1.74177 | 0.67863 |
| 1.50 | 2.59169 | 1.76254 | 0.68007 |

TABLE D.XIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/7$ and $x$ from 0.00 to 1.50.

$\alpha = 3/7$

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 1.50 | 2.59169 | 1.76254 | 0.68007 |
| 1.51 | 2.61706 | 1.78350 | 0.68149 |
| 1.52 | 2.64273 | 1.80466 | 0.68285 |
| 1.53 | 2.66868 | 1.82602 | 0.68424 |
| 1.54 | 2.69492 | 1.84759 | 0.68558 |
| 1.55 | 2.72145 | 1.86936 | 0.68690 |
| 1.56 | 2.74828 | 1.89133 | 0.68819 |
| 1.57 | 2.77542 | 1.91352 | 0.68945 |
| 1.58 | 2.80285 | 1.93591 | 0.69069 |
| 1.59 | 2.83059 | 1.95852 | 0.69191 |
| 1.60 | 2.85864 | 1.98135 | 0.69311 |
| 1.61 | 2.88699 | 2.00440 | 0.69428 |
| 1.62 | 2.91567 | 2.02766 | 0.69544 |
| 1.63 | 2.94466 | 2.05115 | 0.69657 |
| 1.64 | 2.97397 | 2.07487 | 0.69768 |
| 1.65 | 3.00360 | 2.09881 | 0.69877 |
| 1.66 | 3.03356 | 2.12299 | 0.69983 |
| 1.67 | 3.06384 | 2.14739 | 0.70088 |
| 1.68 | 3.09446 | 2.17204 | 0.70191 |
| 1.69 | 3.12542 | 2.19692 | 0.70292 |
| 1.70 | 3.15672 | 2.22204 | 0.70391 |
| 1.71 | 3.18835 | 2.24741 | 0.70488 |
| 1.72 | 3.22033 | 2.27302 | 0.70583 |
| 1.73 | 3.25267 | 2.29888 | 0.70677 |
| 1.74 | 3.28535 | 2.32499 | 0.70769 |
| 1.75 | 3.31839 | 2.35136 | 0.70858 |
| 1.76 | 3.35178 | 2.37798 | 0.70947 |
| 1.77 | 3.38555 | 2.40486 | 0.71033 |
| 1.78 | 3.41967 | 2.43201 | 0.71118 |
| 1.79 | 3.45417 | 2.45942 | 0.71201 |
| 1.80 | 3.48904 | 2.48710 | 0.71283 |
| 1.81 | 3.52428 | 2.51504 | 0.71363 |
| 1.82 | 3.55991 | 2.54327 | 0.71442 |
| 1.83 | 3.59592 | 2.57177 | 0.71519 |
| 1.84 | 3.63231 | 2.60054 | 0.71595 |
| 1.85 | 3.66910 | 2.62960 | 0.71669 |
| 1.86 | 3.70629 | 2.65895 | 0.71742 |
| 1.87 | 3.74387 | 2.68859 | 0.71813 |
| 1.88 | 3.78186 | 2.71851 | 0.71883 |
| 1.89 | 3.82025 | 2.74873 | 0.71952 |
| 1.90 | 3.85906 | 2.77925 | 0.72019 |
| 1.91 | 3.89828 | 2.81007 | 0.72085 |
| 1.92 | 3.93792 | 2.84120 | 0.72150 |
| 1.93 | 3.97799 | 2.87263 | 0.72213 |
| 1.94 | 4.01848 | 2.90437 | 0.72275 |
| 1.95 | 4.05940 | 2.93643 | 0.72336 |
| 1.96 | 4.10076 | 2.96880 | 0.72396 |
| 1.97 | 4.14256 | 3.00149 | 0.72455 |
| 1.98 | 4.18480 | 3.03451 | 0.72513 |
| 1.99 | 4.22750 | 3.06786 | 0.72569 |
| 2.00 | 4.27065 | 3.10153 | 0.72624 |

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 2.0 | 4.27065 | 3.10153 | 0.72624 |
| 2.1 | 4.72807 | 3.45728 | 0.73122 |
| 2.2 | 5.23611 | 3.85017 | 0.73531 |
| 2.3 | 5.80002 | 4.28427 | 0.73866 |
| 2.4 | 6.42568 | 4.76410 | 0.74142 |
| 2.5 | 7.11957 | 5.29460 | 0.74367 |
| 2.6 | 7.88889 | 5.88129 | 0.74552 |
| 2.7 | 8.74163 | 6.53024 | 0.74703 |
| 2.8 | 9.69661 | 7.24815 | 0.74827 |
| 2.9 | 10.73361 | 8.04247 | 0.74928 |
| 3.0 | 11.89347 | 8.92139 | 0.75011 |
| 3.1 | 13.17819 | 9.89402 | 0.75079 |
| 3.2 | 14.60103 | 10.97038 | 0.75134 |
| 3.3 | 16.17669 | 12.16159 | 0.75180 |
| 3.4 | 17.92145 | 13.47996 | 0.75217 |
| 3.5 | 19.85328 | 14.93907 | 0.75247 |
| 3.6 | 21.99212 | 16.55396 | 0.75272 |
| 3.7 | 24.36000 | 18.34129 | 0.75293 |
| 3.8 | 26.98131 | 20.31944 | 0.75309 |
| 3.9 | 29.88304 | 22.50879 | 0.75323 |
| 4.0 | 33.09505 | 24.93186 | 0.75334 |
| 4.1 | 36.65035 | 27.61356 | 0.75343 |
| 4.2 | 40.58552 | 30.58148 | 0.75351 |
| 4.3 | 44.94097 | 33.86609 | 0.75357 |
| 4.4 | 49.76144 | 37.50113 | 0.75362 |
| 4.5 | 55.09643 | 41.52393 | 0.75366 |
| 4.6 | 61.00067 | 45.97576 | 0.75369 |
| 4.7 | 67.53473 | 50.90229 | 0.75372 |
| 4.8 | 74.76562 | 56.35404 | 0.75374 |
| 4.9 | 82.76745 | 62.38688 | 0.75376 |
| 5.0 | 91.62221 | 69.06263 | 0.75378 |
| 5.1 | 101.42059 | 76.44966 | 0.75379 |
| 5.2 | 112.26292 | 84.62362 | 0.75380 |
| 5.3 | 124.26019 | 93.66816 | 0.75381 |
| 5.4 | 137.53514 | 103.67584 | 0.75381 |
| 5.5 | 152.22357 | 114.74901 | 0.75382 |
| 5.6 | 168.47567 | 127.00092 | 0.75382 |
| 5.7 | 186.45759 | 140.55680 | 0.75383 |
| 5.8 | 206.35310 | 155.55520 | 0.75383 |
| 5.9 | 228.36545 | 172.14934 | 0.75383 |
| 6.0 | 252.71948 | 190.50872 | 0.75383 |

| x | $F_{3/7}(x)$ | $H_{4/7}(x)$ | $T_{3/7}(x)$ |
|---|---|---|---|
| 6.0 | 252.71948 | 190.50872 | 0.75383 |
| 6.1 | 279.66387 | 210.82079 | 0.75384 |
| 6.2 | 309.47366 | 233.29292 | 0.75384 |
| 6.3 | 342.45309 | 258.15445 | 0.75384 |
| 6.4 | 378.93865 | 285.65902 | 0.75384 |
| 6.5 | 419.30250 | 316.08721 | 0.75384 |
| 6.6 | 463.95628 | 349.74932 | 0.75384 |
| 6.7 | 513.35525 | 386.98855 | 0.75384 |
| 6.8 | 568.00295 | 428.18449 | 0.75384 |
| 6.9 | 628.45630 | 473.75698 | 0.75384 |
| 7.0 | 695.33122 | 524.17032 | 0.75384 |
| 7.1 | 769.30895 | 579.93806 | 0.75384 |
| 7.2 | 851.14290 | 641.62815 | 0.75384 |
| 7.3 | 941.66633 | 709.86875 | 0.75384 |
| 7.4 | 1041.80078 | 785.35455 | 0.75384 |
| 7.5 | 1152.56546 | 868.85388 | 0.75384 |
| 7.6 | 1275.08753 | 961.21644 | 0.75384 |
| 7.7 | 1410.61358 | 1063.38196 | 0.75384 |
| 7.8 | 1560.52225 | 1176.38974 | 0.75384 |
| 7.9 | 1726.33822 | 1301.38913 | 0.75384 |
| 8.0 | 1909.74766 | 1439.65124 | 0.75384 |
| 8.1 | 2112.61531 | 1592.58180 | 0.75384 |
| 8.2 | 2337.00342 | 1761.73543 | 0.75384 |
| 8.3 | 2585.19263 | 1948.83137 | 0.75384 |
| 8.4 | 2859.70505 | 2155.77090 | 0.75384 |
| 8.5 | 3163.32988 | 2384.65663 | 0.75384 |
| 8.6 | 3499.15164 | 2637.81382 | 0.75384 |
| 8.7 | 3870.58138 | 2917.81387 | 0.75384 |
| 8.8 | 4281.39132 | 3227.50044 | 0.75384 |
| 8.9 | 4735.75299 | 3570.01821 | 0.75384 |
| 9.0 | 5238.27950 | 3948.84475 | 0.75384 |
| 9.1 | 5794.07225 | 4367.82572 | 0.75384 |
| 9.2 | 6408.77262 | 4831.21383 | 0.75384 |
| 9.3 | 7088.61905 | 5343.71190 | 0.75384 |
| 9.4 | 7840.51021 | 5910.52045 | 0.75384 |
| 9.5 | 8672.07486 | 6537.39037 | 0.75384 |
| 9.6 | 9591.74906 | 7230.68112 | 0.75384 |
| 9.7 | 10608.86152 | 7997.42512 | 0.75384 |
| 9.8 | 11733.72807 | 8845.39888 | 0.75384 |
| 9.9 | 12977.75603 | 9783.20173 | 0.75384 |
| 10.0 | 14353.55964 | 10820.34285 | 0.75384 |

TABLE D.XIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/7$ and $x$ from 1.50 to 10.0.

$\alpha = 4/7$

| x | $F_{4/7}(x)$ | $F_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00004 | 0.02487 | 0.02487 |
| 0.02 | 1.00018 | 0.04505 | 0.04504 |
| 0.03 | 1.00039 | 0.06378 | 0.06376 |
| 0.04 | 1.00070 | 0.08163 | 0.08157 |
| 0.05 | 1.00109 | 0.09885 | 0.09874 |
| 0.06 | 1.00158 | 0.11559 | 0.11541 |
| 0.07 | 1.00214 | 0.13195 | 0.13167 |
| 0.08 | 1.00280 | 0.14799 | 0.14758 |
| 0.09 | 1.00355 | 0.16376 | 0.16318 |
| 0.10 | 1.00438 | 0.17929 | 0.17851 |
| 0.11 | 1.00530 | 0.19463 | 0.19360 |
| 0.12 | 1.00631 | 0.20978 | 0.20847 |
| 0.13 | 1.00740 | 0.22478 | 0.22313 |
| 0.14 | 1.00859 | 0.23963 | 0.23759 |
| 0.15 | 1.00986 | 0.25436 | 0.25180 |
| 0.16 | 1.01122 | 0.26896 | 0.26599 |
| 0.17 | 1.01267 | 0.28348 | 0.27994 |
| 0.18 | 1.01421 | 0.29790 | 0.29373 |
| 0.19 | 1.01584 | 0.31223 | 0.30737 |
| 0.20 | 1.01756 | 0.32649 | 0.32086 |
| 0.21 | 1.01936 | 0.34068 | 0.33421 |
| 0.22 | 1.02126 | 0.35480 | 0.34742 |
| 0.23 | 1.02324 | 0.36887 | 0.36049 |
| 0.24 | 1.02532 | 0.38289 | 0.37343 |
| 0.25 | 1.02748 | 0.39686 | 0.38625 |
| 0.26 | 1.02973 | 0.41079 | 0.39893 |
| 0.27 | 1.03208 | 0.42469 | 0.41149 |
| 0.28 | 1.03451 | 0.43856 | 0.42393 |
| 0.29 | 1.03704 | 0.45240 | 0.43624 |
| 0.30 | 1.03966 | 0.46622 | 0.44843 |
| 0.31 | 1.04237 | 0.48001 | 0.46050 |
| 0.32 | 1.04517 | 0.49380 | 0.47246 |
| 0.33 | 1.04806 | 0.50757 | 0.48429 |
| 0.34 | 1.05104 | 0.52133 | 0.49601 |
| 0.35 | 1.05412 | 0.53508 | 0.50761 |
| 0.36 | 1.05729 | 0.54884 | 0.51910 |
| 0.37 | 1.06055 | 0.56259 | 0.53047 |
| 0.38 | 1.06390 | 0.57635 | 0.54173 |
| 0.39 | 1.06735 | 0.59011 | 0.55287 |
| 0.40 | 1.07090 | 0.60388 | 0.56390 |
| 0.41 | 1.07453 | 0.61766 | 0.57482 |
| 0.42 | 1.07826 | 0.63146 | 0.58562 |
| 0.43 | 1.08209 | 0.64527 | 0.59632 |
| 0.44 | 1.08601 | 0.65910 | 0.60690 |
| 0.45 | 1.09003 | 0.67295 | 0.61737 |
| 0.46 | 1.09414 | 0.68683 | 0.62773 |
| 0.47 | 1.09835 | 0.70073 | 0.63798 |
| 0.48 | 1.10266 | 0.71466 | 0.64812 |
| 0.49 | 1.10707 | 0.72861 | 0.65815 |
| 0.50 | 1.11157 | 0.74260 | 0.66807 |

| x | $F_{4/7}(x)$ | $H_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 0.50 | 1.11157 | 0.74260 | 0.66807 |
| 0.51 | 1.11617 | 0.75663 | 0.67788 |
| 0.52 | 1.12087 | 0.77069 | 0.68758 |
| 0.53 | 1.12566 | 0.78479 | 0.69718 |
| 0.54 | 1.13056 | 0.79893 | 0.70666 |
| 0.55 | 1.13556 | 0.81311 | 0.71604 |
| 0.56 | 1.14066 | 0.82734 | 0.72532 |
| 0.57 | 1.14586 | 0.84161 | 0.73448 |
| 0.58 | 1.15116 | 0.85593 | 0.74354 |
| 0.59 | 1.15656 | 0.87030 | 0.75249 |
| 0.60 | 1.16206 | 0.88473 | 0.76134 |
| 0.61 | 1.16767 | 0.39920 | 0.77008 |
| 0.62 | 1.17338 | 0.91374 | 0.77872 |
| 0.63 | 1.17920 | 0.92833 | 0.78726 |
| 0.64 | 1.18512 | 0.94298 | 0.79569 |
| 0.65 | 1.19114 | 0.95769 | 0.80401 |
| 0.66 | 1.19727 | 0.97247 | 0.81224 |
| 0.67 | 1.20351 | 0.98731 | 0.82036 |
| 0.68 | 1.20985 | 1.00222 | 0.82838 |
| 0.69 | 1.21630 | 1.01720 | 0.83630 |
| 0.70 | 1.22286 | 1.03224 | 0.84412 |
| 0.71 | 1.22953 | 1.04736 | 0.85184 |
| 0.72 | 1.23631 | 1.06256 | 0.85946 |
| 0.73 | 1.24320 | 1.07783 | 0.86698 |
| 0.74 | 1.25020 | 1.09318 | 0.87440 |
| 0.75 | 1.25731 | 1.10861 | 0.88173 |
| 0.76 | 1.26453 | 1.12411 | 0.88896 |
| 0.77 | 1.27186 | 1.13971 | 0.89609 |
| 0.78 | 1.27931 | 1.15538 | 0.90313 |
| 0.79 | 1.28688 | 1.17115 | 0.91007 |
| 0.80 | 1.29455 | 1.18700 | 0.91692 |
| 0.81 | 1.30235 | 1.20294 | 0.92367 |
| 0.82 | 1.31026 | 1.21898 | 0.93034 |
| 0.83 | 1.31828 | 1.23511 | 0.93691 |
| 0.84 | 1.32643 | 1.25133 | 0.94339 |
| 0.85 | 1.33469 | 1.26766 | 0.94977 |
| 0.86 | 1.34307 | 1.28408 | 0.95607 |
| 0.87 | 1.35158 | 1.30060 | 0.96228 |
| 0.88 | 1.36020 | 1.31722 | 0.96840 |
| 0.89 | 1.36895 | 1.33395 | 0.97444 |
| 0.90 | 1.37782 | 1.35079 | 0.98039 |
| 0.91 | 1.38681 | 1.36774 | 0.98625 |
| 0.92 | 1.39593 | 1.38479 | 0.99202 |
| 0.93 | 1.40517 | 1.40196 | 0.99771 |
| 0.94 | 1.41454 | 1.41924 | 1.00332 |
| 0.95 | 1.42403 | 1.43663 | 1.00885 |
| 0.96 | 1.43366 | 1.45415 | 1.01429 |
| 0.97 | 1.44341 | 1.47178 | 1.01966 |
| 0.98 | 1.45329 | 1.48953 | 1.02494 |
| 0.99 | 1.46330 | 1.50741 | 1.03014 |
| 1.00 | 1.47345 | 1.52541 | 1.03527 |

| x | $F_{4/7}(x)$ | $H_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 1.00 | 1.47345 | 1.52541 | 1.03527 |
| 1.01 | 1.48373 | 1.54354 | 1.04031 |
| 1.02 | 1.49414 | 1.56180 | 1.04528 |
| 1.03 | 1.50468 | 1.58018 | 1.05018 |
| 1.04 | 1.51537 | 1.59870 | 1.05499 |
| 1.05 | 1.52619 | 1.61736 | 1.05974 |
| 1.06 | 1.53714 | 1.63615 | 1.06441 |
| 1.07 | 1.54824 | 1.65508 | 1.06901 |
| 1.08 | 1.55947 | 1.67415 | 1.07353 |
| 1.09 | 1.57085 | 1.69336 | 1.07799 |
| 1.10 | 1.58237 | 1.71271 | 1.08237 |
| 1.11 | 1.59403 | 1.73221 | 1.08669 |
| 1.12 | 1.60584 | 1.75186 | 1.09093 |
| 1.13 | 1.61779 | 1.77166 | 1.09511 |
| 1.14 | 1.62989 | 1.79161 | 1.09923 |
| 1.15 | 1.64213 | 1.81172 | 1.10327 |
| 1.16 | 1.65453 | 1.83198 | 1.10725 |
| 1.17 | 1.66707 | 1.85240 | 1.11117 |
| 1.18 | 1.67977 | 1.87298 | 1.11502 |
| 1.19 | 1.69262 | 1.89372 | 1.11881 |
| 1.20 | 1.70562 | 1.91463 | 1.12254 |
| 1.21 | 1.71877 | 1.93570 | 1.12621 |
| 1.22 | 1.73209 | 1.95694 | 1.12982 |
| 1.23 | 1.74556 | 1.97835 | 1.13336 |
| 1.24 | 1.75919 | 1.99994 | 1.13685 |
| 1.25 | 1.77298 | 2.02170 | 1.14028 |
| 1.26 | 1.78693 | 2.04364 | 1.14366 |
| 1.27 | 1.80105 | 2.06575 | 1.14697 |
| 1.28 | 1.81532 | 2.08805 | 1.15024 |
| 1.29 | 1.82977 | 2.11053 | 1.15344 |
| 1.30 | 1.84438 | 2.13320 | 1.15660 |
| 1.31 | 1.85916 | 2.15606 | 1.15970 |
| 1.32 | 1.87411 | 2.17910 | 1.16274 |
| 1.33 | 1.88923 | 2.20234 | 1.16574 |
| 1.34 | 1.90452 | 2.22578 | 1.16868 |
| 1.35 | 1.91999 | 2.24941 | 1.17158 |
| 1.36 | 1.93563 | 2.27324 | 1.17442 |
| 1.37 | 1.95145 | 2.29728 | 1.17722 |
| 1.38 | 1.96744 | 2.32152 | 1.17997 |
| 1.39 | 1.98362 | 2.34596 | 1.18267 |
| 1.40 | 1.99998 | 2.37062 | 1.18532 |
| 1.41 | 2.01652 | 2.39549 | 1.18793 |
| 1.42 | 2.03325 | 2.42057 | 1.19049 |
| 1.43 | 2.05017 | 2.44587 | 1.19301 |
| 1.44 | 2.06727 | 2.47138 | 1.19548 |
| 1.45 | 2.08456 | 2.49712 | 1.19791 |
| 1.46 | 2.10204 | 2.52308 | 1.20030 |
| 1.47 | 2.11972 | 2.54928 | 1.20265 |
| 1.48 | 2.13759 | 2.57569 | 1.20495 |
| 1.49 | 2.15566 | 2.60235 | 1.20722 |
| 1.50 | 2.17392 | 2.62923 | 1.20944 |

TABLE D.XIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/7$ and $x$ from 0.00 to 1.50.

$\alpha = 4/7$

| x | $F_{4/7}(x)$ | $H_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 1.50 | 2.17392 | 2.62923 | 1.20944 |
| 1.51 | 2.19239 | 2.65636 | 1.21163 |
| 1.52 | 2.21105 | 2.68372 | 1.21377 |
| 1.53 | 2.22992 | 2.71132 | 1.21588 |
| 1.54 | 2.24900 | 2.73917 | 1.21795 |
| 1.55 | 2.26828 | 2.76727 | 1.21999 |
| 1.56 | 2.28777 | 2.79562 | 1.22198 |
| 1.57 | 2.30747 | 2.82423 | 1.22395 |
| 1.58 | 2.32739 | 2.85309 | 1.22587 |
| 1.59 | 2.34752 | 2.88220 | 1.22777 |
| 1.60 | 2.36787 | 2.91158 | 1.22962 |
| 1.61 | 2.38843 | 2.94123 | 1.23145 |
| 1.62 | 2.40922 | 2.97114 | 1.23324 |
| 1.63 | 2.43022 | 3.00133 | 1.23500 |
| 1.64 | 2.45146 | 3.03178 | 1.23673 |
| 1.65 | 2.47292 | 3.06252 | 1.23842 |
| 1.66 | 2.49460 | 3.09353 | 1.24009 |
| 1.67 | 2.51652 | 3.12483 | 1.24172 |
| 1.68 | 2.53867 | 3.15641 | 1.24333 |
| 1.69 | 2.56106 | 3.18828 | 1.24491 |
| 1.70 | 2.58368 | 3.22044 | 1.24645 |
| 1.71 | 2.60654 | 3.25289 | 1.24797 |
| 1.72 | 2.62965 | 3.28565 | 1.24946 |
| 1.73 | 2.65300 | 3.31871 | 1.25093 |
| 1.74 | 2.67659 | 3.35207 | 1.25237 |
| 1.75 | 2.70043 | 3.38573 | 1.25378 |
| 1.76 | 2.72452 | 3.41971 | 1.25516 |
| 1.77 | 2.74887 | 3.45401 | 1.25652 |
| 1.78 | 2.77347 | 3.48862 | 1.25785 |
| 1.79 | 2.79832 | 3.52355 | 1.25916 |
| 1.80 | 2.82344 | 3.55880 | 1.26045 |
| 1.81 | 2.84882 | 3.59439 | 1.26171 |
| 1.82 | 2.87447 | 3.63030 | 1.26295 |
| 1.83 | 2.90038 | 3.66655 | 1.26416 |
| 1.84 | 2.92656 | 3.70314 | 1.26536 |
| 1.85 | 2.95301 | 3.74007 | 1.26653 |
| 1.86 | 2.97974 | 3.77735 | 1.26768 |
| 1.87 | 3.00675 | 3.81497 | 1.26880 |
| 1.88 | 3.03403 | 3.85295 | 1.26991 |
| 1.89 | 3.06160 | 3.89128 | 1.27100 |
| 1.90 | 3.08945 | 3.92997 | 1.27206 |
| 1.91 | 3.11760 | 3.96903 | 1.27311 |
| 1.92 | 3.14603 | 4.00846 | 1.27413 |
| 1.93 | 3.17475 | 4.04825 | 1.27514 |
| 1.94 | 3.20377 | 4.08842 | 1.27613 |
| 1.95 | 3.23309 | 4.12897 | 1.27710 |
| 1.96 | 3.26272 | 4.16991 | 1.27805 |
| 1.97 | 3.29264 | 4.21123 | 1.27898 |
| 1.98 | 3.32288 | 4.25294 | 1.27990 |
| 1.99 | 3.35342 | 4.29504 | 1.28079 |
| 2.00 | 3.38428 | 4.33755 | 1.28168 |

| x | $F_{4/7}(x)$ | $H_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 2.0 | 3.38428 | 4.33755 | 1.28168 |
| 2.1 | 3.71076 | 4.78548 | 1.28962 |
| 2.2 | 4.07203 | 5.27811 | 1.29619 |
| 2.3 | 4.47160 | 5.82021 | 1.30159 |
| 2.4 | 4.91330 | 6.41702 | 1.30605 |
| 2.5 | 5.40141 | 7.07432 | 1.30972 |
| 2.6 | 5.94065 | 7.79848 | 1.31273 |
| 2.7 | 6.53623 | 8.59651 | 1.31521 |
| 2.8 | 7.19391 | 9.47614 | 1.31723 |
| 2.9 | 7.92006 | 10.44590 | 1.31892 |
| 3.0 | 8.72171 | 11.51518 | 1.32029 |
| 3.1 | 9.60664 | 12.69435 | 1.32141 |
| 3.2 | 10.58343 | 13.99486 | 1.32234 |
| 3.3 | 11.66155 | 15.42934 | 1.32309 |
| 3.4 | 12.85148 | 17.01171 | 1.32372 |
| 3.5 | 14.16478 | 18.75735 | 1.32422 |
| 3.6 | 15.61420 | 20.68323 | 1.32464 |
| 3.7 | 17.21385 | 22.80807 | 1.32498 |
| 3.8 | 18.97929 | 25.15257 | 1.32526 |
| 3.9 | 20.92769 | 27.73953 | 1.32549 |
| 4.0 | 23.07805 | 30.59416 | 1.32568 |
| 4.1 | 25.45130 | 33.74426 | 1.32584 |
| 4.2 | 28.07058 | 37.22054 | 1.32596 |
| 4.3 | 30.96144 | 41.05691 | 1.32607 |
| 4.4 | 34.15206 | 45.29079 | 1.32615 |
| 4.5 | 37.67360 | 49.96350 | 1.32622 |
| 4.6 | 41.56043 | 55.12066 | 1.32628 |
| 4.7 | 45.85052 | 60.81264 | 1.32632 |
| 4.8 | 50.58579 | 67.09506 | 1.32636 |
| 4.9 | 55.81251 | 74.02933 | 1.32639 |
| 5.0 | 61.58180 | 81.68325 | 1.32642 |
| 5.1 | 67.95008 | 90.13169 | 1.32644 |
| 5.2 | 74.97967 | 99.45731 | 1.32646 |
| 5.3 | 82.73938 | 109.75138 | 1.32647 |
| 5.4 | 91.30517 | 121.11471 | 1.32648 |
| 5.5 | 100.76094 | 133.65857 | 1.32649 |
| 5.6 | 111.19932 | 147.50587 | 1.32650 |
| 5.7 | 122.72263 | 162.79230 | 1.32651 |
| 5.8 | 135.44380 | 179.66772 | 1.32651 |
| 5.9 | 149.48759 | 198.29760 | 1.32652 |
| 6.0 | 164.99174 | 218.86467 | 1.32652 |

| x | $F_{4/7}(x)$ | $H_{3/7}(x)$ | $T_{4/7}(x)$ |
|---|---|---|---|
| 6.0 | 164.99174 | 218.86467 | 1.32652 |
| 6.1 | 182.10835 | 241.57070 | 1.32652 |
| 6.2 | 201.00541 | 266.63852 | 1.32652 |
| 6.3 | 221.86839 | 294.31420 | 1.32653 |
| 6.4 | 244.90215 | 324.86946 | 1.32653 |
| 6.5 | 270.33290 | 358.60439 | 1.32653 |
| 6.6 | 298.41044 | 395.85037 | 1.32653 |
| 6.7 | 329.41044 | 436.97335 | 1.32653 |
| 6.8 | 363.63818 | 482.37748 | 1.32653 |
| 6.9 | 401.42949 | 532.50907 | 1.32653 |
| 7.0 | 443.15614 | 587.86102 | 1.32653 |
| 7.1 | 489.22847 | 648.97766 | 1.32653 |
| 7.2 | 540.09965 | 716.46013 | 1.32653 |
| 7.3 | 596.27016 | 790.97233 | 1.32653 |
| 7.4 | 658.29272 | 873.24746 | 1.32653 |
| 7.5 | 726.77774 | 964.09527 | 1.32653 |
| 7.6 | 802.39937 | 1064.41005 | 1.32653 |
| 7.7 | 885.90216 | 1175.17944 | 1.32653 |
| 7.8 | 978.10839 | 1297.49424 | 1.32653 |
| 7.9 | 1079.92621 | 1432.55915 | 1.32653 |
| 8.0 | 1192.35862 | 1581.70466 | 1.32653 |
| 8.1 | 1316.51334 | 1746.40024 | 1.32653 |
| 8.2 | 1453.61383 | 1928.26881 | 1.32653 |
| 8.3 | 1605.01131 | 2129.10284 | 1.32653 |
| 8.4 | 1772.19815 | 2350.88200 | 1.32653 |
| 8.5 | 1956.82264 | 2595.79281 | 1.32653 |
| 8.6 | 2160.70525 | 2866.25017 | 1.32653 |
| 8.7 | 2385.85663 | 3164.92129 | 1.32653 |
| 8.8 | 2634.49754 | 3494.75209 | 1.32653 |
| 8.9 | 2909.08076 | 3858.99625 | 1.32653 |
| 9.0 | 3212.31536 | 4261.24747 | 1.32653 |
| 9.1 | 3547.19354 | 4705.47498 | 1.32653 |
| 9.2 | 3917.02016 | 5196.06282 | 1.32653 |
| 9.3 | 4325.44549 | 5737.85318 | 1.32653 |
| 9.4 | 4776.50131 | 6336.19435 | 1.32653 |
| 9.5 | 5274.64079 | 6996.99364 | 1.32653 |
| 9.6 | 5824.78258 | 7726.77579 | 1.32653 |
| 9.7 | 6432.35948 | 8532.74759 | 1.32653 |
| 9.8 | 7103.37220 | 9422.86922 | 1.32653 |
| 9.9 | 7844.44879 | 10405.93299 | 1.32653 |
| 10.0 | 8662.91020 | 11491.65041 | 1.32653 |

TABLE D.XIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/7$ and x from 1.50 to 10.0.

| $x$ | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ | $x$ | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ | $x$ | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.08911 | 1.66313 | 1.52705 | 1.00 | 1.37632 | 2.83900 | 2.06275 |
| 0.01 | 1.00004 | 0.16951 | 0.16951 | 0.51 | 1.09278 | 1.68529 | 1.54221 | 1.01 | 1.38444 | 2.86550 | 2.06980 |
| 0.02 | 1.00014 | 0.25191 | 0.25187 | 0.52 | 1.09652 | 1.70744 | 1.55714 | 1.02 | 1.39266 | 2.89219 | 2.07673 |
| 0.03 | 1.00032 | 0.31762 | 0.31752 | 0.53 | 1.10035 | 1.72957 | 1.57184 | 1.03 | 1.40099 | 2.91905 | 2.08356 |
| 0.04 | 1.00056 | 0.37442 | 0.37421 | 0.54 | 1.10425 | 1.75170 | 1.58632 | 1.04 | 1.40943 | 2.94609 | 2.09027 |
| 0.05 | 1.00088 | 0.42542 | 0.42505 | 0.55 | 1.10823 | 1.77382 | 1.60059 | 1.05 | 1.41797 | 2.97331 | 2.09688 |
| 0.06 | 1.00126 | 0.47223 | 0.47164 | 0.56 | 1.11229 | 1.79596 | 1.61464 | 1.06 | 1.42661 | 3.00072 | 2.10339 |
| 0.07 | 1.00172 | 0.51585 | 0.51496 | 0.57 | 1.11644 | 1.81810 | 1.62849 | 1.07 | 1.43537 | 3.02831 | 2.10978 |
| 0.08 | 1.00224 | 0.55691 | 0.55547 | 0.58 | 1.12066 | 1.84026 | 1.64212 | 1.08 | 1.44423 | 3.05610 | 2.11608 |
| 0.09 | 1.00284 | 0.59588 | 0.59419 | 0.59 | 1.12496 | 1.86244 | 1.65556 | 1.09 | 1.45320 | 3.08408 | 2.12227 |
| 0.10 | 1.00350 | 0.63309 | 0.63088 | 0.60 | 1.12934 | 1.88464 | 1.66879 | 1.10 | 1.46228 | 3.11227 | 2.12836 |
| 0.11 | 1.00424 | 0.66880 | 0.66598 | 0.61 | 1.13381 | 1.90687 | 1.68183 | 1.11 | 1.47147 | 3.14065 | 2.13436 |
| 0.12 | 1.00505 | 0.70321 | 0.59968 | 0.62 | 1.13836 | 1.92914 | 1.69467 | 1.12 | 1.48078 | 3.16924 | 2.14025 |
| 0.13 | 1.00592 | 0.73648 | 0.73214 | 0.63 | 1.14298 | 1.95144 | 1.70732 | 1.13 | 1.49019 | 3.19803 | 2.14605 |
| 0.14 | 1.00687 | 0.76874 | 0.76349 | 0.64 | 1.14770 | 1.97379 | 1.71979 | 1.14 | 1.49972 | 3.22703 | 2.15175 |
| 0.15 | 1.00789 | 0.80010 | 0.79384 | 0.65 | 1.15249 | 1.99619 | 1.73207 | 1.15 | 1.50936 | 3.25625 | 2.15737 |
| 0.16 | 1.00898 | 0.83066 | 0.82327 | 0.66 | 1.15737 | 2.01864 | 1.74416 | 1.16 | 1.51912 | 3.28569 | 2.16288 |
| 0.17 | 1.01014 | 0.86049 | 0.85185 | 0.67 | 1.16233 | 2.04114 | 1.75608 | 1.17 | 1.52900 | 3.31534 | 2.16831 |
| 0.18 | 1.01137 | 0.88966 | 0.87966 | 0.68 | 1.16737 | 2.06371 | 1.76782 | 1.18 | 1.53898 | 3.34521 | 2.17365 |
| 0.19 | 1.01267 | 0.91824 | 0.90675 | 0.69 | 1.17251 | 2.08634 | 1.77938 | 1.19 | 1.54909 | 3.37531 | 2.17890 |
| 0.20 | 1.01404 | 0.94626 | 0.93316 | 0.70 | 1.17772 | 2.10903 | 1.79078 | 1.20 | 1.55932 | 3.40564 | 2.18406 |
| 0.21 | 1.01548 | 0.97379 | 0.95894 | 0.71 | 1.18302 | 2.13180 | 1.80200 | 1.21 | 1.56966 | 3.43621 | 2.18914 |
| 0.22 | 1.01700 | 1.00086 | 0.98413 | 0.72 | 1.18841 | 2.15465 | 1.81305 | 1.22 | 1.58013 | 3.46700 | 2.19413 |
| 0.23 | 1.01859 | 1.02750 | 1.00875 | 0.73 | 1.19388 | 2.17757 | 1.82394 | 1.23 | 1.59071 | 3.49804 | 2.19904 |
| 0.24 | 1.02024 | 1.05375 | 1.03284 | 0.74 | 1.19944 | 2.20057 | 1.83466 | 1.24 | 1.60142 | 3.52931 | 2.20386 |
| 0.25 | 1.02197 | 1.07964 | 1.05643 | 0.75 | 1.20509 | 2.22367 | 1.84523 | 1.25 | 1.61225 | 3.56083 | 2.20861 |
| 0.26 | 1.02378 | 1.10520 | 1.07954 | 0.76 | 1.21083 | 2.24685 | 1.85563 | 1.26 | 1.62321 | 3.59260 | 2.21328 |
| 0.27 | 1.02565 | 1.13045 | 1.10218 | 0.77 | 1.21665 | 2.27013 | 1.86588 | 1.27 | 1.63428 | 3.62462 | 2.21786 |
| 0.28 | 1.02760 | 1.15542 | 1.12439 | 0.78 | 1.22256 | 2.29350 | 1.87597 | 1.28 | 1.64549 | 3.65689 | 2.22237 |
| 0.29 | 1.02962 | 1.18012 | 1.14618 | 0.79 | 1.22857 | 2.31697 | 1.88591 | 1.29 | 1.65682 | 3.68942 | 2.22681 |
| 0.30 | 1.03171 | 1.20458 | 1.16756 | 0.80 | 1.23466 | 2.34055 | 1.89570 | 1.30 | 1.66828 | 3.72221 | 2.23117 |
| 0.31 | 1.03387 | 1.22880 | 1.18855 | 0.81 | 1.24084 | 2.36424 | 1.90535 | 1.31 | 1.67987 | 3.75527 | 2.23545 |
| 0.32 | 1.03611 | 1.25282 | 1.20916 | 0.82 | 1.24712 | 2.38803 | 1.91484 | 1.32 | 1.69159 | 3.78859 | 2.23966 |
| 0.33 | 1.03842 | 1.27664 | 1.22941 | 0.83 | 1.25349 | 2.41194 | 1.92419 | 1.33 | 1.70344 | 3.82218 | 2.24380 |
| 0.34 | 1.04080 | 1.30029 | 1.24931 | 0.84 | 1.25994 | 2.43597 | 1.93340 | 1.34 | 1.71542 | 3.85605 | 2.24787 |
| 0.35 | 1.04326 | 1.32376 | 1.26887 | 0.85 | 1.26650 | 2.46012 | 1.94246 | 1.35 | 1.72753 | 3.89019 | 2.25187 |
| 0.36 | 1.04579 | 1.34708 | 1.28810 | 0.86 | 1.27314 | 2.48439 | 1.95139 | 1.36 | 1.73978 | 3.92461 | 2.25580 |
| 0.37 | 1.04840 | 1.37026 | 1.30701 | 0.87 | 1.27989 | 2.50879 | 1.96018 | 1.37 | 1.75217 | 3.95931 | 2.25967 |
| 0.38 | 1.05107 | 1.39331 | 1.32560 | 0.88 | 1.28671 | 2.53332 | 1.96883 | 1.38 | 1.76469 | 3.99431 | 2.26347 |
| 0.39 | 1.05383 | 1.41624 | 1.34390 | 0.89 | 1.29364 | 2.55799 | 1.97735 | 1.39 | 1.77734 | 4.02959 | 2.26720 |
| 0.40 | 1.05666 | 1.43905 | 1.36189 | 0.90 | 1.30067 | 2.58279 | 1.98574 | 1.40 | 1.79014 | 4.06517 | 2.27087 |
| 0.41 | 1.05956 | 1.46177 | 1.37960 | 0.91 | 1.30779 | 2.60773 | 1.99400 | 1.41 | 1.80307 | 4.10104 | 2.27447 |
| 0.42 | 1.06254 | 1.48440 | 1.39703 | 0.92 | 1.31500 | 2.63281 | 2.00213 | 1.42 | 1.81615 | 4.13721 | 2.27801 |
| 0.43 | 1.06559 | 1.50694 | 1.41418 | 0.93 | 1.32232 | 2.65804 | 2.01013 | 1.43 | 1.82937 | 4.17369 | 2.28149 |
| 0.44 | 1.06872 | 1.52942 | 1.43137 | 0.94 | 1.32973 | 2.68342 | 2.01801 | 1.44 | 1.84273 | 4.21048 | 2.28491 |
| 0.45 | 1.07193 | 1.55182 | 1.44769 | 0.95 | 1.33725 | 2.70895 | 2.02577 | 1.45 | 1.85624 | 4.24757 | 2.28827 |
| 0.46 | 1.07521 | 1.57417 | 1.46406 | 0.96 | 1.34486 | 2.73464 | 2.03340 | 1.46 | 1.86989 | 4.28499 | 2.29158 |
| 0.47 | 1.07857 | 1.59647 | 1.48017 | 0.97 | 1.35257 | 2.76048 | 2.04091 | 1.47 | 1.88368 | 4.32272 | 2.29482 |
| 0.48 | 1.08200 | 1.61872 | 1.49604 | 0.98 | 1.36039 | 2.78649 | 2.04831 | 1.48 | 1.89763 | 4.36077 | 2.29801 |
| 0.49 | 1.08552 | 1.64094 | 1.51167 | 0.99 | 1.36830 | 2.81266 | 2.05558 | 1.49 | 1.91173 | 4.39914 | 2.30114 |
| 0.50 | 1.08911 | 1.66313 | 1.52705 | 1.00 | 1.37632 | 2.83900 | 2.06275 | 1.50 | 1.92597 | 4.43785 | 2.30422 |

TABLE D.XIIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/7$ and $x$ from 0.00 to 1.50.

$\alpha = 5/7$

| x | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ |
|---|---|---|---|
| 1.50 | 1.92597 | 4.43785 | 2.30422 |
| 1.51 | 1.94037 | 4.47689 | 2.30724 |
| 1.52 | 1.95492 | 4.51627 | 2.31021 |
| 1.53 | 1.96962 | 4.55598 | 2.31313 |
| 1.54 | 1.98448 | 4.59604 | 2.31599 |
| 1.55 | 1.99950 | 4.63645 | 2.31881 |
| 1.56 | 2.01467 | 4.67721 | 2.32157 |
| 1.57 | 2.03001 | 4.71832 | 2.32429 |
| 1.58 | 2.04550 | 4.75979 | 2.32696 |
| 1.59 | 2.06116 | 4.80163 | 2.32958 |
| 1.60 | 2.07698 | 4.84383 | 2.33215 |
| 1.61 | 2.09297 | 4.88640 | 2.33468 |
| 1.62 | 2.10912 | 4.92935 | 2.33716 |
| 1.63 | 2.12544 | 4.97268 | 2.33960 |
| 1.64 | 2.14193 | 5.01639 | 2.34199 |
| 1.65 | 2.15859 | 5.06049 | 2.34435 |
| 1.66 | 2.17543 | 5.10497 | 2.34665 |
| 1.67 | 2.19243 | 5.14986 | 2.34892 |
| 1.68 | 2.20962 | 5.19514 | 2.35115 |
| 1.69 | 2.22698 | 5.24082 | 2.35334 |
| 1.70 | 2.24451 | 5.28692 | 2.35548 |
| 1.71 | 2.26223 | 5.33342 | 2.35759 |
| 1.72 | 2.28013 | 5.38035 | 2.35966 |
| 1.73 | 2.29821 | 5.42769 | 2.36170 |
| 1.74 | 2.31648 | 5.47546 | 2.36369 |
| 1.75 | 2.33494 | 5.52365 | 2.36565 |
| 1.76 | 2.35358 | 5.57228 | 2.36758 |
| 1.77 | 2.37241 | 5.62135 | 2.36947 |
| 1.78 | 2.39143 | 5.67086 | 2.37132 |
| 1.79 | 2.41065 | 5.72082 | 2.37314 |
| 1.80 | 2.43006 | 5.77123 | 2.37493 |
| 1.81 | 2.44967 | 5.82210 | 2.37669 |
| 1.82 | 2.46948 | 5.87343 | 2.37841 |
| 1.83 | 2.48948 | 5.92522 | 2.38010 |
| 1.84 | 2.50969 | 5.97749 | 2.38176 |
| 1.85 | 2.53010 | 6.03023 | 2.38339 |
| 1.86 | 2.55072 | 6.08345 | 2.38499 |
| 1.87 | 2.57154 | 6.13716 | 2.38657 |
| 1.88 | 2.59258 | 6.19135 | 2.38811 |
| 1.89 | 2.61382 | 6.24604 | 2.38962 |
| 1.90 | 2.63528 | 6.30124 | 2.39111 |
| 1.91 | 2.65695 | 6.35693 | 2.39257 |
| 1.92 | 2.67884 | 6.41314 | 2.39400 |
| 1.93 | 2.70095 | 6.46986 | 2.39540 |
| 1.94 | 2.72328 | 6.52710 | 2.39678 |
| 1.95 | 2.74583 | 6.58487 | 2.39813 |
| 1.96 | 2.76861 | 6.64317 | 2.39946 |
| 1.97 | 2.79161 | 6.70200 | 2.40077 |
| 1.98 | 2.81484 | 6.76138 | 2.40205 |
| 1.99 | 2.83831 | 6.82130 | 2.40330 |
| 2.00 | 2.86200 | 6.88178 | 2.40453 |

| x | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ |
|---|---|---|---|
| 2.0 | 2.86200 | 6.88178 | 2.40453 |
| 2.1 | 3.11226 | 7.51817 | 2.41566 |
| 2.2 | 3.38830 | 8.21618 | 2.42487 |
| 2.3 | 3.69260 | 8.98218 | 2.43248 |
| 2.4 | 4.02789 | 9.82310 | 2.43877 |
| 2.5 | 4.39721 | 10.74658 | 2.44395 |
| 2.6 | 4.80389 | 11.76102 | 2.44823 |
| 2.7 | 5.25161 | 12.87562 | 2.45175 |
| 2.8 | 5.74443 | 14.10052 | 2.45464 |
| 2.9 | 6.28682 | 15.44638 | 2.45703 |
| 3.0 | 6.88371 | 16.92695 | 2.45898 |
| 3.1 | 7.54055 | 18.55424 | 2.46060 |
| 3.2 | 8.26332 | 20.34362 | 2.46192 |
| 3.3 | 9.05863 | 22.31145 | 2.46301 |
| 3.4 | 9.93375 | 24.47575 | 2.46390 |
| 3.5 | 10.89672 | 26.85638 | 2.46463 |
| 3.6 | 11.95636 | 29.47520 | 2.46523 |
| 3.7 | 13.12242 | 32.35628 | 2.46572 |
| 3.8 | 14.40564 | 35.52616 | 2.46613 |
| 3.9 | 15.81783 | 39.01407 | 2.46646 |
| 4.0 | 17.37204 | 42.85220 | 2.46673 |
| 4.1 | 19.08263 | 47.07602 | 2.46696 |
| 4.2 | 20.96542 | 51.72464 | 2.46714 |
| 4.3 | 23.03787 | 56.84113 | 2.46729 |
| 4.4 | 25.31920 | 62.47294 | 2.46741 |
| 4.5 | 27.83060 | 68.67241 | 2.46751 |
| 4.6 | 30.59541 | 75.49716 | 2.46760 |
| 4.7 | 33.63938 | 83.01073 | 2.46767 |
| 4.8 | 36.99088 | 91.28315 | 2.46772 |
| 4.9 | 40.68116 | 100.39153 | 2.46777 |
| 5.0 | 44.74468 | 110.42108 | 2.46780 |
| 5.1 | 49.21945 | 121.46542 | 2.46783 |
| 5.2 | 54.14731 | 133.62794 | 2.46786 |
| 5.3 | 59.57444 | 147.02256 | 2.46788 |
| 5.4 | 65.55171 | 161.77484 | 2.46790 |
| 5.5 | 72.13521 | 178.02323 | 2.46791 |
| 5.6 | 79.38677 | 195.92032 | 2.46792 |
| 5.7 | 87.37456 | 215.63436 | 2.46793 |
| 5.8 | 96.17372 | 237.35081 | 2.46794 |
| 5.9 | 105.86711 | 261.27416 | 2.46794 |
| 6.0 | 116.54607 | 287.62985 | 2.46795 |

| x | $F_{5/7}(x)$ | $H_{2/7}(x)$ | $T_{5/7}(x)$ |
|---|---|---|---|
| 6.0 | 116.54607 | 287.62985 | 2.46795 |
| 6.1 | 128.31133 | 316.65644 | 2.46795 |
| 6.2 | 141.27392 | 348.65799 | 2.46796 |
| 6.3 | 155.55627 | 383.90646 | 2.46796 |
| 6.4 | 171.29339 | 422.74545 | 2.46796 |
| 6.5 | 188.63415 | 465.54233 | 2.46796 |
| 6.6 | 207.74267 | 512.70180 | 2.46797 |
| 6.7 | 228.79997 | 564.67077 | 2.46797 |
| 6.8 | 252.00563 | 621.94183 | 2.46797 |
| 6.9 | 277.57975 | 685.05819 | 2.46797 |
| 7.0 | 305.76505 | 754.61886 | 2.46797 |
| 7.1 | 336.82921 | 831.28444 | 2.46797 |
| 7.2 | 371.06741 | 915.78346 | 2.46797 |
| 7.3 | 408.80519 | 1008.91936 | 2.46797 |
| 7.4 | 450.40159 | 1111.57821 | 2.46797 |
| 7.5 | 496.25257 | 1224.73724 | 2.46797 |
| 7.6 | 546.79485 | 1349.47425 | 2.46797 |
| 7.7 | 602.51007 | 1486.97796 | 2.46797 |
| 7.8 | 663.92949 | 1638.55946 | 2.46797 |
| 7.9 | 731.63934 | 1805.66482 | 2.46797 |
| 8.0 | 806.28500 | 1989.88904 | 2.46797 |
| 8.1 | 888.58021 | 2192.99140 | 2.46797 |
| 8.2 | 979.31095 | 2416.91242 | 2.46797 |
| 8.3 | 1079.34451 | 2663.79257 | 2.46797 |
| 8.4 | 1189.63759 | 2935.99288 | 2.46797 |
| 8.5 | 1311.24544 | 3236.11779 | 2.46797 |
| 8.6 | 1445.33217 | 3567.04018 | 2.46797 |
| 8.7 | 1593.18185 | 3931.92918 | 2.46797 |
| 8.8 | 1756.21102 | 4334.28075 | 2.46797 |
| 8.9 | 1935.98231 | 4777.95140 | 2.46797 |
| 9.0 | 2134.21954 | 5267.19550 | 2.46797 |
| 9.1 | 2352.82441 | 5806.70635 | 2.46797 |
| 9.2 | 2593.89483 | 6401.66157 | 2.46797 |
| 9.3 | 2859.74524 | 7057.77317 | 2.46797 |
| 9.4 | 3152.92902 | 7781.34273 | 2.46797 |
| 9.5 | 3476.26309 | 8579.32242 | 2.46797 |
| 9.6 | 3832.85530 | 9459.38226 | 2.46797 |
| 9.7 | 4226.13439 | 10429.98434 | 2.46797 |
| 9.8 | 4659.88325 | 11500.46471 | 2.46797 |
| 9.9 | 5134.27555 | 12681.12386 | 2.46797 |
| 10.0 | 5665.91613 | 13983.32643 | 2.46797 |

TABLE D.XIIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/7$ and $x$ from 1.50 to 10.0.

$\alpha = 4/9$

| x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ | x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ | x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.14369 | 0.40145 | 0.35101 | 1.00 | 1.61287 | 0.97391 | 0.60384 |
| 0.01 | 1.00006 | 0.00500 | 0.00500 | 0.51 | 1.14963 | 0.41104 | 0.35754 | 1.01 | 1.62626 | 0.98774 | 0.60737 |
| 0.02 | 1.00023 | 0.01079 | 0.01079 | 0.52 | 1.15569 | 0.42069 | 0.36401 | 1.02 | 1.63982 | 1.00165 | 0.61084 |
| 0.03 | 1.00051 | 0.01693 | 0.01693 | 0.53 | 1.16138 | 0.43040 | 0.37043 | 1.03 | 1.65356 | 1.01574 | 0.61427 |
| 0.04 | 1.00090 | 0.02332 | 0.02329 | 0.54 | 1.16821 | 0.44017 | 0.37679 | 1.04 | 1.66749 | 1.02992 | 0.61764 |
| 0.05 | 1.00141 | 0.02988 | 0.02984 | 0.55 | 1.17466 | 0.45001 | 0.38310 | 1.05 | 1.68159 | 1.04421 | 0.62096 |
| 0.06 | 1.00203 | 0.03660 | 0.03652 | 0.56 | 1.18124 | 0.45992 | 0.38935 | 1.06 | 1.69588 | 1.05862 | 0.62423 |
| 0.07 | 1.00276 | 0.04344 | 0.04332 | 0.57 | 1.18795 | 0.46989 | 0.39554 | 1.07 | 1.71036 | 1.07315 | 0.62744 |
| 0.08 | 1.00360 | 0.05040 | 0.05022 | 0.58 | 1.19480 | 0.47992 | 0.40168 | 1.08 | 1.72502 | 1.08761 | 0.63061 |
| 0.09 | 1.00456 | 0.05747 | 0.05720 | 0.59 | 1.20178 | 0.49003 | 0.40775 | 1.09 | 1.73986 | 1.10259 | 0.63372 |
| 0.10 | 1.00563 | 0.06462 | 0.06426 | 0.60 | 1.20839 | 0.50020 | 0.41377 | 1.10 | 1.75490 | 1.11749 | 0.63678 |
| 0.11 | 1.00681 | 0.07187 | 0.07138 | 0.61 | 1.21613 | 0.51044 | 0.41973 | 1.11 | 1.77013 | 1.13252 | 0.63980 |
| 0.12 | 1.00811 | 0.07919 | 0.07855 | 0.62 | 1.22351 | 0.52075 | 0.42562 | 1.12 | 1.78555 | 1.14768 | 0.64276 |
| 0.13 | 1.00952 | 0.08659 | 0.08577 | 0.63 | 1.23103 | 0.53113 | 0.43146 | 1.13 | 1.80116 | 1.16297 | 0.64568 |
| 0.14 | 1.01104 | 0.09406 | 0.09303 | 0.64 | 1.23868 | 0.54159 | 0.43723 | 1.14 | 1.81697 | 1.17839 | 0.64855 |
| 0.15 | 1.01268 | 0.10160 | 0.10033 | 0.65 | 1.24647 | 0.55211 | 0.44294 | 1.15 | 1.83298 | 1.19395 | 0.65137 |
| 0.16 | 1.01443 | 0.10921 | 0.10766 | 0.66 | 1.25440 | 0.56271 | 0.44859 | 1.16 | 1.84918 | 1.20964 | 0.65415 |
| 0.17 | 1.01630 | 0.11688 | 0.11501 | 0.67 | 1.26247 | 0.57338 | 0.45418 | 1.17 | 1.86559 | 1.22546 | 0.65688 |
| 0.18 | 1.01828 | 0.12462 | 0.12238 | 0.68 | 1.27067 | 0.58413 | 0.45970 | 1.18 | 1.88220 | 1.24142 | 0.65956 |
| 0.19 | 1.02037 | 0.13241 | 0.12977 | 0.69 | 1.27902 | 0.59496 | 0.46517 | 1.19 | 1.89901 | 1.25752 | 0.66220 |
| 0.20 | 1.02258 | 0.14027 | 0.13717 | 0.70 | 1.28751 | 0.60586 | 0.47057 | 1.20 | 1.91603 | 1.27377 | 0.66480 |
| 0.21 | 1.02490 | 0.14818 | 0.14458 | 0.71 | 1.29614 | 0.61684 | 0.47590 | 1.21 | 1.93325 | 1.29015 | 0.66735 |
| 0.22 | 1.02734 | 0.15614 | 0.15199 | 0.72 | 1.30491 | 0.62789 | 0.48118 | 1.22 | 1.95068 | 1.30668 | 0.66986 |
| 0.23 | 1.02989 | 0.16417 | 0.15940 | 0.73 | 1.31383 | 0.63903 | 0.48639 | 1.23 | 1.96833 | 1.32335 | 0.67232 |
| 0.24 | 1.03256 | 0.17225 | 0.16682 | 0.74 | 1.32290 | 0.65025 | 0.49153 | 1.24 | 1.98619 | 1.34018 | 0.67475 |
| 0.25 | 1.03535 | 0.18038 | 0.17422 | 0.75 | 1.33211 | 0.66155 | 0.49662 | 1.25 | 2.00426 | 1.35715 | 0.67713 |
| 0.26 | 1.03825 | 0.18857 | 0.18162 | 0.76 | 1.34146 | 0.67293 | 0.50164 | 1.26 | 2.02255 | 1.37427 | 0.67947 |
| 0.27 | 1.04127 | 0.19681 | 0.18901 | 0.77 | 1.35097 | 0.68440 | 0.50660 | 1.27 | 2.04106 | 1.39154 | 0.68178 |
| 0.28 | 1.04440 | 0.20511 | 0.19639 | 0.78 | 1.36062 | 0.69595 | 0.51150 | 1.28 | 2.05979 | 1.40897 | 0.68404 |
| 0.29 | 1.04765 | 0.21346 | 0.20375 | 0.79 | 1.37042 | 0.70759 | 0.51633 | 1.29 | 2.07874 | 1.42656 | 0.68626 |
| 0.30 | 1.05102 | 0.22186 | 0.21109 | 0.80 | 1.38038 | 0.71932 | 0.52110 | 1.30 | 2.09791 | 1.44430 | 0.68845 |
| 0.31 | 1.05451 | 0.23032 | 0.21841 | 0.81 | 1.39048 | 0.73113 | 0.52581 | 1.31 | 2.11732 | 1.46220 | 0.69059 |
| 0.32 | 1.05811 | 0.23883 | 0.22571 | 0.82 | 1.40074 | 0.74304 | 0.53046 | 1.32 | 2.13695 | 1.48027 | 0.69270 |
| 0.33 | 1.06184 | 0.24739 | 0.23299 | 0.83 | 1.41116 | 0.75503 | 0.53504 | 1.33 | 2.15681 | 1.49850 | 0.69478 |
| 0.34 | 1.06568 | 0.25601 | 0.24023 | 0.84 | 1.42173 | 0.76712 | 0.53957 | 1.34 | 2.17690 | 1.51689 | 0.69681 |
| 0.35 | 1.06964 | 0.26468 | 0.24745 | 0.85 | 1.43245 | 0.77929 | 0.54403 | 1.35 | 2.19722 | 1.53545 | 0.69881 |
| 0.36 | 1.07372 | 0.27341 | 0.25463 | 0.86 | 1.44333 | 0.79157 | 0.54843 | 1.36 | 2.21779 | 1.55418 | 0.70078 |
| 0.37 | 1.07792 | 0.28218 | 0.26179 | 0.87 | 1.45437 | 0.80393 | 0.55277 | 1.37 | 2.23859 | 1.57308 | 0.70271 |
| 0.38 | 1.08225 | 0.29102 | 0.26890 | 0.88 | 1.46557 | 0.81640 | 0.55705 | 1.38 | 2.25963 | 1.59215 | 0.70460 |
| 0.39 | 1.08669 | 0.29991 | 0.27598 | 0.89 | 1.47693 | 0.82896 | 0.56127 | 1.39 | 2.28092 | 1.61140 | 0.70647 |
| 0.40 | 1.09125 | 0.30885 | 0.28303 | 0.90 | 1.48846 | 0.84162 | 0.56543 | 1.40 | 2.30245 | 1.63082 | 0.70830 |
| 0.41 | 1.09594 | 0.31785 | 0.29003 | 0.91 | 1.50014 | 0.85438 | 0.56953 | 1.41 | 2.32423 | 1.65042 | 0.71009 |
| 0.42 | 1.10075 | 0.32691 | 0.29699 | 0.92 | 1.51199 | 0.86724 | 0.57357 | 1.42 | 2.34626 | 1.67020 | 0.71186 |
| 0.43 | 1.10568 | 0.33602 | 0.30391 | 0.93 | 1.52401 | 0.88020 | 0.57756 | 1.43 | 2.36854 | 1.69017 | 0.71359 |
| 0.44 | 1.11074 | 0.34519 | 0.31078 | 0.94 | 1.53619 | 0.89327 | 0.58148 | 1.44 | 2.39107 | 1.71032 | 0.71529 |
| 0.45 | 1.11592 | 0.35442 | 0.31761 | 0.95 | 1.54854 | 0.90644 | 0.58535 | 1.45 | 2.41386 | 1.73065 | 0.71696 |
| 0.46 | 1.12122 | 0.36371 | 0.32439 | 0.96 | 1.56107 | 0.91972 | 0.58916 | 1.46 | 2.43691 | 1.75118 | 0.71861 |
| 0.47 | 1.12665 | 0.37305 | 0.33112 | 0.97 | 1.57376 | 0.93310 | 0.59291 | 1.47 | 2.46022 | 1.77189 | 0.72022 |
| 0.48 | 1.13220 | 0.38246 | 0.33780 | 0.98 | 1.58662 | 0.94660 | 0.59661 | 1.48 | 2.48380 | 1.79280 | 0.72180 |
| 0.49 | 1.13789 | 0.39192 | 0.34443 | 0.99 | 1.59966 | 0.96020 | 0.60025 | 1.49 | 2.50764 | 1.81391 | 0.72335 |
| 0.50 | 1.14369 | 0.40145 | 0.35101 | 1.00 | 1.61287 | 0.97391 | 0.60384 | 1.50 | 2.53175 | 1.83521 | 0.72488 |

TABLE D.XIVA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/9$ and $x$ from 0.00 to 1.50.

$\alpha = 4/9$

| x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ |
|---|---|---|---|
| 1.50 | 2.53175 | 1.83521 | 0.72488 |
| 1.51 | 2.55613 | 1.85671 | 0.72637 |
| 1.52 | 2.58079 | 1.87841 | 0.72784 |
| 1.53 | 2.60572 | 1.90032 | 0.72929 |
| 1.54 | 2.63093 | 1.92243 | 0.73070 |
| 1.55 | 2.65642 | 1.94475 | 0.73209 |
| 1.56 | 2.68219 | 1.96728 | 0.73346 |
| 1.57 | 2.70826 | 1.99002 | 0.73480 |
| 1.58 | 2.73461 | 2.01298 | 0.73611 |
| 1.59 | 2.76125 | 2.03616 | 0.73740 |
| 1.60 | 2.78819 | 2.05955 | 0.73867 |
| 1.61 | 2.81542 | 2.08317 | 0.73991 |
| 1.62 | 2.84296 | 2.10701 | 0.74113 |
| 1.63 | 2.87080 | 2.13108 | 0.74233 |
| 1.64 | 2.89895 | 2.15538 | 0.74351 |
| 1.65 | 2.92740 | 2.17991 | 0.74466 |
| 1.66 | 2.95617 | 2.20468 | 0.74579 |
| 1.67 | 2.98525 | 2.22968 | 0.74690 |
| 1.68 | 3.01465 | 2.25493 | 0.74799 |
| 1.69 | 3.04437 | 2.28041 | 0.74906 |
| 1.70 | 3.07441 | 2.30614 | 0.75011 |
| 1.71 | 3.10479 | 2.33212 | 0.75114 |
| 1.72 | 3.13549 | 2.35835 | 0.75215 |
| 1.73 | 3.16652 | 2.38483 | 0.75314 |
| 1.74 | 3.19790 | 2.41157 | 0.75411 |
| 1.75 | 3.22961 | 2.43856 | 0.75506 |
| 1.76 | 3.26166 | 2.46582 | 0.75600 |
| 1.77 | 3.29407 | 2.49334 | 0.75692 |
| 1.78 | 3.32682 | 2.52112 | 0.75782 |
| 1.79 | 3.35992 | 2.54918 | 0.75870 |
| 1.80 | 3.39339 | 2.57751 | 0.75957 |
| 1.81 | 3.42721 | 2.60611 | 0.76042 |
| 1.82 | 3.46139 | 2.63499 | 0.76125 |
| 1.83 | 3.49595 | 2.66416 | 0.76207 |
| 1.84 | 3.53087 | 2.69360 | 0.76287 |
| 1.85 | 3.56617 | 2.72334 | 0.76366 |
| 1.86 | 3.60184 | 2.75336 | 0.76443 |
| 1.87 | 3.63790 | 2.78368 | 0.76519 |
| 1.88 | 3.67434 | 2.81430 | 0.76593 |
| 1.89 | 3.71117 | 2.84521 | 0.76666 |
| 1.90 | 3.74840 | 2.87643 | 0.76738 |
| 1.91 | 3.78602 | 2.90795 | 0.76808 |
| 1.92 | 3.82404 | 2.93978 | 0.76876 |
| 1.93 | 3.86247 | 2.97193 | 0.76944 |
| 1.94 | 3.90130 | 3.00439 | 0.77010 |
| 1.95 | 3.94055 | 3.03717 | 0.77075 |
| 1.96 | 3.98021 | 3.07027 | 0.77138 |
| 1.97 | 4.02030 | 3.10370 | 0.77201 |
| 1.98 | 4.06081 | 3.13746 | 0.77262 |
| 1.99 | 4.10175 | 3.17155 | 0.77322 |
| 2.00 | 4.14312 | 3.20597 | 0.77381 |

| x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ |
|---|---|---|---|
| 2.0 | 4.14312 | 3.20597 | 0.77381 |
| 2.1 | 4.58162 | 3.56953 | 0.77910 |
| 2.2 | 5.06842 | 3.97083 | 0.78345 |
| 2.3 | 5.60854 | 4.41401 | 0.78702 |
| 2.4 | 6.20754 | 4.90362 | 0.78995 |
| 2.5 | 6.87159 | 5.44468 | 0.79235 |
| 2.6 | 7.60752 | 6.04277 | 0.79432 |
| 2.7 | 8.42290 | 6.70403 | 0.79593 |
| 2.8 | 9.32612 | 7.43525 | 0.79725 |
| 2.9 | 10.32646 | 8.24394 | 0.79833 |
| 3.0 | 11.43418 | 9.13839 | 0.79922 |
| 3.1 | 12.66066 | 10.12780 | 0.79994 |
| 3.2 | 14.01848 | 11.22230 | 0.80054 |
| 3.3 | 15.52156 | 12.43311 | 0.80102 |
| 3.4 | 17.18533 | 13.77266 | 0.80142 |
| 3.5 | 19.02681 | 15.25465 | 0.80175 |
| 3.6 | 21.06487 | 16.89427 | 0.80201 |
| 3.7 | 23.32039 | 18.70830 | 0.80223 |
| 3.8 | 25.81643 | 20.71531 | 0.80241 |
| 3.9 | 28.57854 | 22.93582 | 0.80255 |
| 4.0 | 31.63496 | 25.39254 | 0.80267 |
| 4.1 | 35.01693 | 28.11058 | 0.80277 |
| 4.2 | 38.75901 | 31.11772 | 0.80285 |
| 4.3 | 42.89942 | 34.44467 | 0.80292 |
| 4.4 | 47.48046 | 38.12540 | 0.80297 |
| 4.5 | 52.54887 | 42.19749 | 0.80301 |
| 4.6 | 58.15639 | 46.70249 | 0.80305 |
| 4.7 | 64.36021 | 51.68637 | 0.80308 |
| 4.8 | 71.22362 | 57.19994 | 0.80310 |
| 4.9 | 78.81658 | 63.29943 | 0.80312 |
| 5.0 | 87.21650 | 70.04700 | 0.80314 |
| 5.1 | 96.50895 | 77.51141 | 0.80315 |
| 5.2 | 106.78862 | 85.76871 | 0.80316 |
| 5.3 | 118.16019 | 94.90298 | 0.80317 |
| 5.4 | 130.73944 | 105.00723 | 0.80318 |
| 5.5 | 144.65443 | 116.18432 | 0.80319 |
| 5.6 | 160.04676 | 128.54799 | 0.80319 |
| 5.7 | 177.07305 | 142.22404 | 0.80319 |
| 5.8 | 195.90649 | 157.35158 | 0.80320 |
| 5.9 | 216.73860 | 174.08446 | 0.80320 |
| 6.0 | 239.78122 | 192.59281 | 0.80320 |

| x | $F_{4/9}(x)$ | $H_{5/9}(x)$ | $T_{4/9}(x)$ |
|---|---|---|---|
| 6.0 | 239.78122 | 152.59281 | 0.80320 |
| 6.1 | 265.26857 | 213.06478 | 0.80320 |
| 6.2 | 293.45969 | 235.70843 | 0.80321 |
| 6.3 | 324.64105 | 260.75386 | 0.80321 |
| 6.4 | 359.12941 | 288.45349 | 0.80321 |
| 6.5 | 397.27510 | 319.09472 | 0.80321 |
| 6.6 | 439.46550 | 352.98269 | 0.80321 |
| 6.7 | 486.12900 | 390.46350 | 0.80321 |
| 6.8 | 537.73930 | 431.91763 | 0.80321 |
| 6.9 | 594.82027 | 477.76585 | 0.80321 |
| 7.0 | 657.95116 | 528.47343 | 0.80321 |
| 7.1 | 727.77254 | 584.55489 | 0.80321 |
| 7.2 | 804.99274 | 646.57917 | 0.80321 |
| 7.3 | 890.39504 | 715.17539 | 0.80321 |
| 7.4 | 984.84556 | 791.03924 | 0.80321 |
| 7.5 | 1089.30204 | 874.94000 | 0.80321 |
| 7.6 | 1204.82353 | 967.72831 | 0.80321 |
| 7.7 | 1332.58110 | 1070.34478 | 0.80321 |
| 7.8 | 1473.86967 | 1183.82951 | 0.80321 |
| 7.9 | 1630.12111 | 1309.33258 | 0.80321 |
| 8.0 | 1802.91869 | 1448.12572 | 0.80321 |
| 8.1 | 1994.01315 | 1601.61512 | 0.80321 |
| 8.2 | 2205.34032 | 1771.35568 | 0.80321 |
| 8.3 | 2439.04073 | 1959.06669 | 0.80321 |
| 8.4 | 2697.48122 | 2166.64924 | 0.80321 |
| 8.5 | 2983.27890 | 2396.20540 | 0.80321 |
| 8.6 | 3299.32752 | 2650.05949 | 0.80321 |
| 8.7 | 3648.82678 | 2930.78153 | 0.80321 |
| 8.8 | 4035.31463 | 3241.21324 | 0.80321 |
| 8.9 | 4462.70298 | 3584.49673 | 0.80321 |
| 9.0 | 4935.31726 | 3964.10624 | 0.80321 |
| 9.1 | 5457.94009 | 4383.88321 | 0.80321 |
| 9.2 | 6035.85958 | 4848.07513 | 0.80321 |
| 9.3 | 6674.92275 | 5361.37839 | 0.80321 |
| 9.4 | 7381.59456 | 5928.98572 | 0.80321 |
| 9.5 | 8163.02320 | 6556.63863 | 0.80321 |
| 9.6 | 9027.11224 | 7250.68538 | 0.80321 |
| 9.7 | 9982.60048 | 8018.14507 | 0.80321 |
| 9.8 | 11039.15012 | 8866.77849 | 0.80321 |
| 9.9 | 12207.44434 | 9805.16650 | 0.80321 |
| 10.0 | 13499.29508 | 10842.79661 | 0.80321 |

TABLE D.XIVB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 4/9$ and x from 1.50 to 10.0.

$\alpha = 5/9$

| $x$ | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ | $x$ | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ | $x$ | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.11478 | 0.68493 | 0.61441 | 1.00 | 1.48736 | 1.43639 | 0.96573 |
| 0.01 | 1.00005 | 0.02027 | 0.02027 | 0.51 | 1.11951 | 0.69829 | 0.62375 | 1.01 | 1.49795 | 1.45387 | 0.97057 |
| 0.02 | 1.00018 | 0.03753 | 0.03753 | 0.52 | 1.12435 | 0.71169 | 0.63298 | 1.02 | 1.50867 | 1.47148 | 0.97534 |
| 0.03 | 1.00041 | 0.05393 | 0.05380 | 0.53 | 1.12928 | 0.72514 | 0.64212 | 1.03 | 1.51954 | 1.48921 | 0.98004 |
| 0.04 | 1.00072 | 0.06952 | 0.06947 | 0.54 | 1.13432 | 0.73863 | 0.65116 | 1.04 | 1.53054 | 1.50708 | 0.98467 |
| 0.05 | 1.00113 | 0.08478 | 0.08469 | 0.55 | 1.13947 | 0.75216 | 0.66010 | 1.05 | 1.54169 | 1.52508 | 0.98922 |
| 0.06 | 1.00162 | 0.09972 | 0.09956 | 0.56 | 1.14471 | 0.76574 | 0.66894 | 1.06 | 1.55298 | 1.54321 | 0.99371 |
| 0.07 | 1.00221 | 0.11439 | 0.11414 | 0.57 | 1.15006 | 0.77938 | 0.67768 | 1.07 | 1.56441 | 1.56147 | 0.99812 |
| 0.08 | 1.00288 | 0.12884 | 0.12847 | 0.58 | 1.15552 | 0.79306 | 0.68632 | 1.08 | 1.57599 | 1.57988 | 1.00247 |
| 0.09 | 1.00365 | 0.14310 | 0.14258 | 0.59 | 1.16108 | 0.80680 | 0.69487 | 1.09 | 1.58771 | 1.59842 | 1.00675 |
| 0.10 | 1.00450 | 0.15720 | 0.15650 | 0.60 | 1.16674 | 0.82059 | 0.70331 | 1.10 | 1.59958 | 1.61710 | 1.01096 |
| 0.11 | 1.00545 | 0.17116 | 0.17024 | 0.61 | 1.17251 | 0.83443 | 0.71166 | 1.11 | 1.61157 | 1.63593 | 1.01510 |
| 0.12 | 1.00649 | 0.18500 | 0.18381 | 0.62 | 1.17839 | 0.84834 | 0.71991 | 1.12 | 1.62376 | 1.65490 | 1.01918 |
| 0.13 | 1.00762 | 0.19873 | 0.19723 | 0.63 | 1.18438 | 0.86231 | 0.72807 | 1.13 | 1.63607 | 1.67402 | 1.02319 |
| 0.14 | 1.00883 | 0.21236 | 0.21050 | 0.64 | 1.19047 | 0.87633 | 0.73612 | 1.14 | 1.64854 | 1.69328 | 1.02714 |
| 0.15 | 1.01014 | 0.22591 | 0.22364 | 0.65 | 1.19667 | 0.89042 | 0.74409 | 1.15 | 1.66116 | 1.71270 | 1.03102 |
| 0.16 | 1.01154 | 0.23937 | 0.23664 | 0.66 | 1.20298 | 0.90458 | 0.75195 | 1.16 | 1.67394 | 1.73227 | 1.03485 |
| 0.17 | 1.01304 | 0.25277 | 0.24952 | 0.67 | 1.20940 | 0.91880 | 0.75972 | 1.17 | 1.68686 | 1.75199 | 1.03861 |
| 0.18 | 1.01462 | 0.26610 | 0.26227 | 0.68 | 1.21593 | 0.93309 | 0.76739 | 1.18 | 1.69995 | 1.77187 | 1.04231 |
| 0.19 | 1.01629 | 0.27938 | 0.27491 | 0.65 | 1.22257 | 0.94745 | 0.77497 | 1.19 | 1.71319 | 1.79191 | 1.04595 |
| 0.20 | 1.01806 | 0.29261 | 0.28742 | 0.70 | 1.22932 | 0.96188 | 0.78245 | 1.20 | 1.72659 | 1.81211 | 1.04953 |
| 0.21 | 1.01992 | 0.30580 | 0.29983 | 0.71 | 1.23619 | 0.97639 | 0.78984 | 1.21 | 1.74016 | 1.83247 | 1.05305 |
| 0.22 | 1.02186 | 0.31895 | 0.31212 | 0.72 | 1.24316 | 0.99097 | 0.79713 | 1.22 | 1.75388 | 1.85299 | 1.05651 |
| 0.23 | 1.02391 | 0.33206 | 0.32431 | 0.73 | 1.25025 | 1.00562 | 0.80434 | 1.23 | 1.76777 | 1.87368 | 1.05992 |
| 0.24 | 1.02604 | 0.34514 | 0.33638 | 0.74 | 1.25746 | 1.02036 | 0.81145 | 1.24 | 1.78182 | 1.89454 | 1.06327 |
| 0.25 | 1.02827 | 0.35820 | 0.34835 | 0.75 | 1.26478 | 1.03517 | 0.81846 | 1.25 | 1.79603 | 1.91558 | 1.06656 |
| 0.26 | 1.03059 | 0.37123 | 0.36021 | 0.76 | 1.27221 | 1.05007 | 0.82539 | 1.26 | 1.81041 | 1.93678 | 1.06980 |
| 0.27 | 1.03300 | 0.38425 | 0.37197 | 0.77 | 1.27976 | 1.06505 | 0.83222 | 1.27 | 1.82497 | 1.95816 | 1.07298 |
| 0.28 | 1.03550 | 0.39725 | 0.38363 | 0.78 | 1.28743 | 1.08011 | 0.83897 | 1.28 | 1.83969 | 1.97971 | 1.07611 |
| 0.29 | 1.03810 | 0.41024 | 0.39518 | 0.79 | 1.29522 | 1.09526 | 0.84562 | 1.29 | 1.85458 | 2.00145 | 1.07919 |
| 0.30 | 1.04079 | 0.42322 | 0.40663 | 0.80 | 1.30312 | 1.11050 | 0.85219 | 1.30 | 1.86964 | 2.02336 | 1.08222 |
| 0.31 | 1.04358 | 0.43619 | 0.41798 | 0.81 | 1.31115 | 1.12583 | 0.85866 | 1.31 | 1.88488 | 2.04546 | 1.08520 |
| 0.32 | 1.04646 | 0.44916 | 0.42922 | 0.82 | 1.31929 | 1.14125 | 0.86505 | 1.32 | 1.90029 | 2.06775 | 1.08812 |
| 0.33 | 1.04944 | 0.46213 | 0.44036 | 0.83 | 1.32756 | 1.15676 | 0.87135 | 1.33 | 1.91589 | 2.09022 | 1.09100 |
| 0.34 | 1.05251 | 0.47511 | 0.45141 | 0.84 | 1.33594 | 1.17237 | 0.87756 | 1.34 | 1.93165 | 2.11289 | 1.09382 |
| 0.35 | 1.05567 | 0.48809 | 0.46235 | 0.85 | 1.34445 | 1.18807 | 0.88369 | 1.35 | 1.94760 | 2.13574 | 1.09660 |
| 0.36 | 1.05893 | 0.50107 | 0.47319 | 0.86 | 1.35308 | 1.20388 | 0.88973 | 1.36 | 1.96374 | 2.15879 | 1.09933 |
| 0.37 | 1.06229 | 0.51407 | 0.48393 | 0.87 | 1.36184 | 1.21978 | 0.89569 | 1.37 | 1.98005 | 2.18204 | 1.10201 |
| 0.38 | 1.06574 | 0.52708 | 0.49457 | 0.88 | 1.37072 | 1.23578 | 0.90156 | 1.38 | 1.99655 | 2.20549 | 1.10465 |
| 0.39 | 1.06929 | 0.54010 | 0.50510 | 0.89 | 1.37973 | 1.25189 | 0.90735 | 1.39 | 2.01323 | 2.22914 | 1.10724 |
| 0.40 | 1.07293 | 0.55314 | 0.51554 | 0.90 | 1.38886 | 1.26810 | 0.91305 | 1.40 | 2.03011 | 2.25299 | 1.10979 |
| 0.41 | 1.07667 | 0.56620 | 0.52588 | 0.91 | 1.39812 | 1.28442 | 0.91868 | 1.41 | 2.04717 | 2.27705 | 1.11229 |
| 0.42 | 1.08051 | 0.57928 | 0.53612 | 0.92 | 1.40751 | 1.30085 | 0.92422 | 1.42 | 2.06442 | 2.30132 | 1.11475 |
| 0.43 | 1.08445 | 0.59238 | 0.54625 | 0.93 | 1.41703 | 1.31739 | 0.92968 | 1.43 | 2.08187 | 2.32580 | 1.11717 |
| 0.44 | 1.08848 | 0.60551 | 0.55629 | 0.94 | 1.42668 | 1.33404 | 0.93507 | 1.44 | 2.09951 | 2.35049 | 1.11954 |
| 0.45 | 1.09262 | 0.61867 | 0.56623 | 0.95 | 1.43646 | 1.35080 | 0.94037 | 1.45 | 2.11735 | 2.37540 | 1.12187 |
| 0.46 | 1.09685 | 0.63185 | 0.57606 | 0.96 | 1.44637 | 1.36768 | 0.94559 | 1.46 | 2.13538 | 2.40052 | 1.12416 |
| 0.47 | 1.10118 | 0.64507 | 0.58580 | 0.97 | 1.45642 | 1.38468 | 0.95074 | 1.47 | 2.15362 | 2.42587 | 1.12642 |
| 0.48 | 1.10561 | 0.65832 | 0.59544 | 0.98 | 1.46660 | 1.40179 | 0.95581 | 1.48 | 2.17205 | 2.45144 | 1.12863 |
| 0.49 | 1.11015 | 0.67161 | 0.60497 | 0.99 | 1.47691 | 1.41903 | 0.96081 | 1.49 | 2.19069 | 2.47724 | 1.13080 |
| 0.50 | 1.11478 | 0.68493 | 0.61441 | 1.00 | 1.48736 | 1.43639 | 0.96573 | 1.50 | 2.20954 | 2.50326 | 1.13293 |

TABLE D.XVA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/9$ and $x$ from 0.00 to 1.50.

$\alpha = 5/9$

| x | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ |
|---|---|---|---|
| 1.50 | 2.20954 | 2.50326 | 1.13293 |
| 1.51 | 2.22859 | 2.52952 | 1.13503 |
| 1.52 | 2.24785 | 2.55601 | 1.13709 |
| 1.53 | 2.26732 | 2.58273 | 1.13911 |
| 1.54 | 2.28701 | 2.60970 | 1.14110 |
| 1.55 | 2.30691 | 2.63690 | 1.14305 |
| 1.56 | 2.32702 | 2.66435 | 1.14496 |
| 1.57 | 2.34735 | 2.69205 | 1.14684 |
| 1.58 | 2.36791 | 2.71999 | 1.14869 |
| 1.59 | 2.38868 | 2.74819 | 1.15051 |
| 1.60 | 2.40968 | 2.77665 | 1.15229 |
| 1.61 | 2.43091 | 2.80536 | 1.15404 |
| 1.62 | 2.45236 | 2.83433 | 1.15575 |
| 1.63 | 2.47405 | 2.86356 | 1.15744 |
| 1.64 | 2.49596 | 2.89307 | 1.15910 |
| 1.65 | 2.51812 | 2.92284 | 1.16072 |
| 1.66 | 2.54051 | 2.95288 | 1.16232 |
| 1.67 | 2.56313 | 2.98320 | 1.16389 |
| 1.68 | 2.58600 | 3.01379 | 1.16543 |
| 1.69 | 2.60911 | 3.04467 | 1.16694 |
| 1.70 | 2.63247 | 3.07583 | 1.16842 |
| 1.71 | 2.65608 | 3.10728 | 1.16988 |
| 1.72 | 2.67993 | 3.13902 | 1.17130 |
| 1.73 | 2.70404 | 3.17105 | 1.17271 |
| 1.74 | 2.72841 | 3.20338 | 1.17408 |
| 1.75 | 2.75303 | 3.23601 | 1.17544 |
| 1.76 | 2.77791 | 3.26894 | 1.17676 |
| 1.77 | 2.80305 | 3.30217 | 1.17806 |
| 1.78 | 2.82846 | 3.33572 | 1.17934 |
| 1.79 | 2.85413 | 3.36958 | 1.18060 |
| 1.80 | 2.88007 | 3.40375 | 1.18183 |
| 1.81 | 2.90629 | 3.43825 | 1.18304 |
| 1.82 | 2.93278 | 3.47306 | 1.18422 |
| 1.83 | 2.95954 | 3.50821 | 1.18539 |
| 1.84 | 2.98659 | 3.54368 | 1.18653 |
| 1.85 | 3.01392 | 3.57948 | 1.18765 |
| 1.86 | 3.04153 | 3.61562 | 1.18875 |
| 1.87 | 3.06943 | 3.65210 | 1.18983 |
| 1.88 | 3.09762 | 3.68893 | 1.19089 |
| 1.89 | 3.12611 | 3.72610 | 1.19193 |
| 1.90 | 3.15489 | 3.76362 | 1.19295 |
| 1.91 | 3.18397 | 3.80150 | 1.19395 |
| 1.92 | 3.21335 | 3.83973 | 1.19493 |
| 1.93 | 3.24303 | 3.87833 | 1.19590 |
| 1.94 | 3.27302 | 3.91729 | 1.19684 |
| 1.95 | 3.30332 | 3.95662 | 1.19777 |
| 1.96 | 3.33394 | 3.99632 | 1.19868 |
| 1.97 | 3.36487 | 4.03640 | 1.19957 |
| 1.98 | 3.39612 | 4.07686 | 1.20045 |
| 1.99 | 3.42769 | 4.11771 | 1.20131 |
| 2.00 | 3.45958 | 4.15895 | 1.20215 |

| x | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ |
|---|---|---|---|
| 2.0 | 3.45958 | 4.15895 | 1.20215 |
| 2.1 | 3.79712 | 4.59359 | 1.20976 |
| 2.2 | 4.17078 | 5.07180 | 1.21603 |
| 2.3 | 4.58419 | 5.59823 | 1.22120 |
| 2.4 | 5.04139 | 6.17801 | 1.22546 |
| 2.5 | 5.54681 | 6.81681 | 1.22896 |
| 2.6 | 6.10538 | 7.52085 | 1.23184 |
| 2.7 | 6.72255 | 8.29700 | 1.23421 |
| 2.8 | 7.40432 | 9.15284 | 1.23615 |
| 2.9 | 8.15736 | 10.09671 | 1.23774 |
| 3.0 | 8.98900 | 11.13782 | 1.23905 |
| 3.1 | 9.90737 | 12.28636 | 1.24012 |
| 3.2 | 10.92143 | 13.55353 | 1.24100 |
| 3.3 | 12.04109 | 14.95173 | 1.24173 |
| 3.4 | 13.27730 | 16.49462 | 1.24232 |
| 3.5 | 14.64214 | 18.19729 | 1.24280 |
| 3.6 | 16.14898 | 20.07640 | 1.24320 |
| 3.7 | 17.81254 | 22.15035 | 1.24353 |
| 3.8 | 19.64913 | 24.43943 | 1.24379 |
| 3.9 | 21.67672 | 26.96607 | 1.24401 |
| 4.0 | 23.91518 | 29.75503 | 1.24419 |
| 4.1 | 26.38646 | 32.83364 | 1.24434 |
| 4.2 | 29.11478 | 36.23209 | 1.24446 |
| 4.3 | 32.12689 | 39.98370 | 1.24456 |
| 4.4 | 35.45236 | 44.12529 | 1.24464 |
| 4.5 | 39.12382 | 48.69750 | 1.24470 |
| 4.6 | 43.17730 | 53.74522 | 1.24476 |
| 4.7 | 47.65262 | 59.31802 | 1.24480 |
| 4.8 | 52.59374 | 65.47063 | 1.24484 |
| 4.9 | 58.04920 | 72.26352 | 1.24487 |
| 5.0 | 64.07262 | 79.76344 | 1.24489 |
| 5.1 | 70.72323 | 88.04413 | 1.24491 |
| 5.2 | 78.06642 | 97.18702 | 1.24493 |
| 5.3 | 86.17442 | 107.28205 | 1.24494 |
| 5.4 | 95.12701 | 118.42853 | 1.24495 |
| 5.5 | 105.01228 | 130.73616 | 1.24496 |
| 5.6 | 115.92754 | 144.32608 | 1.24497 |
| 5.7 | 127.98027 | 159.33211 | 1.24497 |
| 5.8 | 141.28915 | 175.90202 | 1.24498 |
| 5.9 | 155.98528 | 194.19902 | 1.24498 |
| 6.0 | 172.21345 | 214.40339 | 1.24499 |

| x | $F_{5/9}(x)$ | $H_{4/9}(x)$ | $T_{5/9}(x)$ |
|---|---|---|---|
| 6.0 | 172.21345 | 214.40339 | 1.24499 |
| 6.1 | 190.13359 | 236.71422 | 1.24499 |
| 6.2 | 209.92230 | 261.35142 | 1.24499 |
| 6.3 | 231.77469 | 288.55787 | 1.24499 |
| 6.4 | 255.90621 | 318.60182 | 1.24499 |
| 6.5 | 282.55485 | 351.77958 | 1.24500 |
| 6.6 | 311.98348 | 388.41841 | 1.24500 |
| 6.7 | 344.48243 | 428.87978 | 1.24500 |
| 6.8 | 380.37238 | 473.56293 | 1.24500 |
| 6.9 | 420.00753 | 522.90885 | 1.24500 |
| 7.0 | 463.77911 | 577.40463 | 1.24500 |
| 7.1 | 512.11927 | 637.58826 | 1.24500 |
| 7.2 | 565.50530 | 704.05402 | 1.24500 |
| 7.3 | 624.46443 | 777.45826 | 1.24500 |
| 7.4 | 689.57900 | 858.52607 | 1.24500 |
| 7.5 | 761.49225 | 948.05822 | 1.24500 |
| 7.6 | 840.91470 | 1046.93931 | 1.24500 |
| 7.7 | 928.63114 | 1156.14644 | 1.24500 |
| 7.8 | 1025.50848 | 1276.75889 | 1.24500 |
| 7.9 | 1132.50427 | 1409.96882 | 1.24500 |
| 8.0 | 1250.67622 | 1557.09308 | 1.24500 |
| 8.1 | 1381.19269 | 1719.58627 | 1.24500 |
| 8.2 | 1525.34424 | 1899.05517 | 1.24500 |
| 8.3 | 1684.55646 | 2097.27460 | 1.24500 |
| 8.4 | 1860.40405 | 2316.20509 | 1.24500 |
| 8.5 | 2054.62647 | 2558.01226 | 1.24500 |
| 8.6 | 2269.14518 | 2825.08834 | 1.24500 |
| 8.7 | 2506.08267 | 3120.07581 | 1.24500 |
| 8.8 | 2767.78351 | 3445.89369 | 1.24500 |
| 8.9 | 3056.83761 | 3805.76642 | 1.24500 |
| 9.0 | 3376.10590 | 4203.25584 | 1.24500 |
| 9.1 | 3728.74870 | 4642.29657 | 1.24500 |
| 9.2 | 4118.25705 | 5127.23496 | 1.24500 |
| 9.3 | 4546.48737 | 5662.87224 | 1.24500 |
| 9.4 | 5023.69967 | 6254.51215 | 1.24500 |
| 9.5 | 5548.59986 | 6908.01353 | 1.24500 |
| 9.6 | 6128.38637 | 7629.84845 | 1.24500 |
| 9.7 | 6768.80181 | 8427.16646 | 1.24500 |
| 9.8 | 7476.18986 | 9307.86545 | 1.24500 |
| 9.9 | 8257.55829 | 10280.67011 | 1.24500 |
| 10.0 | 9120.64845 | 11355.21641 | 1.24500 |

TABLE D.XVB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/9$ and $x$ from 1.50 to 10.0.

| $x$ | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ | $x$ | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ | $x$ | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.23485 | 0.18975 | 0.15367 | 1.00 | 2.01006 | 0.57771 | 0.28741 |
| 0.01 | 1.00009 | 0.00062 | 0.00062 | 0.51 | 1.24457 | 0.19558 | 0.15715 | 1.01 | 2.03235 | 0.58776 | 0.28920 |
| 0.02 | 1.00037 | 0.00170 | 0.00169 | 0.52 | 1.25451 | 0.20148 | 0.16060 | 1.02 | 2.05494 | 0.59791 | 0.29096 |
| 0.03 | 1.00083 | 0.00306 | 0.00306 | 0.53 | 1.26467 | 0.20745 | 0.16404 | 1.03 | 2.07784 | 0.60816 | 0.29269 |
| 0.04 | 1.00147 | 0.00465 | 0.00464 | 0.54 | 1.27504 | 0.21349 | 0.16744 | 1.04 | 2.10105 | 0.61852 | 0.29439 |
| 0.05 | 1.00229 | 0.00643 | 0.00641 | 0.55 | 1.28562 | 0.21961 | 0.17082 | 1.05 | 2.12457 | 0.62899 | 0.29605 |
| 0.06 | 1.00330 | 0.00838 | 0.00836 | 0.56 | 1.29642 | 0.22580 | 0.17417 | 1.06 | 2.14841 | 0.63957 | 0.29769 |
| 0.07 | 1.00449 | 0.01049 | 0.01045 | 0.57 | 1.30744 | 0.23206 | 0.17749 | 1.07 | 2.17257 | 0.65025 | 0.29930 |
| 0.08 | 1.00587 | 0.01274 | 0.01267 | 0.58 | 1.31868 | 0.23839 | 0.18078 | 1.08 | 2.19704 | 0.66105 | 0.30088 |
| 0.09 | 1.00743 | 0.01513 | 0.01502 | 0.59 | 1.33014 | 0.24480 | 0.18404 | 1.09 | 2.22184 | 0.67195 | 0.30243 |
| 0.10 | 1.00918 | 0.01764 | 0.01748 | 0.60 | 1.34182 | 0.25129 | 0.18727 | 1.10 | 2.24696 | 0.68298 | 0.30396 |
| 0.11 | 1.01110 | 0.02027 | 0.02005 | 0.61 | 1.35373 | 0.25784 | 0.19047 | 1.11 | 2.27242 | 0.69411 | 0.30545 |
| 0.12 | 1.01322 | 0.02301 | 0.02271 | 0.62 | 1.36586 | 0.26448 | 0.19363 | 1.12 | 2.29820 | 0.70537 | 0.30692 |
| 0.13 | 1.01552 | 0.02586 | 0.02547 | 0.63 | 1.37822 | 0.27118 | 0.19676 | 1.13 | 2.32433 | 0.71673 | 0.30836 |
| 0.14 | 1.01800 | 0.02882 | 0.02831 | 0.64 | 1.39080 | 0.27797 | 0.19986 | 1.14 | 2.35079 | 0.72822 | 0.30978 |
| 0.15 | 1.02067 | 0.03187 | 0.03123 | 0.65 | 1.40361 | 0.28483 | 0.20292 | 1.15 | 2.37759 | 0.73982 | 0.31116 |
| 0.16 | 1.02353 | 0.03503 | 0.03422 | 0.66 | 1.41666 | 0.29176 | 0.20595 | 1.16 | 2.40473 | 0.75154 | 0.31253 |
| 0.17 | 1.02657 | 0.03827 | 0.03728 | 0.67 | 1.42993 | 0.29878 | 0.20894 | 1.17 | 2.43222 | 0.76339 | 0.31386 |
| 0.18 | 1.02979 | 0.04161 | 0.04041 | 0.68 | 1.44345 | 0.30587 | 0.21190 | 1.18 | 2.46007 | 0.77536 | 0.31518 |
| 0.19 | 1.03321 | 0.04504 | 0.04359 | 0.69 | 1.45719 | 0.31304 | 0.21482 | 1.19 | 2.48826 | 0.78745 | 0.31646 |
| 0.20 | 1.03681 | 0.04856 | 0.04683 | 0.70 | 1.47117 | 0.32029 | 0.21771 | 1.20 | 2.51682 | 0.79966 | 0.31773 |
| 0.21 | 1.04060 | 0.05216 | 0.05013 | 0.71 | 1.48540 | 0.32761 | 0.22056 | 1.21 | 2.54573 | 0.81201 | 0.31897 |
| 0.22 | 1.04458 | 0.05585 | 0.05346 | 0.72 | 1.49986 | 0.33502 | 0.22337 | 1.22 | 2.57501 | 0.82448 | 0.32018 |
| 0.23 | 1.04874 | 0.05962 | 0.05685 | 0.73 | 1.51450 | 0.34251 | 0.22614 | 1.23 | 2.60466 | 0.83708 | 0.32138 |
| 0.24 | 1.05310 | 0.06347 | 0.06027 | 0.74 | 1.52951 | 0.35008 | 0.22888 | 1.24 | 2.63468 | 0.84981 | 0.32255 |
| 0.25 | 1.05764 | 0.06740 | 0.06372 | 0.75 | 1.54470 | 0.35773 | 0.23159 | 1.25 | 2.66507 | 0.86267 | 0.32370 |
| 0.26 | 1.06238 | 0.07141 | 0.06721 | 0.76 | 1.56015 | 0.36546 | 0.23425 | 1.26 | 2.69584 | 0.87567 | 0.32482 |
| 0.27 | 1.06730 | 0.07549 | 0.07073 | 0.77 | 1.57584 | 0.37328 | 0.23686 | 1.27 | 2.72699 | 0.88880 | 0.32593 |
| 0.28 | 1.07242 | 0.07966 | 0.07428 | 0.78 | 1.59178 | 0.38118 | 0.23947 | 1.28 | 2.75852 | 0.90206 | 0.32701 |
| 0.29 | 1.07775 | 0.08390 | 0.07785 | 0.79 | 1.60797 | 0.38917 | 0.24202 | 1.29 | 2.79044 | 0.91547 | 0.32807 |
| 0.30 | 1.08323 | 0.08821 | 0.08144 | 0.80 | 1.62442 | 0.39724 | 0.24454 | 1.30 | 2.82276 | 0.92901 | 0.32911 |
| 0.31 | 1.08893 | 0.09260 | 0.08504 | 0.81 | 1.64112 | 0.40539 | 0.24702 | 1.31 | 2.85547 | 0.94270 | 0.33014 |
| 0.32 | 1.09481 | 0.09707 | 0.08866 | 0.82 | 1.65809 | 0.41363 | 0.24946 | 1.32 | 2.88858 | 0.95652 | 0.33114 |
| 0.33 | 1.10090 | 0.10161 | 0.09229 | 0.83 | 1.67531 | 0.42196 | 0.25187 | 1.33 | 2.92210 | 0.97049 | 0.33212 |
| 0.34 | 1.10717 | 0.10622 | 0.09594 | 0.84 | 1.69280 | 0.43038 | 0.25424 | 1.34 | 2.95602 | 0.98460 | 0.33308 |
| 0.35 | 1.11365 | 0.11090 | 0.09959 | 0.85 | 1.71055 | 0.43889 | 0.25658 | 1.35 | 2.99035 | 0.99887 | 0.33403 |
| 0.36 | 1.12032 | 0.11566 | 0.10324 | 0.86 | 1.72857 | 0.44748 | 0.25888 | 1.36 | 3.02510 | 1.01327 | 0.33496 |
| 0.37 | 1.12719 | 0.12049 | 0.10689 | 0.87 | 1.74686 | 0.45617 | 0.26114 | 1.37 | 3.06027 | 1.02783 | 0.33586 |
| 0.38 | 1.13425 | 0.12539 | 0.11055 | 0.88 | 1.76541 | 0.46495 | 0.26336 | 1.38 | 3.09586 | 1.04254 | 0.33675 |
| 0.39 | 1.14152 | 0.13036 | 0.11420 | 0.89 | 1.78424 | 0.47382 | 0.26556 | 1.39 | 3.13188 | 1.05741 | 0.33763 |
| 0.40 | 1.14898 | 0.13540 | 0.11785 | 0.90 | 1.80335 | 0.48278 | 0.26771 | 1.40 | 3.16833 | 1.07242 | 0.33848 |
| 0.41 | 1.15665 | 0.14052 | 0.12149 | 0.91 | 1.82273 | 0.49184 | 0.26983 | 1.41 | 3.20522 | 1.08760 | 0.33932 |
| 0.42 | 1.16452 | 0.14571 | 0.12512 | 0.92 | 1.84240 | 0.50099 | 0.27192 | 1.42 | 3.24255 | 1.10293 | 0.34014 |
| 0.43 | 1.17259 | 0.15096 | 0.12874 | 0.93 | 1.86234 | 0.51023 | 0.27397 | 1.43 | 3.28032 | 1.11842 | 0.34095 |
| 0.44 | 1.18087 | 0.15629 | 0.13235 | 0.94 | 1.88257 | 0.51958 | 0.27599 | 1.44 | 3.31854 | 1.13407 | 0.34174 |
| 0.45 | 1.18934 | 0.16169 | 0.13595 | 0.95 | 1.90309 | 0.52902 | 0.27798 | 1.45 | 3.35721 | 1.14989 | 0.34251 |
| 0.46 | 1.19803 | 0.16716 | 0.13953 | 0.96 | 1.92390 | 0.53856 | 0.27993 | 1.46 | 3.39634 | 1.16587 | 0.34327 |
| 0.47 | 1.20692 | 0.17270 | 0.14309 | 0.97 | 1.94499 | 0.54819 | 0.28185 | 1.47 | 3.43594 | 1.18202 | 0.34402 |
| 0.48 | 1.21602 | 0.17831 | 0.14664 | 0.98 | 1.96639 | 0.55793 | 0.28374 | 1.48 | 3.47600 | 1.19833 | 0.34474 |
| 0.49 | 1.22533 | 0.18400 | 0.15016 | 0.99 | 1.98807 | 0.56777 | 0.28559 | 1.49 | 3.51653 | 1.21482 | 0.34546 |
| 0.50 | 1.23485 | 0.18975 | 0.15367 | 1.00 | 2.01006 | 0.57771 | 0.28741 | 1.50 | 3.55754 | 1.23148 | 0.34616 |

TABLE D.XVIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/11$ and $x$ from 0.00 to 1.50.

α = 3/11

| x | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ |
|---|---|---|---|
| 1.50 | 3.55754 | 1.23148 | 0.34616 |
| 1.51 | 3.59902 | 1.24831 | 0.34685 |
| 1.52 | 3.64100 | 1.26532 | 0.34752 |
| 1.53 | 3.68346 | 1.28250 | 0.34818 |
| 1.54 | 3.72642 | 1.29987 | 0.34882 |
| 1.55 | 3.76988 | 1.31741 | 0.34946 |
| 1.56 | 3.81385 | 1.33514 | 0.35008 |
| 1.57 | 3.85832 | 1.35306 | 0.35069 |
| 1.58 | 3.90331 | 1.37116 | 0.35128 |
| 1.59 | 3.94882 | 1.38945 | 0.35186 |
| 1.60 | 3.99486 | 1.40793 | 0.35244 |
| 1.61 | 4.04143 | 1.42661 | 0.35300 |
| 1.62 | 4.08853 | 1.44548 | 0.35355 |
| 1.63 | 4.13618 | 1.46456 | 0.35408 |
| 1.64 | 4.18437 | 1.48383 | 0.35461 |
| 1.65 | 4.23312 | 1.50330 | 0.35513 |
| 1.66 | 4.28242 | 1.52298 | 0.35563 |
| 1.67 | 4.33229 | 1.54286 | 0.35613 |
| 1.68 | 4.38273 | 1.56295 | 0.35662 |
| 1.69 | 4.43374 | 1.58325 | 0.35709 |
| 1.70 | 4.48533 | 1.60377 | 0.35756 |
| 1.71 | 4.53752 | 1.62450 | 0.35602 |
| 1.72 | 4.59029 | 1.64545 | 0.35846 |
| 1.73 | 4.64367 | 1.66662 | 0.35890 |
| 1.74 | 4.69765 | 1.68801 | 0.35933 |
| 1.75 | 4.75224 | 1.70963 | 0.35975 |
| 1.76 | 4.80745 | 1.73148 | 0.36017 |
| 1.77 | 4.86328 | 1.75355 | 0.36057 |
| 1.78 | 4.91975 | 1.77586 | 0.36097 |
| 1.79 | 4.97685 | 1.79840 | 0.36135 |
| 1.80 | 5.03459 | 1.82118 | 0.36173 |
| 1.81 | 5.09299 | 1.84420 | 0.36211 |
| 1.82 | 5.15204 | 1.86746 | 0.36247 |
| 1.83 | 5.21175 | 1.89097 | 0.36283 |
| 1.84 | 5.27214 | 1.91472 | 0.36318 |
| 1.85 | 5.33320 | 1.93872 | 0.36352 |
| 1.86 | 5.39495 | 1.96298 | 0.36386 |
| 1.87 | 5.45739 | 1.98750 | 0.36418 |
| 1.88 | 5.52052 | 2.01227 | 0.36451 |
| 1.89 | 5.58436 | 2.03730 | 0.36482 |
| 1.90 | 5.64892 | 2.06260 | 0.36513 |
| 1.91 | 5.71419 | 2.08817 | 0.36544 |
| 1.92 | 5.78020 | 2.11400 | 0.36573 |
| 1.93 | 5.84693 | 2.14011 | 0.36602 |
| 1.94 | 5.91441 | 2.16650 | 0.36631 |
| 1.95 | 5.98265 | 2.19316 | 0.36659 |
| 1.96 | 6.05163 | 2.22010 | 0.36686 |
| 1.97 | 6.12139 | 2.24733 | 0.36713 |
| 1.98 | 6.19192 | 2.27485 | 0.36739 |
| 1.99 | 6.26323 | 2.30266 | 0.36765 |
| 2.00 | 6.33533 | 2.33076 | 0.36790 |

| x | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ |
|---|---|---|---|
| 2.0 | 6.33533 | 2.33076 | 0.36790 |
| 2.1 | 7.10165 | 2.62868 | 0.37015 |
| 2.2 | 7.95658 | 2.95971 | 0.37198 |
| 2.3 | 8.90980 | 3.32757 | 0.37347 |
| 2.4 | 9.97211 | 3.73641 | 0.37469 |
| 2.5 | 11.15547 | 4.19082 | 0.37567 |
| 2.6 | 12.47321 | 4.69588 | 0.37648 |
| 2.7 | 13.94011 | 5.25726 | 0.37713 |
| 2.8 | 15.57260 | 5.88123 | 0.37767 |
| 2.9 | 17.38893 | 6.57475 | 0.37810 |
| 3.0 | 19.40935 | 7.34555 | 0.37845 |
| 3.1 | 21.65633 | 8.20218 | 0.37874 |
| 3.2 | 24.15483 | 9.15415 | 0.37898 |
| 3.3 | 26.93254 | 10.21201 | 0.37917 |
| 3.4 | 30.02017 | 11.38745 | 0.37933 |
| 3.5 | 33.45181 | 12.69344 | 0.37945 |
| 3.6 | 37.26528 | 14.14437 | 0.37956 |
| 3.7 | 41.50251 | 15.75618 | 0.37964 |
| 3.8 | 46.21003 | 17.54657 | 0.37971 |
| 3.9 | 51.43944 | 19.53517 | 0.37977 |
| 4.0 | 57.24797 | 21.74373 | 0.37982 |
| 4.1 | 63.69909 | 24.19638 | 0.37985 |
| 4.2 | 70.86318 | 26.91988 | 0.37989 |
| 4.3 | 78.81826 | 29.94390 | 0.37991 |
| 4.4 | 87.65088 | 33.30131 | 0.37993 |
| 4.5 | 97.45698 | 37.02860 | 0.37995 |
| 4.6 | 108.34293 | 41.16619 | 0.37996 |
| 4.7 | 120.42666 | 45.75890 | 0.37997 |
| 4.8 | 133.83889 | 50.85642 | 0.37998 |
| 4.9 | 148.72458 | 56.51384 | 0.37999 |
| 5.0 | 165.24436 | 62.79221 | 0.38000 |
| 5.1 | 183.57632 | 69.75921 | 0.38000 |
| 5.2 | 203.91787 | 77.48985 | 0.38001 |
| 5.3 | 226.48778 | 86.06730 | 0.38001 |
| 5.4 | 251.52855 | 95.58370 | 0.38001 |
| 5.5 | 279.30894 | 106.14119 | 0.38001 |
| 5.6 | 310.12681 | 117.85297 | 0.38002 |
| 5.7 | 344.31227 | 130.84549 | 0.38002 |
| 5.8 | 382.23113 | 145.25477 | 0.38002 |
| 5.9 | 424.28878 | 161.23786 | 0.38002 |
| 6.0 | 470.93444 | 178.96450 | 0.38002 |

| x | $F_{3/11}(x)$ | $H_{8/11}(x)$ | $T_{3/11}(x)$ |
|---|---|---|---|
| 6.0 | 470.93444 | 178.96450 | 0.38002 |
| 6.1 | 522.66591 | 198.62384 | 0.38002 |
| 6.2 | 580.03475 | 220.42550 | 0.38002 |
| 6.3 | 643.65215 | 244.60175 | 0.38002 |
| 6.4 | 714.19529 | 271.40993 | 0.38002 |
| 6.5 | 792.41443 | 301.13518 | 0.38002 |
| 6.6 | 879.14094 | 334.09340 | 0.38002 |
| 6.7 | 975.29576 | 370.63459 | 0.38002 |
| 6.8 | 1081.89926 | 411.14653 | 0.38002 |
| 6.9 | 1200.08186 | 456.05879 | 0.38002 |
| 7.0 | 1331.09587 | 505.84728 | 0.38002 |
| 7.1 | 1476.32861 | 561.03923 | 0.38002 |
| 7.2 | 1637.31692 | 622.21865 | 0.38002 |
| 7.3 | 1815.76322 | 690.03752 | 0.38002 |
| 7.4 | 2013.55334 | 765.19749 | 0.38002 |
| 7.5 | 2232.77622 | 848.50742 | 0.38002 |
| 7.6 | 2475.74571 | 940.84162 | 0.38002 |
| 7.7 | 2745.02478 | 1043.17407 | 0.38002 |
| 7.8 | 3043.45221 | 1156.58359 | 0.38002 |
| 7.9 | 3374.17228 | 1282.26508 | 0.38002 |
| 8.0 | 3740.66754 | 1421.54198 | 0.38002 |
| 8.1 | 4146.79510 | 1575.88009 | 0.38002 |
| 8.2 | 4596.82667 | 1746.90284 | 0.38002 |
| 8.3 | 5095.49407 | 1936.40820 | 0.38002 |
| 8.4 | 5648.03650 | 2146.38742 | 0.38002 |
| 8.5 | 6260.25709 | 2379.04575 | 0.38002 |
| 8.6 | 6938.58234 | 2636.82542 | 0.38002 |
| 8.7 | 7690.12918 | 2922.43103 | 0.38002 |
| 8.8 | 8522.77697 | 3238.85767 | 0.38002 |
| 8.9 | 9445.25952 | 3589.42213 | 0.38002 |
| 9.0 | 10467.23577 | 3977.79731 | 0.38002 |
| 9.1 | 11599.41029 | 4408.05044 | 0.38002 |
| 9.2 | 12853.63442 | 4884.68531 | 0.38002 |
| 9.3 | 14243.03139 | 5412.68907 | 0.38002 |
| 9.4 | 15782.13262 | 5997.58398 | 0.38002 |
| 9.5 | 17487.02849 | 6645.48478 | 0.38002 |
| 9.6 | 19375.55536 | 7363.16211 | 0.38002 |
| 9.7 | 21467.38047 | 8158.11278 | 0.38002 |
| 9.8 | 23784.40639 | 9038.63749 | 0.38002 |
| 9.9 | 26350.79765 | 10013.92693 | 0.38002 |
| 10.0 | 29193.33137 | 11054.15705 | 0.38002 |

TABLE D.XVIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/11$ and x from 1.50 to 10.0.

$\alpha = 5/11$

| $x$ | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00006 | 0.00566 | 0.00566 |
| 0.02 | 1.00022 | 0.01206 | 0.01206 |
| 0.03 | 1.00050 | 0.01878 | 0.01877 |
| 0.04 | 1.00088 | 0.02570 | 0.02568 |
| 0.05 | 1.00138 | 0.03279 | 0.03276 |
| 0.06 | 1.00198 | 0.04001 | 0.03993 |
| 0.07 | 1.00270 | 0.04735 | 0.04722 |
| 0.08 | 1.00352 | 0.05479 | 0.05459 |
| 0.09 | 1.00446 | 0.06231 | 0.06204 |
| 0.10 | 1.00550 | 0.06993 | 0.06954 |
| 0.11 | 1.00666 | 0.07761 | 0.07710 |
| 0.12 | 1.00793 | 0.08537 | 0.08470 |
| 0.13 | 1.00931 | 0.09320 | 0.09236 |
| 0.14 | 1.01080 | 0.10109 | 0.10001 |
| 0.15 | 1.01240 | 0.10905 | 0.10771 |
| 0.16 | 1.01411 | 0.11706 | 0.11543 |
| 0.17 | 1.01593 | 0.12513 | 0.12317 |
| 0.18 | 1.01787 | 0.13326 | 0.13092 |
| 0.19 | 1.01992 | 0.14144 | 0.13868 |
| 0.20 | 1.02208 | 0.14967 | 0.14644 |
| 0.21 | 1.02435 | 0.15796 | 0.15420 |
| 0.22 | 1.02673 | 0.16630 | 0.16197 |
| 0.23 | 1.02923 | 0.17469 | 0.16973 |
| 0.24 | 1.03184 | 0.18313 | 0.17748 |
| 0.25 | 1.03455 | 0.19162 | 0.18522 |
| 0.26 | 1.03740 | 0.20016 | 0.19294 |
| 0.27 | 1.04035 | 0.20875 | 0.20065 |
| 0.28 | 1.04341 | 0.21739 | 0.20835 |
| 0.29 | 1.04659 | 0.22608 | 0.21602 |
| 0.30 | 1.04988 | 0.23482 | 0.22366 |
| 0.31 | 1.05329 | 0.24361 | 0.23129 |
| 0.32 | 1.05682 | 0.25245 | 0.23888 |
| 0.33 | 1.06046 | 0.26134 | 0.24644 |
| 0.34 | 1.06421 | 0.27029 | 0.25398 |
| 0.35 | 1.06809 | 0.27928 | 0.26148 |
| 0.36 | 1.07208 | 0.28832 | 0.26894 |
| 0.37 | 1.07618 | 0.29742 | 0.27636 |
| 0.38 | 1.08041 | 0.30657 | 0.28375 |
| 0.39 | 1.08475 | 0.31577 | 0.29110 |
| 0.40 | 1.08922 | 0.32502 | 0.29840 |
| 0.41 | 1.09380 | 0.33433 | 0.30566 |
| 0.42 | 1.09850 | 0.34369 | 0.31287 |
| 0.43 | 1.10332 | 0.35311 | 0.32004 |
| 0.44 | 1.10826 | 0.36258 | 0.32716 |
| 0.45 | 1.11333 | 0.37211 | 0.33423 |
| 0.46 | 1.11851 | 0.38169 | 0.34125 |
| 0.47 | 1.12382 | 0.39133 | 0.34822 |
| 0.48 | 1.12925 | 0.40103 | 0.35513 |
| 0.49 | 1.13480 | 0.41079 | 0.36199 |
| 0.50 | 1.14048 | 0.42061 | 0.36880 |

| $x$ | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 0.50 | 1.14048 | 0.42061 | 0.36880 |
| 0.51 | 1.14628 | 0.43048 | 0.37555 |
| 0.52 | 1.15221 | 0.44042 | 0.38224 |
| 0.53 | 1.15826 | 0.45042 | 0.38887 |
| 0.54 | 1.16444 | 0.46048 | 0.39545 |
| 0.55 | 1.17074 | 0.47060 | 0.40197 |
| 0.56 | 1.17718 | 0.48079 | 0.40843 |
| 0.57 | 1.18374 | 0.49104 | 0.41482 |
| 0.58 | 1.19043 | 0.50136 | 0.42116 |
| 0.59 | 1.19725 | 0.51175 | 0.42744 |
| 0.60 | 1.20420 | 0.52220 | 0.43365 |
| 0.61 | 1.21128 | 0.53272 | 0.43980 |
| 0.62 | 1.21850 | 0.54331 | 0.44588 |
| 0.63 | 1.22584 | 0.55397 | 0.45191 |
| 0.64 | 1.23332 | 0.56470 | 0.45787 |
| 0.65 | 1.24093 | 0.57550 | 0.46376 |
| 0.66 | 1.24868 | 0.58637 | 0.46959 |
| 0.67 | 1.25657 | 0.59732 | 0.47536 |
| 0.68 | 1.26459 | 0.60834 | 0.48106 |
| 0.69 | 1.27274 | 0.61944 | 0.48670 |
| 0.70 | 1.28104 | 0.63062 | 0.49227 |
| 0.71 | 1.28947 | 0.64187 | 0.49778 |
| 0.72 | 1.29805 | 0.65320 | 0.50322 |
| 0.73 | 1.30676 | 0.66461 | 0.50859 |
| 0.74 | 1.31562 | 0.67610 | 0.51390 |
| 0.75 | 1.32462 | 0.68767 | 0.51915 |
| 0.76 | 1.33376 | 0.69933 | 0.52433 |
| 0.77 | 1.34305 | 0.71107 | 0.52945 |
| 0.78 | 1.35248 | 0.72290 | 0.53450 |
| 0.79 | 1.36206 | 0.73481 | 0.53948 |
| 0.80 | 1.37179 | 0.74680 | 0.54440 |
| 0.81 | 1.38166 | 0.75889 | 0.54926 |
| 0.82 | 1.39168 | 0.77107 | 0.55405 |
| 0.83 | 1.40186 | 0.78333 | 0.55878 |
| 0.84 | 1.41218 | 0.79569 | 0.56345 |
| 0.85 | 1.42266 | 0.80814 | 0.56805 |
| 0.86 | 1.43329 | 0.82069 | 0.57259 |
| 0.87 | 1.44408 | 0.83333 | 0.57707 |
| 0.88 | 1.45502 | 0.84607 | 0.58148 |
| 0.89 | 1.46612 | 0.85891 | 0.58584 |
| 0.90 | 1.47738 | 0.87184 | 0.59013 |
| 0.91 | 1.48679 | 0.88487 | 0.59436 |
| 0.92 | 1.50037 | 0.89801 | 0.59853 |
| 0.93 | 1.51211 | 0.91125 | 0.60264 |
| 0.94 | 1.52401 | 0.92459 | 0.60669 |
| 0.95 | 1.53607 | 0.93804 | 0.61068 |
| 0.96 | 1.54830 | 0.95160 | 0.61461 |
| 0.97 | 1.56070 | 0.96526 | 0.61848 |
| 0.98 | 1.57326 | 0.97904 | 0.62230 |
| 0.99 | 1.58600 | 0.99292 | 0.62605 |
| 1.00 | 1.59890 | 1.00692 | 0.62976 |

| $x$ | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 1.00 | 1.59890 | 1.00692 | 0.62976 |
| 1.01 | 1.61198 | 1.02102 | 0.63340 |
| 1.02 | 1.62522 | 1.03525 | 0.63699 |
| 1.03 | 1.63865 | 1.04959 | 0.64052 |
| 1.04 | 1.65224 | 1.06405 | 0.64400 |
| 1.05 | 1.66602 | 1.07862 | 0.64743 |
| 1.06 | 1.67997 | 1.09332 | 0.65080 |
| 1.07 | 1.69411 | 1.10814 | 0.65412 |
| 1.08 | 1.70843 | 1.12309 | 0.65738 |
| 1.09 | 1.72292 | 1.13815 | 0.66059 |
| 1.10 | 1.73761 | 1.15335 | 0.66376 |
| 1.11 | 1.75248 | 1.16867 | 0.66687 |
| 1.12 | 1.76753 | 1.18412 | 0.66993 |
| 1.13 | 1.78278 | 1.19971 | 0.67294 |
| 1.14 | 1.79822 | 1.21542 | 0.67590 |
| 1.15 | 1.81384 | 1.23127 | 0.67882 |
| 1.16 | 1.82967 | 1.24726 | 0.68169 |
| 1.17 | 1.84569 | 1.26338 | 0.68450 |
| 1.18 | 1.86190 | 1.27964 | 0.68728 |
| 1.19 | 1.87831 | 1.29604 | 0.69000 |
| 1.20 | 1.89493 | 1.31259 | 0.69268 |
| 1.21 | 1.91174 | 1.32928 | 0.69532 |
| 1.22 | 1.92877 | 1.34611 | 0.69791 |
| 1.23 | 1.94599 | 1.36309 | 0.70046 |
| 1.24 | 1.96342 | 1.38022 | 0.70297 |
| 1.25 | 1.98107 | 1.39750 | 0.70543 |
| 1.26 | 1.99892 | 1.41493 | 0.70785 |
| 1.27 | 2.01699 | 1.43252 | 0.71023 |
| 1.28 | 2.03527 | 1.45026 | 0.71257 |
| 1.29 | 2.05377 | 1.46816 | 0.71486 |
| 1.30 | 2.07248 | 1.48623 | 0.71712 |
| 1.31 | 2.09142 | 1.50445 | 0.71934 |
| 1.32 | 2.11058 | 1.52283 | 0.72152 |
| 1.33 | 2.12996 | 1.54138 | 0.72367 |
| 1.34 | 2.14957 | 1.56010 | 0.72577 |
| 1.35 | 2.16941 | 1.57898 | 0.72784 |
| 1.36 | 2.18948 | 1.59804 | 0.72987 |
| 1.37 | 2.20978 | 1.61727 | 0.73187 |
| 1.38 | 2.23031 | 1.63667 | 0.73383 |
| 1.39 | 2.25109 | 1.65625 | 0.73576 |
| 1.40 | 2.27210 | 1.67601 | 0.73765 |
| 1.41 | 2.29335 | 1.69595 | 0.73951 |
| 1.42 | 2.31484 | 1.71607 | 0.74133 |
| 1.43 | 2.33658 | 1.73637 | 0.74312 |
| 1.44 | 2.35857 | 1.75686 | 0.74487 |
| 1.45 | 2.38081 | 1.77755 | 0.74661 |
| 1.46 | 2.40330 | 1.79842 | 0.74831 |
| 1.47 | 2.42604 | 1.81948 | 0.74998 |
| 1.48 | 2.44904 | 1.84074 | 0.75162 |
| 1.49 | 2.47230 | 1.86220 | 0.75322 |
| 1.50 | 2.49582 | 1.88385 | 0.75480 |

TABLE D.XVIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 5/11$ and $x$ from 0.00 to 1.50.

| x | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 1.50 | 2.49582 | 1.88385 | 0.75480 |
| 1.51 | 2.51961 | 1.90571 | 0.75635 |
| 1.52 | 2.54366 | 1.92777 | 0.75787 |
| 1.53 | 2.56798 | 1.95004 | 0.75937 |
| 1.54 | 2.59257 | 1.97252 | 0.76083 |
| 1.55 | 2.61744 | 1.99520 | 0.76227 |
| 1.56 | 2.64258 | 2.01810 | 0.76369 |
| 1.57 | 2.66800 | 2.04122 | 0.76507 |
| 1.58 | 2.69370 | 2.06455 | 0.76643 |
| 1.59 | 2.71969 | 2.08810 | 0.76777 |
| 1.60 | 2.74597 | 2.11188 | 0.76908 |
| 1.61 | 2.77253 | 2.13588 | 0.77037 |
| 1.62 | 2.79939 | 2.16010 | 0.77163 |
| 1.63 | 2.82654 | 2.18456 | 0.77287 |
| 1.64 | 2.85398 | 2.20925 | 0.77409 |
| 1.65 | 2.88173 | 2.23417 | 0.77529 |
| 1.66 | 2.90979 | 2.25933 | 0.77646 |
| 1.67 | 2.93815 | 2.28473 | 0.77761 |
| 1.68 | 2.96682 | 2.31037 | 0.77874 |
| 1.69 | 2.99580 | 2.33626 | 0.77985 |
| 1.70 | 3.02509 | 2.36239 | 0.78093 |
| 1.71 | 3.05471 | 2.38878 | 0.78200 |
| 1.72 | 3.08464 | 2.41542 | 0.78305 |
| 1.73 | 3.11490 | 2.44231 | 0.78407 |
| 1.74 | 3.14549 | 2.46946 | 0.78508 |
| 1.75 | 3.17641 | 2.49688 | 0.78607 |
| 1.76 | 3.20766 | 2.52455 | 0.78704 |
| 1.77 | 3.23925 | 2.55250 | 0.78799 |
| 1.78 | 3.27118 | 2.58071 | 0.78892 |
| 1.79 | 3.30345 | 2.60920 | 0.78984 |
| 1.80 | 3.33607 | 2.63796 | 0.79074 |
| 1.81 | 3.36904 | 2.66700 | 0.79162 |
| 1.82 | 3.40237 | 2.69632 | 0.79248 |
| 1.83 | 3.43605 | 2.72593 | 0.79333 |
| 1.84 | 3.47009 | 2.75582 | 0.79416 |
| 1.85 | 3.50449 | 2.78601 | 0.79498 |
| 1.86 | 3.53927 | 2.81648 | 0.79578 |
| 1.87 | 3.57441 | 2.84726 | 0.79657 |
| 1.88 | 3.60993 | 2.87833 | 0.79734 |
| 1.89 | 3.64582 | 2.90971 | 0.79809 |
| 1.90 | 3.68210 | 2.94139 | 0.79883 |
| 1.91 | 3.71876 | 2.97338 | 0.79956 |
| 1.92 | 3.75582 | 3.00569 | 0.80027 |
| 1.93 | 3.79326 | 3.03831 | 0.80097 |
| 1.94 | 3.83111 | 3.07125 | 0.80166 |
| 1.95 | 3.86935 | 3.10451 | 0.80233 |
| 1.96 | 3.90800 | 3.13810 | 0.80299 |
| 1.97 | 3.94706 | 3.17202 | 0.80364 |
| 1.98 | 3.98653 | 3.20627 | 0.80428 |
| 1.99 | 4.02642 | 3.24086 | 0.80490 |
| 2.00 | 4.06673 | 3.27578 | 0.80551 |

| x | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 2.0 | 4.06673 | 3.27578 | 0.80551 |
| 2.1 | 4.49390 | 3.64456 | 0.81109 |
| 2.2 | 4.96800 | 4.05149 | 0.81552 |
| 2.3 | 5.49388 | 4.50074 | 0.81923 |
| 2.4 | 6.07694 | 4.99690 | 0.82227 |
| 2.5 | 6.72315 | 5.54504 | 0.82477 |
| 2.6 | 7.43912 | 6.15079 | 0.82682 |
| 2.7 | 8.23218 | 6.82031 | 0.82849 |
| 2.8 | 9.11044 | 7.56048 | 0.82987 |
| 2.9 | 10.08289 | 8.37883 | 0.83100 |
| 3.0 | 11.15945 | 9.28375 | 0.83192 |
| 3.1 | 12.35115 | 10.28447 | 0.83267 |
| 3.2 | 13.67013 | 11.39120 | 0.83329 |
| 3.3 | 15.12958 | 12.61526 | 0.83380 |
| 3.4 | 16.74529 | 13.96912 | 0.83421 |
| 3.5 | 18.53284 | 15.46660 | 0.83455 |
| 3.6 | 20.51075 | 17.12296 | 0.83483 |
| 3.7 | 22.69921 | 18.95511 | 0.83506 |
| 3.8 | 25.12051 | 20.98170 | 0.83524 |
| 3.9 | 27.79932 | 23.22339 | 0.83539 |
| 4.0 | 30.76293 | 25.70300 | 0.83552 |
| 4.1 | 34.04152 | 28.44579 | 0.83562 |
| 4.2 | 37.66846 | 31.47968 | 0.83570 |
| 4.3 | 41.68066 | 34.83555 | 0.83577 |
| 4.4 | 46.11854 | 38.54752 | 0.83583 |
| 4.5 | 51.02846 | 42.65337 | 0.83587 |
| 4.6 | 56.45912 | 47.19484 | 0.83591 |
| 4.7 | 62.46615 | 52.21810 | 0.83594 |
| 4.8 | 69.11059 | 57.77420 | 0.83597 |
| 4.9 | 76.45996 | 63.91960 | 0.83599 |
| 5.0 | 84.58892 | 70.71674 | 0.83600 |
| 5.1 | 93.58004 | 78.23464 | 0.83602 |
| 5.2 | 103.52460 | 86.54965 | 0.83603 |
| 5.3 | 114.52358 | 95.74618 | 0.83604 |
| 5.4 | 126.68861 | 105.91758 | 0.83605 |
| 5.5 | 140.14315 | 117.16706 | 0.83605 |
| 5.6 | 155.02369 | 129.60876 | 0.83606 |
| 5.7 | 171.48119 | 143.36890 | 0.83606 |
| 5.8 | 189.68257 | 158.58702 | 0.83607 |
| 5.9 | 209.81238 | 175.41744 | 0.83607 |
| 6.0 | 232.07472 | 194.03080 | 0.83607 |

$\alpha = 5/11$

| x | $F_{5/11}(x)$ | $H_{6/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|
| 6.0 | 232.07472 | 194.03080 | 0.83607 |
| 6.1 | 256.69525 | 214.61577 | 0.83607 |
| 6.2 | 283.92350 | 237.38098 | 0.83607 |
| 6.3 | 314.03538 | 262.55713 | 0.83607 |
| 6.4 | 347.33557 | 290.39927 | 0.83608 |
| 6.5 | 384.16261 | 321.16946 | 0.83608 |
| 6.6 | 424.88830 | 355.23956 | 0.83608 |
| 6.7 | 469.92547 | 392.89439 | 0.83608 |
| 6.8 | 519.73014 | 434.53522 | 0.83608 |
| 6.9 | 574.80653 | 480.58362 | 0.83608 |
| 7.0 | 635.71217 | 531.50574 | 0.83608 |
| 7.1 | 703.06350 | 587.81696 | 0.83608 |
| 7.2 | 777.54210 | 650.08715 | 0.83608 |
| 7.3 | 859.90160 | 718.94640 | 0.83608 |
| 7.4 | 950.97525 | 795.09138 | 0.83608 |
| 7.5 | 1051.68456 | 879.29237 | 0.83608 |
| 7.6 | 1163.04758 | 972.40105 | 0.83608 |
| 7.7 | 1286.19120 | 1075.35909 | 0.83608 |
| 7.8 | 1422.36050 | 1189.20764 | 0.83608 |
| 7.9 | 1572.93233 | 1315.09787 | 0.83608 |
| 8.0 | 1739.42900 | 1454.30254 | 0.83608 |
| 8.1 | 1923.53362 | 1608.22887 | 0.83608 |
| 8.2 | 2127.10712 | 1778.43276 | 0.83608 |
| 8.3 | 2352.20700 | 1966.63443 | 0.83608 |
| 8.4 | 2601.10810 | 2174.73584 | 0.83608 |
| 8.5 | 2876.32553 | 2404.83981 | 0.83608 |
| 8.6 | 3180.64004 | 2659.27129 | 0.83608 |
| 8.7 | 3517.12609 | 2940.60078 | 0.83608 |
| 8.8 | 3889.18285 | 3251.67024 | 0.83608 |
| 8.9 | 4300.56846 | 3595.62178 | 0.83608 |
| 9.0 | 4755.43794 | 3975.92935 | 0.83608 |
| 9.1 | 5258.38510 | 4396.43373 | 0.83608 |
| 9.2 | 5814.48886 | 4861.38131 | 0.83608 |
| 9.3 | 6429.36442 | 5375.46686 | 0.83608 |
| 9.4 | 7109.21993 | 5943.88088 | 0.83608 |
| 9.5 | 7860.91903 | 6572.36193 | 0.83608 |
| 9.6 | 8692.05006 | 7267.25448 | 0.83608 |
| 9.7 | 9611.00255 | 8035.57285 | 0.83608 |
| 9.8 | 10627.05178 | 8885.07192 | 0.83608 |
| 9.9 | 11750.45229 | 9824.32531 | 0.83608 |
| 10.0 | 12992.54118 | 10862.81176 | 0.83608 |

TABLE D.XVIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/11$ and $x$ from 1.50 to 10.0.

$\alpha = 6/11$

| $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{6/11}(x)$ | $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{6/11}(x)$ | $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{5/11}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.11692 | 0.65102 | 0.58287 | 1.00 | 1.49664 | 1.38341 | 0.92434 |
| 0.01 | 1.00005 | 0.01781 | 0.01781 | 0.51 | 1.12174 | 0.66398 | 0.59192 | 1.01 | 1.50743 | 1.40050 | 0.92906 |
| 0.02 | 1.00018 | 0.03344 | 0.03343 | 0.52 | 1.12667 | 0.67698 | 0.60087 | 1.02 | 1.51837 | 1.41772 | 0.93371 |
| 0.03 | 1.00041 | 0.04835 | 0.04832 | 0.53 | 1.13170 | 0.69003 | 0.60973 | 1.03 | 1.52945 | 1.43506 | 0.93829 |
| 0.04 | 1.00073 | 0.06281 | 0.06276 | 0.54 | 1.13683 | 0.70312 | 0.61849 | 1.04 | 1.54067 | 1.45253 | 0.94279 |
| 0.05 | 1.00115 | 0.07695 | 0.07686 | 0.55 | 1.14207 | 0.71626 | 0.62716 | 1.05 | 1.55203 | 1.47013 | 0.94723 |
| 0.06 | 1.00165 | 0.09084 | 0.09069 | 0.56 | 1.14742 | 0.72945 | 0.63573 | 1.06 | 1.56354 | 1.48786 | 0.95160 |
| 0.07 | 1.00225 | 0.10452 | 0.10429 | 0.57 | 1.15287 | 0.74269 | 0.64421 | 1.07 | 1.57520 | 1.50572 | 0.95590 |
| 0.08 | 1.00293 | 0.11804 | 0.11770 | 0.58 | 1.15842 | 0.75599 | 0.65260 | 1.08 | 1.58700 | 1.52372 | 0.96013 |
| 0.09 | 1.00371 | 0.13142 | 0.13094 | 0.59 | 1.16409 | 0.76934 | 0.66089 | 1.09 | 1.59895 | 1.54186 | 0.96429 |
| 0.10 | 1.00459 | 0.14468 | 0.14402 | 0.60 | 1.16986 | 0.78274 | 0.66909 | 1.10 | 1.61106 | 1.56014 | 0.96839 |
| 0.11 | 1.00555 | 0.15783 | 0.15696 | 0.61 | 1.17574 | 0.79621 | 0.67720 | 1.11 | 1.62331 | 1.57855 | 0.97243 |
| 0.12 | 1.00661 | 0.17089 | 0.16977 | 0.62 | 1.18173 | 0.80973 | 0.68521 | 1.12 | 1.63571 | 1.59711 | 0.97640 |
| 0.13 | 1.00776 | 0.18387 | 0.18246 | 0.63 | 1.18783 | 0.82332 | 0.69313 | 1.13 | 1.64827 | 1.61582 | 0.98031 |
| 0.14 | 1.00900 | 0.19678 | 0.19502 | 0.64 | 1.19404 | 0.83696 | 0.70095 | 1.14 | 1.66099 | 1.63467 | 0.98416 |
| 0.15 | 1.01033 | 0.20962 | 0.20747 | 0.65 | 1.20036 | 0.85068 | 0.70869 | 1.15 | 1.67386 | 1.65367 | 0.98794 |
| 0.16 | 1.01176 | 0.22240 | 0.21982 | 0.66 | 1.20679 | 0.86445 | 0.71633 | 1.16 | 1.68688 | 1.67282 | 0.99166 |
| 0.17 | 1.01328 | 0.23514 | 0.23206 | 0.67 | 1.21333 | 0.87830 | 0.72388 | 1.17 | 1.70007 | 1.69212 | 0.99533 |
| 0.18 | 1.01489 | 0.24783 | 0.24419 | 0.68 | 1.21998 | 0.89221 | 0.73133 | 1.18 | 1.71341 | 1.71158 | 0.99893 |
| 0.19 | 1.01659 | 0.26048 | 0.25623 | 0.69 | 1.22675 | 0.90620 | 0.73870 | 1.19 | 1.72692 | 1.73119 | 1.00247 |
| 0.20 | 1.01839 | 0.27309 | 0.26816 | 0.70 | 1.23363 | 0.92025 | 0.74597 | 1.20 | 1.74059 | 1.75096 | 1.00596 |
| 0.21 | 1.02028 | 0.28568 | 0.28000 | 0.71 | 1.24062 | 0.93438 | 0.75316 | 1.21 | 1.75442 | 1.77089 | 1.00939 |
| 0.22 | 1.02227 | 0.29824 | 0.29174 | 0.72 | 1.24773 | 0.94859 | 0.76025 | 1.22 | 1.76842 | 1.79098 | 1.01276 |
| 0.23 | 1.02435 | 0.31078 | 0.30339 | 0.73 | 1.25496 | 0.96287 | 0.76725 | 1.23 | 1.78258 | 1.81124 | 1.01608 |
| 0.24 | 1.02652 | 0.32330 | 0.31495 | 0.74 | 1.26230 | 0.97723 | 0.77417 | 1.24 | 1.79691 | 1.83166 | 1.01934 |
| 0.25 | 1.02879 | 0.33580 | 0.32641 | 0.75 | 1.26976 | 0.99167 | 0.78099 | 1.25 | 1.81141 | 1.85225 | 1.02255 |
| 0.26 | 1.03115 | 0.34830 | 0.33778 | 0.76 | 1.27734 | 1.00620 | 0.78773 | 1.26 | 1.82608 | 1.87301 | 1.02570 |
| 0.27 | 1.03361 | 0.36078 | 0.34905 | 0.77 | 1.28503 | 1.02080 | 0.79438 | 1.27 | 1.84092 | 1.89395 | 1.02880 |
| 0.28 | 1.03616 | 0.37326 | 0.36023 | 0.78 | 1.29285 | 1.03549 | 0.80094 | 1.28 | 1.85594 | 1.91505 | 1.03185 |
| 0.29 | 1.03881 | 0.38574 | 0.37132 | 0.79 | 1.30078 | 1.05027 | 0.80741 | 1.29 | 1.87113 | 1.93634 | 1.03485 |
| 0.30 | 1.04155 | 0.39821 | 0.38232 | 0.80 | 1.30884 | 1.06513 | 0.81380 | 1.30 | 1.88650 | 1.95780 | 1.03780 |
| 0.31 | 1.04439 | 0.41069 | 0.39323 | 0.81 | 1.31702 | 1.08009 | 0.82010 | 1.31 | 1.90204 | 1.97944 | 1.04069 |
| 0.32 | 1.04732 | 0.42317 | 0.40405 | 0.82 | 1.32532 | 1.09513 | 0.82632 | 1.32 | 1.91777 | 2.00127 | 1.04354 |
| 0.33 | 1.05035 | 0.43565 | 0.41477 | 0.83 | 1.33374 | 1.11027 | 0.83245 | 1.33 | 1.93367 | 2.02328 | 1.04634 |
| 0.34 | 1.05346 | 0.44815 | 0.42540 | 0.84 | 1.34229 | 1.12551 | 0.83850 | 1.34 | 1.94976 | 2.04548 | 1.04909 |
| 0.35 | 1.05670 | 0.46066 | 0.43594 | 0.85 | 1.35096 | 1.14083 | 0.84446 | 1.35 | 1.96603 | 2.06787 | 1.05180 |
| 0.36 | 1.06003 | 0.47318 | 0.44639 | 0.86 | 1.35976 | 1.15626 | 0.85034 | 1.36 | 1.98249 | 2.09045 | 1.05446 |
| 0.37 | 1.06344 | 0.48572 | 0.45674 | 0.87 | 1.36866 | 1.17179 | 0.85614 | 1.37 | 1.99913 | 2.11322 | 1.05707 |
| 0.38 | 1.06696 | 0.49827 | 0.46700 | 0.88 | 1.37774 | 1.18741 | 0.86186 | 1.38 | 2.01597 | 2.13619 | 1.05964 |
| 0.39 | 1.07057 | 0.51085 | 0.47717 | 0.89 | 1.38692 | 1.20314 | 0.86749 | 1.39 | 2.03299 | 2.15936 | 1.06216 |
| 0.40 | 1.07429 | 0.52345 | 0.48725 | 0.90 | 1.39623 | 1.21898 | 0.87305 | 1.40 | 2.05021 | 2.18273 | 1.06464 |
| 0.41 | 1.07810 | 0.53607 | 0.49723 | 0.91 | 1.40567 | 1.23492 | 0.87853 | 1.41 | 2.06762 | 2.20630 | 1.06707 |
| 0.42 | 1.08201 | 0.54871 | 0.50712 | 0.92 | 1.41524 | 1.25096 | 0.88392 | 1.42 | 2.08522 | 2.23008 | 1.06947 |
| 0.43 | 1.08602 | 0.56139 | 0.51692 | 0.93 | 1.42494 | 1.26712 | 0.88924 | 1.43 | 2.10302 | 2.25406 | 1.07182 |
| 0.44 | 1.09013 | 0.57409 | 0.52662 | 0.94 | 1.43478 | 1.28339 | 0.89448 | 1.44 | 2.12102 | 2.27826 | 1.07413 |
| 0.45 | 1.09434 | 0.58682 | 0.53623 | 0.95 | 1.44475 | 1.29976 | 0.89965 | 1.45 | 2.13923 | 2.30266 | 1.07640 |
| 0.46 | 1.09865 | 0.59959 | 0.54575 | 0.96 | 1.45485 | 1.31626 | 0.90474 | 1.46 | 2.15763 | 2.32728 | 1.07863 |
| 0.47 | 1.10307 | 0.61239 | 0.55517 | 0.97 | 1.46509 | 1.33287 | 0.90975 | 1.47 | 2.17624 | 2.35212 | 1.08082 |
| 0.48 | 1.10758 | 0.62523 | 0.56450 | 0.98 | 1.47547 | 1.34960 | 0.91469 | 1.48 | 2.19505 | 2.37718 | 1.08297 |
| 0.49 | 1.11220 | 0.63011 | 0.57373 | 0.99 | 1.48599 | 1.36644 | 0.91955 | 1.49 | 2.21408 | 2.40247 | 1.08509 |
| 0.50 | 1.11692 | 0.65102 | 0.58287 | 1.00 | 1.49664 | 1.38341 | 0.92434 | 1.50 | 2.23331 | 2.42797 | 1.08716 |

TABLE D.XVIIIA. LANCHESTER–CLIFFORD–SCHLÄFLI Functions $F_{\alpha}(x)$, $H_{1-\alpha}(x)$, and $T_{\alpha}(x)$ for $\alpha = 6/11$ and $x$ from 0.00 to 1.50.

$\alpha = 6/11$

| $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{6/11}(x)$ |
|---|---|---|---|
| 1.50 | 2.23331 | 2.42797 | 1.08716 |
| 1.51 | 2.25275 | 2.45371 | 1.08920 |
| 1.52 | 2.27241 | 2.47967 | 1.09121 |
| 1.53 | 2.29228 | 2.50586 | 1.09317 |
| 1.54 | 2.31237 | 2.53230 | 1.09511 |
| 1.55 | 2.33268 | 2.55896 | 1.09700 |
| 1.56 | 2.35321 | 2.58587 | 1.09887 |
| 1.57 | 2.37397 | 2.61302 | 1.10070 |
| 1.58 | 2.39495 | 2.64042 | 1.10250 |
| 1.59 | 2.41615 | 2.66807 | 1.10426 |
| 1.60 | 2.43759 | 2.69596 | 1.10600 |
| 1.61 | 2.45926 | 2.72411 | 1.10770 |
| 1.62 | 2.48116 | 2.75252 | 1.10937 |
| 1.63 | 2.50330 | 2.78118 | 1.11101 |
| 1.64 | 2.52567 | 2.81011 | 1.11262 |
| 1.65 | 2.54829 | 2.83930 | 1.11420 |
| 1.66 | 2.57114 | 2.86876 | 1.11575 |
| 1.67 | 2.59424 | 2.89849 | 1.11728 |
| 1.68 | 2.61759 | 2.92850 | 1.11878 |
| 1.69 | 2.64119 | 2.95678 | 1.12024 |
| 1.70 | 2.66504 | 2.98934 | 1.12169 |
| 1.71 | 2.68914 | 3.02018 | 1.12310 |
| 1.72 | 2.71350 | 3.05131 | 1.12449 |
| 1.73 | 2.73812 | 3.08273 | 1.12586 |
| 1.74 | 2.76300 | 3.11444 | 1.12720 |
| 1.75 | 2.78814 | 3.14644 | 1.12851 |
| 1.76 | 2.81354 | 3.17874 | 1.12980 |
| 1.77 | 2.83922 | 3.21134 | 1.13107 |
| 1.78 | 2.86517 | 3.24425 | 1.13231 |
| 1.79 | 2.89138 | 3.27747 | 1.13353 |
| 1.80 | 2.91788 | 3.31099 | 1.13473 |
| 1.81 | 2.94465 | 3.34483 | 1.13590 |
| 1.82 | 2.97171 | 3.37899 | 1.13705 |
| 1.83 | 2.99904 | 3.41346 | 1.13818 |
| 1.84 | 3.02667 | 3.44827 | 1.13929 |
| 1.85 | 3.05458 | 3.48340 | 1.14038 |
| 1.86 | 3.08279 | 3.51886 | 1.14145 |
| 1.87 | 3.11128 | 3.55465 | 1.14250 |
| 1.88 | 3.14008 | 3.59078 | 1.14353 |
| 1.89 | 3.16918 | 3.62726 | 1.14454 |
| 1.90 | 3.19858 | 3.66408 | 1.14553 |
| 1.91 | 3.22838 | 3.70125 | 1.14651 |
| 1.92 | 3.25829 | 3.73877 | 1.14746 |
| 1.93 | 3.28862 | 3.77664 | 1.14840 |
| 1.94 | 3.31926 | 3.81488 | 1.14932 |
| 1.95 | 3.35021 | 3.85348 | 1.15022 |
| 1.96 | 3.38149 | 3.89244 | 1.15110 |
| 1.97 | 3.41309 | 3.93178 | 1.15197 |
| 1.98 | 3.44502 | 3.97150 | 1.15282 |
| 1.99 | 3.47728 | 4.01159 | 1.15366 |
| 2.00 | 3.50987 | 4.05206 | 1.15448 |

| $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{6/11}(x)$ |
|---|---|---|---|
| 2.0 | 3.50987 | 4.05206 | 1.15448 |
| 2.1 | 3.85480 | 4.47875 | 1.16186 |
| 2.2 | 4.23674 | 4.94833 | 1.16796 |
| 2.3 | 4.65941 | 5.46539 | 1.17298 |
| 2.4 | 5.12697 | 6.03500 | 1.17711 |
| 2.5 | 5.64397 | 6.66275 | 1.18051 |
| 2.6 | 6.21548 | 7.35478 | 1.18330 |
| 2.7 | 6.84709 | 8.11788 | 1.18559 |
| 2.8 | 7.54500 | 8.95952 | 1.18748 |
| 2.9 | 8.31603 | 9.88796 | 1.18902 |
| 3.0 | 9.16775 | 10.91230 | 1.19029 |
| 3.1 | 10.10851 | 12.04260 | 1.19133 |
| 3.2 | 11.14754 | 13.28993 | 1.19219 |
| 3.3 | 12.29503 | 14.66655 | 1.19288 |
| 3.4 | 13.56224 | 16.18597 | 1.19346 |
| 3.5 | 14.96163 | 17.86311 | 1.19393 |
| 3.6 | 16.50694 | 19.71445 | 1.19431 |
| 3.7 | 18.21335 | 21.75818 | 1.19463 |
| 3.8 | 20.09764 | 24.01440 | 1.19489 |
| 3.9 | 22.17833 | 26.50528 | 1.19510 |
| 4.0 | 24.47588 | 29.25533 | 1.19527 |
| 4.1 | 27.01291 | 32.29161 | 1.19541 |
| 4.2 | 29.81437 | 35.64398 | 1.19553 |
| 4.3 | 32.90785 | 39.34547 | 1.19563 |
| 4.4 | 36.32380 | 43.43250 | 1.19570 |
| 4.5 | 40.09587 | 47.94534 | 1.19577 |
| 4.6 | 44.26122 | 52.92845 | 1.19582 |
| 4.7 | 48.86088 | 58.43092 | 1.19586 |
| 4.8 | 53.94021 | 64.50699 | 1.19590 |
| 4.9 | 59.54927 | 71.21657 | 1.19593 |
| 5.0 | 65.74337 | 78.62581 | 1.19595 |
| 5.1 | 72.58361 | 86.80780 | 1.19597 |
| 5.2 | 80.13746 | 95.84324 | 1.19599 |
| 5.3 | 88.47947 | 105.82130 | 1.19600 |
| 5.4 | 97.69195 | 116.84045 | 1.19601 |
| 5.5 | 107.86585 | 129.00947 | 1.19602 |
| 5.6 | 119.10163 | 142.44850 | 1.19602 |
| 5.7 | 131.51022 | 157.29025 | 1.19603 |
| 5.8 | 145.21419 | 173.68131 | 1.19604 |
| 5.9 | 160.34892 | 191.78361 | 1.19604 |
| 6.0 | 177.06395 | 211.77600 | 1.19604 |

| $x$ | $F_{6/11}(x)$ | $H_{5/11}(x)$ | $T_{6/11}(x)$ |
|---|---|---|---|
| 6.0 | 177.06395 | 211.77600 | 1.19604 |
| 6.1 | 195.52444 | 233.85605 | 1.19605 |
| 6.2 | 215.91287 | 258.24199 | 1.19605 |
| 6.3 | 238.43078 | 285.17489 | 1.19605 |
| 6.4 | 263.30060 | 314.92103 | 1.19605 |
| 6.5 | 290.76886 | 347.77456 | 1.19605 |
| 6.6 | 321.10661 | 384.06039 | 1.19605 |
| 6.7 | 354.61415 | 424.13744 | 1.19605 |
| 6.8 | 391.62295 | 468.40220 | 1.19605 |
| 6.9 | 432.49920 | 517.29263 | 1.19605 |
| 7.0 | 477.64739 | 571.29252 | 1.19605 |
| 7.1 | 527.51434 | 630.93633 | 1.19606 |
| 7.2 | 582.59366 | 696.81441 | 1.19606 |
| 7.3 | 643.43061 | 769.57894 | 1.19606 |
| 7.4 | 710.62752 | 849.95034 | 1.19606 |
| 7.5 | 784.84978 | 938.72444 | 1.19606 |
| 7.6 | 866.83243 | 1036.78041 | 1.19606 |
| 7.7 | 957.38745 | 1145.08942 | 1.19606 |
| 7.8 | 1057.41185 | 1264.72434 | 1.19606 |
| 7.9 | 1167.89653 | 1396.87034 | 1.19606 |
| 8.0 | 1289.93614 | 1542.83670 | 1.19606 |
| 8.1 | 1424.73996 | 1704.06977 | 1.19606 |
| 8.2 | 1573.64387 | 1882.16736 | 1.19606 |
| 8.3 | 1738.12366 | 2078.89457 | 1.19606 |
| 8.4 | 1919.80963 | 2296.20135 | 1.19606 |
| 8.5 | 2120.50281 | 2536.24185 | 1.19606 |
| 8.6 | 2342.19287 | 2801.39579 | 1.19606 |
| 8.7 | 2587.07781 | 3094.29214 | 1.19606 |
| 8.8 | 2857.58587 | 3417.83519 | 1.19606 |
| 8.9 | 3156.39958 | 3775.23341 | 1.19606 |
| 9.0 | 3486.48246 | 4170.03133 | 1.19606 |
| 9.1 | 3851.10841 | 4606.14474 | 1.19606 |
| 9.2 | 4253.89429 | 5087.89959 | 1.19606 |
| 9.3 | 4698.83581 | 5620.07497 | 1.19606 |
| 9.4 | 5190.34722 | 6207.95060 | 1.19606 |
| 9.5 | 5733.30521 | 6857.35928 | 1.19606 |
| 9.6 | 6333.09737 | 7574.74484 | 1.19606 |
| 9.7 | 6995.67566 | 8367.22620 | 1.19606 |
| 9.8 | 7727.61565 | 9242.66809 | 1.19606 |
| 9.9 | 8536.18180 | 10209.75923 | 1.19606 |
| 10.0 | 9429.39968 | 11278.09866 | 1.19606 |

TABLE D.XVIIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 6/11$ and $x$ from 1.50 to 10.0.

| x | $F_{8/11}(x)$ | $H_{3/11}(x)$ | $T_{8/11}(x)$ | x | $F_{8/11}(x)$ | $H_{3/11}(x)$ | $T_{8/11}(x)$ | x | $F_{8/11}(x)$ | $H_{3/11}(x)$ | $T_{8/11}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.08750 | 1.80707 | 1.66167 | 1.00 | 1.36940 | 3.03364 | 2.21531 |
| 0.01 | 1.00003 | 0.20379 | 0.20378 | 0.51 | 1.09111 | 1.83025 | 1.67743 | 1.01 | 1.37737 | 3.06128 | 2.22256 |
| 0.02 | 1.00014 | 0.29744 | 0.29740 | 0.52 | 1.09478 | 1.85340 | 1.69294 | 1.02 | 1.38544 | 3.08909 | 2.22969 |
| 0.03 | 1.00031 | 0.37110 | 0.37098 | 0.53 | 1.09854 | 1.87654 | 1.70821 | 1.03 | 1.39361 | 3.11709 | 2.23671 |
| 0.04 | 1.00055 | 0.43421 | 0.43397 | 0.54 | 1.10237 | 1.89966 | 1.72325 | 1.04 | 1.40188 | 3.14528 | 2.24361 |
| 0.05 | [illegible] | 0.49049 | 0.49007 | 0.55 | 1.10628 | 1.92278 | 1.73805 | 1.05 | 1.41026 | 3.17366 | 2.25041 |
| 0.06 | [illegible] | 0.54190 | 0.54123 | 0.56 | 1.11027 | 1.94590 | 1.75263 | 1.06 | 1.41874 | 3.20223 | 2.25709 |
| 0.07 | 1.00168 | 0.58958 | 0.58859 | 0.57 | 1.11434 | 1.96903 | 1.76699 | 1.07 | 1.42733 | 3.23100 | 2.26367 |
| 0.08 | 1.00220 | 0.63431 | 0.63292 | 0.58 | 1.11848 | 1.99216 | 1.78113 | 1.08 | 1.43602 | 3.25998 | 2.27014 |
| 0.09 | 1.00279 | 0.67663 | 0.67475 | 0.59 | 1.12271 | 2.01532 | 1.79506 | 1.09 | 1.44483 | 3.28915 | 2.27650 |
| 0.10 | 1.00344 | 0.71692 | 0.71446 | 0.60 | 1.12701 | 2.03850 | 1.80877 | 1.10 | 1.45373 | 3.31853 | 2.28276 |
| 0.11 | [illegible] | 0.75549 | [illegible] | 0.61 | 1.13139 | 2.06171 | 1.82228 | 1.11 | 1.46275 | 3.34812 | 2.28892 |
| 0.12 | [illegible] | 0.79257 | 0.78864 | 0.62 | 1.13586 | 2.08495 | 1.83558 | 1.12 | 1.47188 | 3.37793 | 2.29498 |
| 0.13 | 1.00582 | 0.82834 | 0.82355 | 0.63 | 1.14040 | 2.10823 | 1.84868 | 1.13 | 1.48111 | 3.40795 | 2.30094 |
| 0.14 | 1.00675 | 0.86297 | 0.85719 | 0.64 | 1.14503 | 2.13155 | 1.86158 | 1.14 | 1.49046 | 3.43819 | 2.30680 |
| 0.15 | 1.00775 | 0.89657 | 0.88968 | 0.65 | 1.14973 | 2.15492 | 1.87428 | 1.15 | 1.49992 | 3.46865 | 2.31256 |
| 0.16 | 1.00882 | 0.92926 | 0.92114 | 0.66 | 1.15452 | 2.17835 | 1.88680 | 1.16 | 1.50949 | 3.49934 | 2.31823 |
| 0.17 | 1.00996 | 0.96113 | 0.95165 | 0.67 | 1.15939 | 2.20182 | 1.89912 | 1.17 | 1.51917 | 3.53026 | 2.32381 |
| 0.18 | 1.01116 | 0.99224 | 0.98129 | 0.68 | 1.16434 | 2.22536 | 1.91126 | 1.18 | 1.52897 | 3.56141 | 2.32929 |
| 0.19 | 1.01244 | 1.02268 | 1.01012 | 0.69 | 1.16938 | 2.24896 | 1.92321 | 1.19 | 1.53888 | 3.59280 | 2.33468 |
| 0.20 | 1.01379 | [illegible] | 1.03819 | 0.70 | 1.17450 | 2.27264 | 1.93498 | 1.20 | 1.54891 | 3.62442 | 2.33998 |
| 0.21 | 1.01521 | 1.08176 | 1.06555 | 0.71 | 1.17970 | 2.29638 | 1.94658 | 1.21 | 1.55905 | 3.65629 | 2.34520 |
| 0.22 | 1.01670 | 1.11049 | 1.09225 | 0.72 | 1.18499 | 2.32021 | 1.95799 | 1.22 | 1.56932 | 3.68840 | 2.35032 |
| 0.23 | [illegible] | 1.13875 | 1.11833 | 0.73 | 1.19036 | 2.34411 | 1.96924 | 1.23 | 1.57970 | 3.72077 | 2.35536 |
| 0.24 | 1.01988 | 1.16656 | 1.14382 | 0.74 | 1.19582 | 2.36810 | 1.98031 | 1.24 | 1.59020 | 3.75338 | 2.36032 |
| 0.25 | 1.02158 | 1.19397 | 1.16875 | 0.75 | 1.20137 | 2.39218 | 1.99122 | 1.25 | 1.60082 | 3.78625 | 2.36519 |
| 0.26 | 1.02335 | 1.22101 | 1.19314 | 0.76 | 1.20700 | 2.41635 | 2.00195 | 1.26 | 1.61156 | 3.81938 | 2.36998 |
| 0.27 | 1.02519 | 1.24769 | 1.21703 | 0.77 | 1.21271 | 2.44062 | 2.01253 | 1.27 | 1.62243 | 3.85277 | 2.37469 |
| 0.28 | 1.02710 | 1.27406 | 1.24044 | 0.78 | 1.21852 | 2.46499 | 2.02294 | 1.28 | 1.63341 | 3.88642 | 2.37933 |
| 0.29 | 1.02909 | 1.30012 | 1.26338 | 0.79 | 1.22441 | 2.48946 | 2.03319 | 1.29 | 1.64453 | 3.92035 | 2.38388 |
| 0.30 | [illegible] | 1.32591 | 1.28587 | 0.80 | 1.23039 | 2.51405 | 2.04329 | 1.30 | 1.65576 | 3.95454 | 2.38835 |
| 0.31 | 1.03325 | 1.35145 | 1.30794 | 0.81 | 1.23646 | 2.53874 | 2.05323 | 1.31 | 1.66713 | 3.98901 | 2.39275 |
| 0.32 | 1.03546 | 1.37674 | 1.32959 | 0.82 | 1.24262 | 2.56355 | 2.06302 | 1.32 | 1.67862 | 4.02376 | 2.39707 |
| 0.33 | 1.03773 | 1.40182 | 1.35085 | 0.83 | 1.24887 | 2.58847 | 2.07266 | 1.33 | 1.69023 | 4.05880 | 2.40132 |
| 0.34 | 1.04007 | 1.42669 | 1.37172 | 0.84 | 1.25521 | 2.61352 | 2.08215 | 1.34 | 1.70198 | 4.09412 | 2.40550 |
| 0.35 | 1.04248 | 1.45137 | 1.39222 | 0.85 | 1.26164 | 2.63870 | 2.09149 | 1.35 | 1.71386 | 4.12972 | 2.40961 |
| 0.36 | 1.04497 | 1.47588 | 1.41237 | 0.86 | 1.26816 | 2.66400 | 2.10069 | 1.36 | 1.72587 | 4.16562 | 2.41364 |
| 0.37 | 1.04753 | 1.50023 | 1.43216 | 0.87 | 1.27477 | 2.68943 | 2.10974 | 1.37 | 1.73801 | 4.20182 | 2.41761 |
| 0.38 | 1.05016 | 1.52443 | 1.45162 | 0.88 | 1.28148 | 2.71500 | 2.11865 | 1.38 | 1.75028 | 4.23831 | 2.42151 |
| 0.39 | 1.05286 | 1.54849 | 1.47074 | 0.89 | 1.28828 | 2.74071 | 2.12743 | 1.39 | 1.76269 | 4.27511 | 2.42534 |
| 0.40 | 1.05564 | 1.57243 | 1.48955 | 0.90 | 1.29517 | 2.76657 | 2.13607 | 1.40 | 1.77523 | 4.31222 | 2.42910 |
| 0.41 | 1.05849 | 1.59625 | 1.50805 | 0.91 | 1.30216 | 2.79256 | 2.14457 | 1.41 | 1.78791 | 4.34963 | 2.43280 |
| 0.42 | 1.06142 | 1.61997 | 1.52624 | 0.92 | 1.30924 | 2.81871 | 2.15294 | 1.42 | 1.80073 | 4.38736 | 2.43644 |
| 0.43 | [illegible] | 1.64360 | 1.54414 | 0.93 | 1.31642 | 2.84501 | 2.16118 | 1.43 | 1.81369 | 4.42541 | 2.44001 |
| 0.44 | [illegible] | 1.66714 | 1.56174 | 0.94 | 1.32369 | 2.87146 | 2.16928 | 1.44 | 1.82678 | 4.46378 | 2.44352 |
| 0.45 | 1.07064 | 1.69061 | 1.57907 | 0.95 | 1.33106 | 2.89808 | 2.17726 | 1.45 | 1.84002 | 4.50247 | 2.44697 |
| 0.46 | 1.07386 | 1.71400 | 1.59612 | 0.96 | 1.33853 | 2.92485 | 2.18512 | 1.46 | 1.85340 | 4.54149 | 2.45036 |
| 0.47 | 1.07716 | 1.73734 | 1.61290 | 0.97 | 1.34610 | 2.95180 | 2.19285 | 1.47 | 1.86692 | 4.58084 | 2.45368 |
| 0.48 | 1.08053 | 1.76063 | 1.62941 | 0.98 | 1.35377 | 2.97891 | 2.20046 | 1.48 | 1.88059 | 4.62053 | 2.45696 |
| 0.49 | 1.08398 | 1.78387 | 1.64567 | 0.99 | 1.36153 | 3.00619 | 2.20794 | 1.49 | 1.89441 | 4.66056 | 2.46017 |
| 0.50 | 1.08750 | 1.80707 | 1.66167 | 1.00 | 1.36940 | 3.03364 | 2.21531 | 1.50 | 1.90837 | 4.70093 | 2.46333 |

TABLE D.XIXA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 8/11$ and x from 0.00 to 1.50.

$\alpha = 8/11$

| $x$ | $F_{8/11}(x)$ | $H_{3/11}(x)$ | $T_{8/11}(x)$ |
|---|---|---|---|
| 1.50 | 1.90837 | 4.70093 | 2.46333 |
| 1.51 | 1.92248 | 4.74165 | 2.46643 |
| 1.52 | 1.93674 | 4.78272 | 2.46948 |
| 1.53 | 1.95115 | 4.82415 | 2.47247 |
| 1.54 | 1.96571 | 4.86593 | 2.47541 |
| 1.55 | 1.98042 | 4.90808 | 2.47830 |
| 1.56 | 1.99529 | 4.95060 | 2.48114 |
| 1.57 | 2.01032 | 4.99348 | 2.48392 |
| 1.58 | 2.02550 | 5.03674 | 2.48666 |
| 1.59 | 2.04084 | 5.08038 | 2.48935 |
| 1.60 | 2.05635 | 5.12440 | 2.49199 |
| 1.61 | 2.07201 | 5.16880 | 2.49458 |
| 1.62 | 2.08784 | 5.21360 | 2.49713 |
| 1.63 | 2.10383 | 5.25880 | 2.49963 |
| 1.64 | 2.11998 | 5.30439 | 2.50209 |
| 1.65 | 2.13630 | 5.35038 | 2.50450 |
| 1.66 | 2.15280 | 5.39679 | 2.50687 |
| 1.67 | 2.16946 | 5.44360 | 2.50920 |
| 1.68 | 2.18629 | 5.49084 | 2.51149 |
| 1.69 | 2.20329 | 5.53849 | 2.51373 |
| 1.70 | 2.22047 | 5.58657 | 2.51593 |
| 1.71 | 2.23783 | 5.63508 | 2.51810 |
| 1.72 | 2.25536 | 5.68402 | 2.52022 |
| 1.73 | 2.27307 | 5.73340 | 2.52231 |
| 1.74 | 2.29097 | 5.78322 | 2.52436 |
| 1.75 | 2.30904 | 5.83349 | 2.52637 |
| 1.76 | 2.32730 | 5.88421 | 2.52834 |
| 1.77 | 2.34574 | 5.93539 | 2.53028 |
| 1.78 | 2.36437 | 5.98704 | 2.53219 |
| 1.79 | 2.38319 | 6.03914 | 2.53406 |
| 1.80 | 2.40220 | 6.09172 | 2.53589 |
| 1.81 | 2.42140 | 6.14478 | 2.53769 |
| 1.82 | 2.44080 | 6.19831 | 2.53946 |
| 1.83 | 2.46039 | 6.25233 | 2.54120 |
| 1.84 | 2.48018 | 6.30684 | 2.54290 |
| 1.85 | 2.50016 | 6.36185 | 2.54457 |
| 1.86 | 2.52035 | 6.41736 | 2.54622 |
| 1.87 | 2.54074 | 6.47337 | 2.54783 |
| 1.88 | 2.56133 | 6.52989 | 2.54941 |
| 1.89 | 2.58213 | 6.58693 | 2.55096 |
| 1.90 | 2.60314 | 6.64449 | 2.55249 |
| 1.91 | 2.62436 | 6.70258 | 2.55398 |
| 1.92 | 2.64579 | 6.76119 | 2.55545 |
| 1.93 | 2.66743 | 6.82035 | 2.55690 |
| 1.94 | 2.68929 | 6.88004 | 2.55831 |
| 1.95 | 2.71139 | 6.94029 | 2.55970 |
| 1.96 | 2.73366 | 7.00108 | 2.56106 |
| 1.97 | 2.75618 | 7.06244 | 2.56240 |
| 1.98 | 2.77892 | 7.12436 | 2.56371 |
| 1.99 | 2.80189 | 7.18684 | 2.56500 |
| 2.00 | 2.82508 | 7.24991 | 2.56627 |
| 2.0 | 2.82508 | 7.24991 | 2.56627 |
| 2.1 | 3.06999 | 7.91347 | 2.57769 |
| 2.2 | 3.34005 | 8.64118 | 2.58714 |
| 2.3 | 3.63768 | 9.43961 | 2.59496 |
| 2.4 | 3.96553 | 10.31597 | 2.60141 |
| 2.5 | 4.32655 | 11.27817 | 2.60673 |
| 2.6 | 4.72398 | 12.33490 | 2.61112 |
| 2.7 | 5.16140 | 13.49570 | 2.61474 |
| 2.8 | 5.64274 | 14.77107 | 2.61771 |
| 2.9 | 6.17235 | 16.17255 | 2.62016 |
| 3.0 | 6.75502 | 17.71284 | 2.62218 |
| 3.1 | 7.39603 | 19.40592 | 2.62383 |
| 3.2 | 8.10119 | 21.26717 | 2.62519 |
| 3.3 | 8.87692 | 23.31353 | 2.62631 |
| 3.4 | 9.73028 | 25.56363 | 2.62722 |
| 3.5 | 10.66904 | 28.03801 | 2.62798 |
| 3.6 | 11.70179 | 30.75927 | 2.62860 |
| 3.7 | 12.83796 | 33.75231 | 2.62910 |
| 3.8 | 14.08796 | 37.04456 | 2.62952 |
| 3.9 | 15.46326 | 40.66622 | 2.62986 |
| 4.0 | 16.97650 | 44.65057 | 2.63014 |
| 4.1 | 18.64158 | 49.03427 | 2.63037 |
| 4.2 | 20.47386 | 53.85769 | 2.63056 |
| 4.3 | 22.49022 | 59.16532 | 2.63071 |
| 4.4 | 24.70929 | 65.00619 | 2.63084 |
| 4.5 | 27.15159 | 71.43430 | 2.63094 |
| 4.6 | 29.83973 | 78.50918 | 2.63103 |
| 4.7 | 32.79861 | 86.29638 | 2.63110 |
| 4.8 | 36.05571 | 94.86817 | 2.63116 |
| 4.9 | 39.64127 | 104.30420 | 2.63120 |
| 5.0 | 43.58864 | 114.69220 | 2.63124 |
| 5.1 | 47.93457 | 126.12890 | 2.63127 |
| 5.2 | 52.71956 | 138.72088 | 2.63130 |
| 5.3 | 57.98826 | 152.58561 | 2.63132 |
| 5.4 | 63.78986 | 167.85258 | 2.63134 |
| 5.5 | 70.17859 | 184.66448 | 2.63135 |
| 5.6 | 77.21423 | 203.17861 | 2.63136 |
| 5.7 | 84.96267 | 223.56837 | 2.63137 |
| 5.8 | 93.49653 | 246.02487 | 2.63138 |
| 5.9 | 102.89588 | 270.75878 | 2.63139 |
| 6.0 | 113.24897 | 298.00235 | 2.63139 |
| 6.0 | 113.24897 | 298.00235 | 2.63139 |
| 6.1 | 124.65310 | 328.01160 | 2.63140 |
| 6.2 | 137.21551 | 361.06677 | 2.63140 |
| 6.3 | 151.05446 | 397.48501 | 2.63140 |
| 6.4 | 166.30031 | 437.60336 | 2.63140 |
| 6.5 | 183.09678 | 481.80202 | 2.63141 |
| 6.6 | 201.60233 | 530.49796 | 2.63141 |
| 6.7 | 221.99168 | 584.15092 | 2.63141 |
| 6.8 | 244.45744 | 643.26779 | 2.63141 |
| 6.9 | 269.21201 | 708.40747 | 2.63141 |
| 7.0 | 296.48960 | 780.18621 | 2.63141 |
| 7.1 | 326.54842 | 859.28354 | 2.63141 |
| 7.2 | 359.67322 | 946.44874 | 2.63141 |
| 7.3 | 396.17800 | 1042.50804 | 2.63141 |
| 7.4 | 436.40900 | 1148.37254 | 2.63141 |
| 7.5 | 480.74802 | 1265.04698 | 2.63141 |
| 7.6 | 529.61612 | 1393.63930 | 2.63141 |
| 7.7 | 583.47764 | 1535.37136 | 2.63141 |
| 7.8 | 642.84465 | 1691.59063 | 2.63141 |
| 7.9 | 708.28186 | 1863.78312 | 2.63141 |
| 8.0 | 780.41208 | 2053.58771 | 2.63141 |
| 8.1 | 859.92219 | 2262.81185 | 2.63141 |
| 8.2 | 947.56974 | 2493.44896 | 2.63141 |
| 8.3 | 1044.19024 | 2747.69760 | 2.63141 |
| 8.4 | 1150.70518 | 3027.98264 | 2.63141 |
| 8.5 | 1268.13095 | 3336.97859 | 2.63141 |
| 8.6 | 1397.58856 | 3677.63533 | 2.63141 |
| 8.7 | 1540.31451 | 4053.20654 | 2.63141 |
| 8.8 | 1697.67262 | 4467.28105 | 2.63141 |
| 8.9 | 1871.16724 | 4923.81741 | 2.63141 |
| 9.0 | 2062.45773 | 5427.18208 | 2.63141 |
| 9.1 | 2273.37443 | 5982.19147 | 2.63141 |
| 9.2 | 2505.93638 | 6594.15848 | 2.63141 |
| 9.3 | 2762.37073 | 7268.94369 | 2.63141 |
| 9.4 | 3045.13429 | 8013.01197 | 2.63141 |
| 9.5 | 3356.93721 | 8833.49486 | 2.63141 |
| 9.6 | 3700.76916 | 9738.25943 | 2.63141 |
| 9.7 | 4079.92823 | 10735.98430 | 2.63141 |
| 9.8 | 4498.05276 | 11836.24347 | 2.63141 |
| 9.9 | 4959.15653 | 13049.59883 | 2.63141 |
| 10.0 | 5467.66752 | 14387.70229 | 2.63141 |

TABLE D.XIXB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 8/11$ and $x$ from 1.50 to 10.0.

$\alpha = 5/13$

| x | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00007 | 0.00239 | 0.00239 |
| 0.02 | 1.00026 | 0.00561 | 0.00561 |
| 0.03 | 1.00059 | 0.00925 | 0.00924 |
| 0.04 | 1.00104 | 0.01318 | 0.01317 |
| 0.05 | 1.00163 | 0.01735 | 0.01732 |
| 0.06 | 1.00234 | 0.02172 | 0.02167 |
| 0.07 | 1.00319 | 0.02626 | 0.02617 |
| 0.08 | 1.00416 | 0.03096 | 0.03083 |
| 0.09 | 1.00527 | 0.03579 | 0.03561 |
| 0.10 | 1.00651 | 0.04076 | 0.04050 |
| 0.11 | 1.00787 | 0.04585 | 0.04549 |
| 0.12 | 1.00937 | 0.05105 | 0.05058 |
| 0.13 | 1.01100 | 0.05636 | 0.05575 |
| 0.14 | 1.01276 | 0.06177 | 0.06099 |
| 0.15 | 1.01465 | 0.06727 | 0.06630 |
| 0.16 | 1.01668 | 0.07287 | 0.07167 |
| 0.17 | 1.01883 | 0.07855 | 0.07710 |
| 0.18 | 1.02112 | 0.08432 | 0.08258 |
| 0.19 | 1.02354 | 0.09018 | 0.08810 |
| 0.20 | 1.02609 | 0.09611 | 0.09367 |
| 0.21 | 1.02878 | 0.10212 | 0.09927 |
| 0.22 | 1.03160 | 0.10821 | 0.10490 |
| 0.23 | 1.03455 | 0.11438 | 0.11056 |
| 0.24 | 1.03764 | 0.12062 | 0.11624 |
| 0.25 | 1.04085 | 0.12693 | 0.12194 |
| 0.26 | 1.04421 | 0.13331 | 0.12766 |
| 0.27 | 1.04770 | 0.13976 | 0.13340 |
| 0.28 | 1.05132 | 0.14628 | 0.13914 |
| 0.29 | 1.05508 | 0.15287 | 0.14489 |
| 0.30 | 1.05898 | 0.15953 | 0.15065 |
| 0.31 | 1.06301 | 0.16626 | 0.15640 |
| 0.32 | 1.06718 | 0.17305 | 0.16216 |
| 0.33 | 1.07148 | 0.17991 | 0.16791 |
| 0.34 | 1.07593 | 0.18683 | 0.17365 |
| 0.35 | 1.08051 | 0.19383 | 0.17938 |
| 0.36 | 1.08523 | 0.20088 | 0.18511 |
| 0.37 | 1.09009 | 0.20801 | 0.19082 |
| 0.38 | 1.09509 | 0.21519 | 0.19651 |
| 0.39 | 1.10023 | 0.22245 | 0.20218 |
| 0.40 | 1.10551 | 0.22977 | 0.20784 |
| 0.41 | 1.11093 | 0.23715 | 0.21347 |
| 0.42 | 1.11650 | 0.24460 | 0.21908 |
| 0.43 | 1.12220 | 0.25212 | 0.22466 |
| 0.44 | 1.12805 | 0.25970 | 0.23022 |
| 0.45 | 1.13405 | 0.26734 | 0.23574 |
| 0.46 | 1.14019 | 0.27506 | 0.24124 |
| 0.47 | 1.14647 | 0.28284 | 0.24670 |
| 0.48 | 1.15290 | 0.29068 | 0.25213 |
| 0.49 | 1.15948 | 0.29859 | 0.25753 |
| 0.50 | 1.16620 | 0.30658 | 0.26288 |

| x | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 0.50 | 1.16620 | 0.30658 | 0.26288 |
| 0.51 | 1.17307 | 0.31462 | 0.26820 |
| 0.52 | 1.18009 | 0.32274 | 0.27349 |
| 0.53 | 1.18726 | 0.33092 | 0.27873 |
| 0.54 | 1.19458 | 0.33918 | 0.28393 |
| 0.55 | 1.20205 | 0.34750 | 0.28909 |
| 0.56 | 1.20967 | 0.35589 | 0.29420 |
| 0.57 | 1.21745 | 0.36435 | 0.29928 |
| 0.58 | 1.22538 | 0.37289 | 0.30430 |
| 0.59 | 1.23346 | 0.38149 | 0.30929 |
| 0.60 | 1.24170 | 0.39017 | 0.31422 |
| 0.61 | 1.25010 | 0.39892 | 0.31911 |
| 0.62 | 1.25865 | 0.40774 | 0.32395 |
| 0.63 | 1.26736 | 0.41664 | 0.32875 |
| 0.64 | 1.27623 | 0.42561 | 0.33349 |
| 0.65 | 1.28526 | 0.43466 | 0.33819 |
| 0.66 | 1.29445 | 0.44378 | 0.34283 |
| 0.67 | 1.30380 | 0.45297 | 0.34743 |
| 0.68 | 1.31331 | 0.46225 | 0.35197 |
| 0.69 | 1.32299 | 0.47160 | 0.35647 |
| 0.70 | 1.33283 | 0.48103 | 0.36091 |
| 0.71 | 1.34284 | 0.49054 | 0.36530 |
| 0.72 | 1.35302 | 0.50014 | 0.36964 |
| 0.73 | 1.36336 | 0.50981 | 0.37393 |
| 0.74 | 1.37388 | 0.51956 | 0.37817 |
| 0.75 | 1.38456 | 0.52940 | 0.38236 |
| 0.76 | 1.39542 | 0.53932 | 0.38649 |
| 0.77 | 1.40645 | [illegible] | 0.39057 |
| 0.78 | 1.41765 | [illegible] | 0.39460 |
| 0.79 | 1.42903 | [illegible] | 0.39858 |
| 0.80 | 1.44058 | 0.57984 | 0.40251 |
| 0.81 | 1.45231 | 0.59019 | 0.40638 |
| 0.82 | 1.46423 | 0.60063 | 0.41020 |
| 0.83 | 1.47631 | 0.61116 | 0.41398 |
| 0.84 | 1.48859 | 0.62177 | 0.41770 |
| 0.85 | 1.50104 | 0.63248 | 0.42136 |
| 0.86 | 1.51368 | 0.64329 | 0.42498 |
| 0.87 | 1.52650 | 0.65418 | 0.42855 |
| 0.88 | 1.53952 | 0.66517 | 0.43206 |
| 0.89 | 1.55272 | 0.67625 | 0.43553 |
| 0.90 | 1.56611 | 0.68744 | 0.43895 |
| 0.91 | 1.57969 | 0.69872 | 0.44231 |
| 0.92 | 1.59346 | 0.71009 | 0.44563 |
| 0.93 | 1.60743 | 0.72157 | 0.44890 |
| 0.94 | 1.62159 | 0.73315 | 0.45212 |
| 0.95 | 1.63595 | 0.74483 | 0.45529 |
| 0.96 | 1.65051 | 0.75661 | 0.45841 |
| 0.97 | 1.66527 | 0.76850 | 0.46148 |
| 0.98 | 1.68023 | 0.78049 | 0.46451 |
| 0.99 | 1.69540 | 0.79259 | 0.46749 |
| 1.00 | 1.71077 | 0.80479 | 0.47043 |

| x | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 1.00 | 1.71077 | 0.80479 | 0.47043 |
| 1.01 | 1.72635 | 0.81711 | 0.47332 |
| 1.02 | 1.74213 | 0.82953 | 0.47616 |
| 1.03 | 1.75813 | 0.84207 | 0.47896 |
| 1.04 | 1.77434 | 0.85472 | 0.48171 |
| 1.05 | 1.79076 | 0.86748 | 0.48442 |
| 1.06 | 1.80739 | 0.88036 | 0.48709 |
| 1.07 | 1.82425 | 0.89335 | 0.48971 |
| 1.08 | 1.84132 | 0.90646 | 0.49229 |
| 1.09 | 1.85861 | 0.91969 | 0.49483 |
| 1.10 | 1.87613 | 0.93304 | 0.49732 |
| 1.11 | 1.89387 | 0.94651 | 0.49978 |
| 1.12 | 1.91184 | 0.96011 | 0.50219 |
| 1.13 | 1.93003 | 0.97382 | 0.50456 |
| 1.14 | 1.94846 | 0.98767 | 0.50690 |
| 1.15 | 1.96712 | 1.00164 | 0.50919 |
| 1.16 | 1.98601 | 1.01574 | 0.51145 |
| 1.17 | 2.00514 | 1.02997 | 0.51366 |
| 1.18 | 2.02451 | 1.04433 | 0.51584 |
| 1.19 | 2.04412 | 1.05882 | 0.51798 |
| 1.20 | 2.06397 | 1.07345 | 0.52009 |
| 1.21 | 2.08406 | 1.08821 | 0.52216 |
| 1.22 | 2.10441 | 1.10311 | 0.52419 |
| 1.23 | 2.12500 | 1.11814 | 0.52619 |
| 1.24 | 2.14584 | 1.13332 | 0.52815 |
| 1.25 | 2.16694 | 1.14364 | 0.53008 |
| 1.26 | 2.18829 | 1.16411 | 0.53197 |
| 1.27 | 2.20990 | 1.17971 | 0.53383 |
| 1.28 | 2.23178 | 1.19547 | 0.53566 |
| 1.29 | 2.25391 | 1.21137 | 0.53745 |
| 1.30 | 2.27631 | 1.22742 | 0.53922 |
| 1.31 | 2.29898 | 1.24363 | 0.54095 |
| 1.32 | 2.32192 | 1.25998 | 0.54265 |
| 1.33 | 2.34513 | 1.27649 | 0.54432 |
| 1.34 | 2.36861 | 1.29316 | 0.54596 |
| 1.35 | 2.39237 | 1.30999 | 0.54757 |
| 1.36 | 2.41642 | 1.32697 | 0.54915 |
| 1.37 | 2.44074 | 1.34412 | 0.55070 |
| 1.38 | 2.46535 | 1.36143 | 0.55223 |
| 1.39 | 2.49025 | 1.37891 | 0.55372 |
| 1.40 | 2.51544 | 1.39655 | 0.55519 |
| 1.41 | 2.54092 | 1.41436 | 0.55663 |
| 1.42 | 2.56670 | 1.43235 | 0.55805 |
| 1.43 | 2.59278 | 1.45050 | 0.55944 |
| 1.44 | 2.61916 | 1.46883 | 0.56080 |
| 1.45 | 2.64584 | 1.48734 | 0.56214 |
| 1.46 | 2.67283 | 1.50602 | 0.56346 |
| 1.47 | 2.70013 | 1.52489 | 0.56475 |
| 1.48 | 2.72775 | 1.54393 | 0.56601 |
| 1.49 | 2.75568 | 1.56317 | 0.56725 |
| 1.50 | 2.78392 | 1.58258 | 0.56847 |

TABLE D.XXA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/13$ and $x$ from 0.00 to 1.50.

$\alpha = 5/13$

| $x$ | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 1.50 | 2.78392 | 1.58256 | 0.56847 |
| 1.51 | 2.81250 | 1.60219 | 0.56967 |
| 1.52 | 2.84139 | 1.62199 | 0.57084 |
| 1.53 | 2.87062 | 1.64197 | 0.57199 |
| 1.54 | 2.90017 | 1.66216 | 0.57312 |
| 1.55 | 2.93006 | 1.68254 | 0.57423 |
| 1.56 | 2.96029 | 1.70311 | 0.57532 |
| 1.57 | 2.99086 | 1.72389 | 0.57639 |
| 1.58 | 3.02177 | 1.74488 | 0.57743 |
| 1.59 | 3.05304 | 1.76606 | 0.57846 |
| 1.60 | 3.08465 | 1.78746 | 0.57947 |
| 1.61 | 3.11661 | 1.80906 | 0.58046 |
| 1.62 | 3.14894 | 1.83088 | 0.58143 |
| 1.63 | 3.18162 | 1.85291 | 0.58238 |
| 1.64 | 3.21467 | 1.87516 | 0.58331 |
| 1.65 | 3.24809 | 1.89763 | 0.58423 |
| 1.66 | 3.28188 | 1.92032 | 0.58513 |
| 1.67 | 3.31605 | 1.94323 | 0.58601 |
| 1.68 | 3.35059 | 1.96637 | 0.58687 |
| 1.69 | 3.38552 | 1.98973 | 0.58772 |
| 1.70 | 3.42083 | 2.01332 | 0.58855 |
| 1.71 | 3.45654 | 2.03716 | 0.58936 |
| 1.72 | 3.49264 | 2.06123 | 0.59016 |
| 1.73 | 3.52713 | 2.06554 | 0.59095 |
| 1.74 | 3.56333 | 2.11008 | 0.59172 |
| 1.75 | 3.60334 | 2.13487 | 0.59247 |
| 1.76 | 3.64105 | 2.15991 | 0.59321 |
| 1.77 | 3.67918 | 2.18520 | 0.59394 |
| 1.78 | 3.71772 | 2.21074 | 0.59465 |
| 1.79 | 3.75669 | 2.23653 | 0.59535 |
| 1.80 | 3.79608 | 2.26258 | 0.59603 |
| 1.81 | 3.83591 | 2.28889 | 0.59670 |
| 1.82 | 3.87617 | 2.31546 | 0.59736 |
| 1.83 | 3.91686 | 2.34229 | 0.59800 |
| 1.84 | 3.95801 | 2.36940 | 0.59863 |
| 1.85 | 3.99959 | 2.39678 | 0.59925 |
| 1.86 | 4.04163 | 2.42443 | 0.59986 |
| 1.87 | 4.08413 | 2.45235 | 0.60046 |
| 1.88 | 4.12709 | 2.48056 | 0.60104 |
| 1.89 | 4.17052 | 2.50905 | 0.60162 |
| 1.90 | 4.21441 | 2.53783 | 0.60218 |
| 1.91 | 4.25878 | 2.56689 | 0.60273 |
| 1.92 | 4.30363 | 2.59625 | 0.60327 |
| 1.93 | 4.34896 | 2.62590 | 0.60380 |
| 1.94 | 4.39479 | 2.65585 | 0.60432 |
| 1.95 | 4.44110 | 2.68610 | 0.60483 |
| 1.96 | 4.48792 | 2.71464 | 0.60533 |
| 1.97 | 4.53524 | 2.74732 | 0.60582 |
| 1.98 | 4.58307 | 2.77870 | 0.60630 |
| 1.99 | 4.63142 | 2.81019 | 0.60677 |
| 2.00 | 4.68028 | 2.84199 | 0.60723 |

| $x$ | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 2.0 | 4.68028 | 2.84199 | 0.60723 |
| 2.1 | 5.19867 | 3.17830 | 0.61137 |
| 2.2 | 5.77510 | 3.55030 | 0.61476 |
| 2.3 | 6.41571 | 3.96194 | 0.61754 |
| 2.4 | 7.12731 | 4.41759 | 0.61981 |
| 2.5 | 7.91747 | 4.92206 | 0.62167 |
| 2.6 | 8.79455 | 5.48070 | 0.62319 |
| 2.7 | 9.76786 | 6.09941 | 0.62444 |
| 2.8 | 10.84770 | 6.78474 | 0.62545 |
| 2.9 | 12.04548 | 7.54392 | 0.62629 |
| 3.0 | 13.37387 | 8.38497 | 0.62697 |
| 3.1 | 14.84689 | 9.31675 | 0.62752 |
| 3.2 | 16.48004 | 10.34909 | 0.62798 |
| 3.3 | 18.29055 | 11.49285 | 0.62835 |
| 3.4 | 20.29746 | 12.76006 | 0.62865 |
| 3.5 | 22.52187 | 14.16403 | 0.62890 |
| 3.6 | 24.98714 | 15.71952 | 0.62910 |
| 3.7 | 27.71914 | 17.44284 | 0.62927 |
| 3.8 | 30.74651 | 19.35205 | 0.62941 |
| 3.9 | 34.10098 | 21.46717 | 0.62952 |
| 4.0 | 37.81766 | 23.81032 | 0.62961 |
| 4.1 | 41.93541 | 26.40601 | 0.62968 |
| 4.2 | 46.49729 | 29.28137 | 0.62974 |
| 4.3 | 51.55094 | 32.46644 | 0.62979 |
| 4.4 | 57.14910 | 35.99445 | 0.62983 |
| 4.5 | 63.35016 | 39.90219 | 0.62987 |
| 4.6 | 70.21874 | 44.23040 | 0.62989 |
| 4.7 | 77.82641 | 49.02416 | 0.62992 |
| 4.8 | 86.25233 | 54.33336 | 0.62994 |
| 4.9 | 95.58417 | 60.21324 | 0.62995 |
| 5.0 | 105.91894 | 66.72492 | 0.62996 |
| 5.1 | 117.36404 | 73.93607 | 0.62997 |
| 5.2 | 130.03830 | 81.92156 | 0.62998 |
| 5.3 | 144.07328 | 90.76428 | 0.62999 |
| 5.4 | 159.61456 | 100.55596 | 0.62999 |
| 5.5 | 176.82328 | 111.39811 | 0.63000 |
| 5.6 | 195.87776 | 123.40309 | 0.63000 |
| 5.7 | 216.97536 | 136.69524 | 0.63000 |
| 5.8 | 240.33449 | 151.41217 | 0.63001 |
| 5.9 | 268.19687 | 167.70615 | 0.63001 |
| 6.0 | 294.82997 | 185.74572 | 0.63001 |

| $x$ | $F_{5/13}(x)$ | $H_{8/13}(x)$ | $T_{5/13}(x)$ |
|---|---|---|---|
| 6.0 | 294.82997 | 185.74572 | 0.63001 |
| 6.1 | 326.52983 | 205.71737 | 0.63001 |
| 6.2 | 361.62402 | 227.82750 | 0.63001 |
| 6.3 | 400.47505 | 252.30447 | 0.63001 |
| 6.4 | 443.48405 | 279.40103 | 0.63001 |
| 6.5 | 491.09493 | 309.39683 | 0.63001 |
| 6.6 | 543.79888 | 342.60134 | 0.63001 |
| 6.7 | 602.13944 | 379.35700 | 0.63002 |
| 6.8 | 666.71805 | 420.04272 | 0.63002 |
| 6.9 | 738.20021 | 465.07779 | 0.63002 |
| 7.0 | 817.32232 | 514.92615 | 0.63002 |
| 7.1 | 904.89918 | 570.10114 | 0.63002 |
| 7.2 | 1001.83236 | 631.17076 | 0.63002 |
| 7.3 | 1109.11941 | 698.26349 | 0.63002 |
| 7.4 | 1227.86410 | 773.57472 | 0.63002 |
| 7.5 | 1359.28765 | 856.37385 | 0.63002 |
| 7.6 | 1504.74128 | 948.01216 | 0.63002 |
| 7.7 | 1665.71999 | 1049.43153 | 0.63002 |
| 7.8 | 1843.87788 | 1161.67407 | 0.63002 |
| 7.9 | 2041.04502 | 1285.89276 | 0.63002 |
| 8.0 | 2259.24617 | 1423.36323 | 0.63002 |
| 8.1 | 2500.72147 | 1575.49680 | 0.63002 |
| 8.2 | 2767.94933 | 1743.85493 | 0.63002 |
| 8.3 | 3063.67175 | 1930.16511 | 0.63002 |
| 8.4 | 3390.92229 | 2136.33854 | 0.63002 |
| 8.5 | 3753.05709 | 2364.48964 | 0.63002 |
| 8.6 | 4153.78911 | 2616.95765 | 0.63002 |
| 8.7 | 4597.22603 | 2896.33048 | 0.63002 |
| 8.8 | 5087.91219 | 3205.47112 | 0.63002 |
| 8.9 | 5630.87494 | 3547.54690 | 0.63002 |
| 9.0 | 6231.67594 | 3926.06177 | 0.63002 |
| 9.1 | 6896.46790 | 4344.89202 | 0.63002 |
| 9.2 | 7632.05733 | 4808.32589 | 0.63002 |
| 9.3 | 8445.97396 | 5321.10723 | 0.63002 |
| 9.4 | 9346.54752 | 5888.48390 | 0.63002 |
| 9.5 | 10342.99263 | 6516.26128 | 0.63002 |
| 9.6 | 11445.50277 | 7210.86144 | 0.63002 |
| 9.7 | 12665.35416 | 7979.38858 | 0.63002 |
| 9.8 | 14015.02067 | 8829.70145 | 0.63002 |
| 9.9 | 15508.30090 | 9770.49341 | 0.63002 |
| 10.0 | 17160.45881 | 10811.38101 | 0.63002 |

TABLE D.XXB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/13$ and $x$ from 1.50 to 10.0.

$\alpha = 8/13$

| x | $F_{8/13}(x)$ | $H_{5/13}(x)$ | $T_{8/13}(x)$ |
|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 |
| 0.01 | 1.00004 | 0.04415 | 0.04415 |
| 0.02 | 1.00016 | 0.07526 | 0.07524 |
| 0.03 | 1.00037 | 0.10281 | 0.10277 |
| 0.04 | 1.00065 | 0.12829 | 0.12821 |
| 0.05 | 1.00102 | 0.15234 | 0.15219 |
| 0.06 | 1.00146 | 0.17531 | 0.17506 |
| 0.07 | 1.00199 | 0.19743 | 0.19704 |
| 0.08 | 1.00260 | 0.21884 | 0.21828 |
| 0.09 | 1.00329 | 0.23967 | 0.23889 |
| 0.10 | 1.00407 | 0.25999 | 0.25894 |
| 0.11 | 1.00492 | 0.27988 | 0.27851 |
| 0.12 | 1.00586 | 0.29938 | 0.29763 |
| 0.13 | 1.00687 | 0.31853 | 0.31636 |
| 0.14 | 1.00797 | 0.33738 | 0.33471 |
| 0.15 | 1.00916 | 0.35596 | 0.35273 |
| 0.16 | 1.01042 | 0.37428 | 0.37042 |
| 0.17 | 1.01177 | 0.39239 | 0.38782 |
| 0.18 | 1.01320 | 0.41028 | 0.40494 |
| 0.19 | 1.01471 | 0.42799 | 0.42179 |
| 0.20 | 1.01630 | 0.44553 | 0.43838 |
| 0.21 | 1.01798 | 0.46291 | 0.45473 |
| 0.22 | 1.01974 | 0.48014 | 0.47085 |
| 0.23 | 1.02158 | 0.49725 | 0.48674 |
| 0.24 | 1.02350 | 0.51423 | 0.50242 |
| 0.25 | 1.02551 | 0.53110 | 0.51789 |
| 0.26 | 1.02761 | 0.54787 | 0.53315 |
| 0.27 | 1.02978 | 0.56455 | 0.54822 |
| 0.28 | 1.03204 | 0.58114 | 0.56309 |
| 0.29 | 1.03439 | 0.59765 | 0.57778 |
| 0.30 | 1.03682 | 0.61409 | 0.59228 |
| 0.31 | 1.03933 | 0.63046 | 0.60660 |
| 0.32 | 1.04193 | 0.64678 | 0.62075 |
| 0.33 | 1.04461 | 0.66304 | 0.63473 |
| 0.34 | 1.04738 | 0.67926 | 0.64853 |
| 0.35 | 1.05024 | 0.69543 | 0.66217 |
| 0.36 | 1.05318 | 0.71157 | 0.67564 |
| 0.37 | 1.05621 | 0.72768 | 0.68895 |
| 0.38 | 1.05932 | 0.74376 | 0.70211 |
| 0.39 | 1.06252 | 0.75981 | 0.71510 |
| 0.40 | 1.06581 | 0.77585 | 0.72794 |
| 0.41 | 1.06918 | 0.79187 | 0.74063 |
| 0.42 | 1.07265 | 0.80788 | 0.75316 |
| 0.43 | 1.07620 | 0.82388 | 0.76555 |
| 0.44 | 1.07984 | 0.83988 | 0.77778 |
| 0.45 | 1.08356 | 0.85588 | 0.78987 |
| 0.46 | 1.08738 | 0.87188 | 0.80182 |
| 0.47 | 1.09129 | 0.88789 | 0.81362 |
| 0.48 | 1.09528 | 0.90391 | 0.82528 |
| 0.49 | 1.09937 | 0.91994 | 0.83679 |
| 0.50 | 1.10354 | 0.93599 | 0.84817 |

| x | $F_{8/13}(x)$ | $H_{5/13}(x)$ | $T_{8/13}(x)$ |
|---|---|---|---|
| 0.50 | 1.10354 | 0.93599 | 0.84817 |
| 0.51 | 1.10781 | 0.95206 | 0.85941 |
| 0.52 | 1.11217 | 0.96815 | 0.87051 |
| 0.53 | 1.11662 | 0.98427 | 0.88147 |
| 0.54 | 1.12116 | 1.00041 | 0.89230 |
| 0.55 | 1.12580 | 1.01659 | 0.90300 |
| 0.56 | 1.13052 | 1.03280 | 0.91356 |
| 0.57 | 1.13534 | 1.04904 | 0.92399 |
| 0.58 | 1.14026 | 1.06533 | 0.93429 |
| 0.59 | 1.14527 | 1.08166 | 0.94446 |
| 0.60 | 1.15037 | 1.09803 | 0.95450 |
| 0.61 | 1.15557 | 1.11444 | 0.96441 |
| 0.62 | 1.16086 | 1.13091 | 0.97420 |
| 0.63 | 1.16626 | 1.14743 | 0.98386 |
| 0.64 | 1.17174 | 1.16400 | 0.99339 |
| 0.65 | 1.17733 | 1.18063 | 1.00281 |
| 0.66 | 1.18301 | 1.19732 | 1.01210 |
| 0.67 | 1.18879 | 1.21407 | 1.02126 |
| 0.68 | 1.19467 | 1.23088 | 1.03031 |
| 0.69 | 1.20065 | 1.24776 | 1.03924 |
| 0.70 | 1.20673 | 1.26471 | 1.04805 |
| 0.71 | 1.21291 | 1.28173 | 1.05674 |
| 0.72 | 1.21919 | 1.29882 | 1.06531 |
| 0.73 | 1.22557 | 1.31598 | 1.07377 |
| 0.74 | 1.23206 | 1.33323 | 1.08211 |
| 0.75 | 1.23864 | 1.35055 | 1.09034 |
| 0.76 | 1.24533 | 1.36795 | 1.09846 |
| 0.77 | 1.25213 | 1.38544 | 1.10647 |
| 0.78 | 1.25903 | 1.40301 | 1.11436 |
| 0.79 | 1.26603 | 1.42067 | 1.12215 |
| 0.80 | 1.27314 | 1.43843 | 1.12982 |
| 0.81 | 1.28036 | 1.45627 | 1.13739 |
| 0.82 | 1.28768 | 1.47421 | 1.14485 |
| 0.83 | 1.29512 | 1.49225 | 1.15221 |
| 0.84 | 1.30266 | 1.51038 | 1.15946 |
| 0.85 | 1.31031 | 1.52862 | 1.16661 |
| 0.86 | 1.31807 | 1.54696 | 1.17365 |
| 0.87 | 1.32594 | 1.56540 | 1.18060 |
| 0.88 | 1.33392 | 1.58395 | 1.18744 |
| 0.89 | 1.34202 | 1.60261 | 1.19418 |
| 0.90 | 1.35023 | 1.62139 | 1.20083 |
| 0.91 | 1.35855 | 1.64027 | 1.20737 |
| 0.92 | 1.36699 | 1.65928 | 1.21382 |
| 0.93 | 1.37554 | 1.67840 | 1.22018 |
| 0.94 | 1.38421 | 1.69764 | 1.22643 |
| 0.95 | 1.39299 | 1.71700 | 1.23260 |
| 0.96 | 1.40190 | 1.73649 | 1.23867 |
| 0.97 | 1.41092 | 1.75611 | 1.24466 |
| 0.98 | 1.42006 | 1.77585 | 1.25055 |
| 0.99 | 1.42932 | 1.79573 | 1.25635 |
| 1.00 | 1.43871 | 1.81573 | 1.26206 |

| x | $F_{8/13}(x)$ | $H_{5/13}(x)$ | $T_{8/13}(x)$ |
|---|---|---|---|
| 1.00 | 1.43871 | 1.81573 | 1.26206 |
| 1.01 | 1.44821 | 1.83588 | 1.26769 |
| 1.02 | 1.45784 | 1.85616 | 1.27323 |
| 1.03 | 1.46759 | 1.87658 | 1.27868 |
| 1.04 | 1.47747 | 1.89714 | 1.28405 |
| 1.05 | 1.48747 | 1.91785 | 1.28933 |
| 1.06 | 1.49760 | 1.93870 | 1.29454 |
| 1.07 | 1.50786 | 1.95970 | 1.29966 |
| 1.08 | 1.51824 | 1.98085 | 1.30470 |
| 1.09 | 1.52875 | 2.00215 | 1.30966 |
| 1.10 | 1.53940 | 2.02361 | 1.31455 |
| 1.11 | 1.55018 | 2.04523 | 1.31935 |
| 1.12 | 1.56109 | 2.06700 | 1.32408 |
| 1.13 | 1.57213 | 2.08894 | 1.32873 |
| 1.14 | 1.58331 | 2.11104 | 1.33331 |
| 1.15 | 1.59462 | 2.13331 | 1.33782 |
| 1.16 | 1.60607 | 2.15574 | 1.34225 |
| 1.17 | 1.61765 | 2.17835 | 1.34661 |
| 1.18 | 1.62938 | 2.20113 | 1.35090 |
| 1.19 | 1.64125 | 2.22408 | 1.35512 |
| 1.20 | 1.65325 | 2.24721 | 1.35927 |
| 1.21 | 1.66540 | 2.27053 | 1.36335 |
| 1.22 | 1.67769 | 2.29402 | 1.36737 |
| 1.23 | 1.69013 | 2.31770 | 1.37132 |
| 1.24 | 1.70271 | 2.34157 | 1.37520 |
| 1.25 | 1.71544 | 2.36562 | 1.37902 |
| 1.26 | 1.72832 | 2.38987 | 1.38278 |
| 1.27 | 1.74134 | 2.41432 | 1.38647 |
| 1.28 | 1.75452 | 2.43895 | 1.39010 |
| 1.29 | 1.76785 | 2.46379 | 1.39367 |
| 1.30 | 1.78133 | 2.48883 | 1.39718 |
| 1.31 | 1.79496 | 2.51408 | 1.40063 |
| 1.32 | 1.80875 | 2.53953 | 1.40403 |
| 1.33 | 1.82270 | 2.56519 | 1.40736 |
| 1.34 | 1.83680 | 2.59106 | 1.41064 |
| 1.35 | 1.85106 | 2.61715 | 1.41386 |
| 1.36 | 1.86549 | 2.64346 | 1.41703 |
| 1.37 | 1.88007 | 2.66998 | 1.42015 |
| 1.38 | 1.89482 | 2.69672 | 1.42321 |
| 1.39 | 1.90974 | 2.72369 | 1.42621 |
| 1.40 | 1.92482 | 2.75089 | 1.42917 |
| 1.41 | 1.94007 | 2.77832 | 1.43208 |
| 1.42 | 1.95548 | 2.80598 | 1.43493 |
| 1.43 | 1.97107 | 2.83388 | 1.43774 |
| 1.44 | 1.98683 | 2.86201 | 1.44049 |
| 1.45 | 2.00276 | 2.89039 | 1.44320 |
| 1.46 | 2.01887 | 2.91901 | 1.44586 |
| 1.47 | 2.03515 | 2.94787 | 1.44848 |
| 1.48 | 2.05161 | 2.97699 | 1.45105 |
| 1.49 | 2.06825 | 3.00635 | 1.45357 |
| 1.50 | 2.08507 | 3.03597 | 1.45605 |

TABLE D.XXIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 8/13$ and x from 0.00 to 1.50.

$\alpha = 8/13$

| x | $F_{8/13}(x)$ | $H_{5/13}(x)$ | $T_{8/13}(x)$ |
|---|---|---|---|
| 1.50 | 2.08507 | 3.03597 | 1.45605 |
| 1.51 | 2.10208 | 3.06585 | 1.45849 |
| 1.52 | 2.11926 | 3.09599 | 1.46084 |
| 1.53 | 2.13664 | 3.12640 | 1.46323 |
| 1.54 | 2.15420 | 3.15707 | 1.46554 |
| 1.55 | 2.17195 | 3.18801 | 1.46781 |
| 1.56 | 2.18989 | 3.21922 | 1.47004 |
| 1.57 | 2.20802 | 3.25071 | 1.47223 |
| 1.58 | 2.22635 | 3.28248 | 1.47438 |
| 1.59 | 2.24487 | 3.31452 | 1.47649 |
| 1.60 | 2.26359 | 3.34686 | 1.47856 |
| 1.61 | 2.28251 | 3.37948 | 1.48060 |
| 1.62 | 2.30163 | 3.41239 | 1.48260 |
| 1.63 | 2.32095 | 3.44560 | 1.48456 |
| 1.64 | 2.34048 | 3.47910 | 1.48649 |
| 1.65 | 2.36021 | 3.51290 | 1.48839 |
| 1.66 | 2.38015 | 3.54701 | 1.49024 |
| 1.67 | 2.40031 | 3.58143 | 1.49207 |
| 1.68 | 2.42067 | 3.61615 | 1.49386 |
| 1.69 | 2.44125 | 3.65119 | 1.49562 |
| 1.70 | 2.46204 | 3.68655 | 1.49735 |
| 1.71 | 2.48305 | 3.72222 | 1.49905 |
| 1.72 | 2.50429 | 3.75822 | 1.50072 |
| 1.73 | 2.52574 | 3.79455 | 1.50235 |
| 1.74 | 2.54742 | 3.83121 | 1.50396 |
| 1.75 | 2.56932 | 3.86820 | 1.50554 |
| 1.76 | 2.59145 | 3.90553 | 1.50708 |
| 1.77 | 2.61381 | 3.94321 | 1.50860 |
| 1.78 | 2.63641 | 3.98122 | 1.51009 |
| 1.79 | 2.65923 | 4.01959 | 1.51156 |
| 1.80 | 2.68230 | 4.05831 | 1.51300 |
| 1.81 | 2.70560 | 4.09738 | 1.51441 |
| 1.82 | 2.72914 | 4.13681 | 1.51579 |
| 1.83 | 2.75293 | 4.17661 | 1.51715 |
| 1.84 | 2.77696 | 4.21677 | 1.51849 |
| 1.85 | 2.80124 | 4.25731 | 1.51980 |
| 1.86 | 2.82577 | 4.29822 | 1.52108 |
| 1.87 | 2.85055 | 4.33951 | 1.52234 |
| 1.88 | 2.87558 | 4.38118 | 1.52358 |
| 1.89 | 2.90087 | 4.42324 | 1.52480 |
| 1.90 | 2.92642 | 4.46569 | 1.52599 |
| 1.91 | 2.95223 | 4.50853 | 1.52716 |
| 1.92 | 2.97831 | 4.55177 | 1.52831 |
| 1.93 | 3.00465 | 4.59542 | 1.52944 |
| 1.94 | 3.03126 | 4.63947 | 1.53054 |
| 1.95 | 3.05814 | 4.68394 | 1.53163 |
| 1.96 | 3.08530 | 4.72882 | 1.53269 |
| 1.97 | 3.11273 | 4.77412 | 1.53374 |
| 1.98 | 3.14044 | 4.81984 | 1.53476 |
| 1.99 | 3.16843 | 4.86599 | 1.53577 |
| 2.00 | 3.19671 | 4.91258 | 1.53676 |
| 2.0 | 3.19671 | 4.91258 | 1.53676 |
| 2.1 | 3.49571 | 5.40323 | 1.54567 |
| 2.2 | 3.82624 | 5.94230 | 1.55304 |
| 2.3 | 4.19142 | 6.53491 | 1.55912 |
| 2.4 | 4.59469 | 7.18668 | 1.56413 |
| 2.5 | 5.03987 | 7.90380 | 1.56826 |
| 2.6 | 5.53117 | 8.69307 | 1.57165 |
| 2.7 | 6.07324 | 9.56199 | 1.57445 |
| 2.8 | 6.67122 | 10.51861 | 1.57674 |
| 2.9 | 7.33080 | 11.57263 | 1.57863 |
| 3.0 | 8.05822 | 12.73345 | 1.58018 |
| 3.1 | 8.86041 | 14.01234 | 1.58145 |
| 3.2 | 9.74501 | 15.42147 | 1.58250 |
| 3.3 | 10.72043 | 16.97426 | 1.58336 |
| 3.4 | 11.79598 | 18.68553 | 1.58406 |
| 3.5 | 12.98191 | 20.57160 | 1.58464 |
| 3.6 | 14.28955 | 22.65049 | 1.58511 |
| 3.7 | 15.73139 | 24.94208 | 1.58550 |
| 3.8 | 17.32123 | 27.46827 | 1.58582 |
| 3.9 | 19.07428 | 30.25325 | 1.58608 |
| 4.0 | 21.00732 | 33.32370 | 1.58629 |
| 4.1 | 23.13890 | 36.70904 | 1.58646 |
| 4.2 | 25.48944 | 40.44175 | 1.58661 |
| 4.3 | 28.08152 | 44.55766 | 1.58673 |
| 4.4 | 30.94002 | 49.09631 | 1.58682 |
| 4.5 | 34.09242 | 54.10130 | 1.58690 |
| 4.6 | 37.56904 | 59.62078 | 1.58697 |
| 4.7 | 41.40333 | 65.70786 | 1.58702 |
| 4.8 | 45.63221 | 72.42114 | 1.58706 |
| 4.9 | 50.29641 | 79.82531 | 1.58710 |
| 5.0 | 55.44091 | 87.99175 | 1.58713 |
| 5.1 | 61.11532 | 96.99921 | 1.58715 |
| 5.2 | 67.37439 | 106.93463 | 1.58717 |
| 5.3 | 74.27857 | 117.89393 | 1.58719 |
| 5.4 | 81.89456 | 129.98298 | 1.58720 |
| 5.5 | 90.29595 | 143.31864 | 1.58721 |
| 5.6 | 99.56399 | 158.02985 | 1.58722 |
| 5.7 | 109.78835 | 174.25893 | 1.58723 |
| 5.8 | 121.06799 | 192.16298 | 1.58723 |
| 5.9 | 133.51213 | 211.91536 | 1.58724 |
| 6.0 | 147.24132 | 233.70743 | 1.58724 |
| 6.0 | 147.24132 | 233.70743 | 1.58724 |
| 6.1 | 162.38864 | 257.75040 | 1.58724 |
| 6.2 | 179.10095 | 284.27738 | 1.58725 |
| 6.3 | 197.54035 | 313.54569 | 1.58725 |
| 6.4 | 217.88578 | 345.83935 | 1.58725 |
| 6.5 | 240.33475 | 381.47183 | 1.58725 |
| 6.6 | 265.10526 | 420.78320 | 1.58725 |
| 6.7 | 292.43796 | 464.17341 | 1.58725 |
| 6.8 | 322.59848 | 512.04609 | 1.58726 |
| 6.9 | 355.88005 | 564.87266 | 1.58726 |
| 7.0 | 392.60634 | 623.16688 | 1.58726 |
| 7.1 | 433.13464 | 687.49587 | 1.58726 |
| 7.2 | 477.85938 | 758.48568 | 1.58726 |
| 7.3 | 527.21594 | 836.82740 | 1.58726 |
| 7.4 | 581.68499 | 923.28393 | 1.58726 |
| 7.5 | 641.79710 | 1018.69746 | 1.58726 |
| 7.6 | 708.13802 | 1123.99772 | 1.58726 |
| 7.7 | 781.35438 | 1240.21108 | 1.58726 |
| 7.8 | 862.16003 | 1368.47060 | 1.58726 |
| 7.9 | 951.34305 | 1510.02717 | 1.58726 |
| 8.0 | 1049.77346 | 1666.26172 | 1.58726 |
| 8.1 | 1158.41174 | 1838.69881 | 1.58726 |
| 8.2 | 1278.31828 | 2029.02154 | 1.58726 |
| 8.3 | 1410.66376 | 2239.08810 | 1.58726 |
| 8.4 | 1556.74063 | 2470.94993 | 1.58726 |
| 8.5 | 1717.97585 | 2726.87193 | 1.58726 |
| 8.6 | 1895.94474 | 3009.35463 | 1.58726 |
| 8.7 | 2092.38662 | 3321.15873 | 1.58726 |
| 8.8 | 2309.22178 | 3665.33224 | 1.58726 |
| 8.9 | 2548.57034 | 4045.24035 | 1.58726 |
| 9.0 | 2812.77312 | 4464.59853 | 1.58726 |
| 9.1 | 3104.41456 | 4927.50896 | 1.58726 |
| 9.2 | 3426.34815 | 5438.50086 | 1.58726 |
| 9.3 | 3781.72443 | 6002.57496 | 1.58726 |
| 9.4 | 4174.02197 | 6625.25267 | 1.58726 |
| 9.5 | 4607.08154 | 7312.63026 | 1.58726 |
| 9.6 | 5085.14387 | 8071.43885 | 1.58726 |
| 9.7 | 5612.89127 | 8909.11053 | 1.58726 |
| 9.8 | 6195.49377 | 9833.85142 | 1.58726 |
| 9.9 | 6838.65982 | 10854.72236 | 1.58726 |
| 10.0 | 7548.69254 | 11981.72798 | 1.58726 |

TABLE D.XXIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 8/13$ and $x$ from 1.50 to 10.0.

$\alpha = 5/17$

| x | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ | x | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ | x | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.21768 | 0.20754 | 0.17044 | 1.00 | 1.93514 | 0.61418 | 0.31738 |
| 0.01 | 1.00009 | 0.00080 | 0.00080 | 0.51 | 1.22669 | 0.21374 | 0.17424 | 1.01 | 1.95575 | 0.62461 | 0.31937 |
| 0.02 | 1.00036 | 0.00213 | 0.00213 | 0.52 | 1.23590 | 0.22001 | 0.17801 | 1.02 | 1.97664 | 0.63515 | 0.32133 |
| 0.03 | 1.00077 | 0.00377 | 0.00377 | 0.53 | 1.24531 | 0.22634 | 0.18176 | 1.03 | 1.99781 | 0.64579 | 0.32325 |
| 0.04 | 1.00136 | 0.00566 | 0.00565 | 0.54 | 1.25492 | 0.23276 | 0.18548 | 1.04 | 2.01926 | 0.65654 | 0.32514 |
| 0.05 | 1.00213 | 0.00776 | 0.00774 | 0.55 | 1.26472 | 0.23924 | 0.18916 | 1.05 | 2.04100 | 0.66740 | 0.32700 |
| 0.06 | 1.00306 | 0.01004 | 0.01001 | 0.56 | 1.27473 | 0.24580 | 0.19282 | 1.06 | 2.06303 | 0.67837 | 0.32882 |
| 0.07 | 1.00417 | 0.01248 | 0.01243 | 0.57 | 1.28493 | 0.25243 | 0.19645 | 1.07 | 2.08536 | 0.68945 | 0.33062 |
| 0.08 | 1.00544 | 0.01507 | 0.01499 | 0.58 | 1.29535 | 0.25913 | 0.20005 | 1.08 | 2.10798 | 0.70064 | 0.33238 |
| 0.09 | 1.00689 | 0.01780 | 0.01768 | 0.59 | 1.30596 | 0.26591 | 0.20361 | 1.09 | 2.13090 | 0.71195 | 0.33411 |
| 0.10 | 1.00851 | 0.02066 | 0.02049 | 0.60 | 1.31678 | 0.27276 | 0.20714 | 1.10 | 2.15411 | 0.72337 | 0.33581 |
| 0.11 | 1.01030 | 0.02364 | 0.02340 | 0.61 | 1.32781 | 0.27959 | 0.21064 | 1.11 | 2.17763 | 0.73490 | 0.33748 |
| 0.12 | 1.01226 | 0.02674 | 0.02642 | 0.62 | 1.33904 | 0.28669 | 0.21410 | 1.12 | 2.20146 | 0.74655 | 0.33912 |
| 0.13 | 1.01439 | 0.02995 | 0.02953 | 0.63 | 1.35049 | 0.29376 | 0.21752 | 1.13 | 2.22559 | 0.75832 | 0.34073 |
| 0.14 | 1.01669 | 0.03327 | 0.03272 | 0.64 | 1.36214 | 0.30091 | 0.22091 | 1.14 | 2.25004 | 0.77021 | 0.34231 |
| 0.15 | 1.01917 | 0.03669 | 0.03600 | 0.65 | 1.37401 | 0.30814 | 0.22427 | 1.15 | 2.27480 | 0.78222 | 0.34386 |
| 0.16 | 1.02181 | 0.04021 | 0.03935 | 0.66 | 1.38609 | 0.31545 | 0.22758 | 1.16 | 2.29987 | 0.79435 | 0.34539 |
| 0.17 | 1.02463 | 0.04382 | 0.04277 | 0.67 | 1.39838 | 0.32283 | 0.23086 | 1.17 | 2.32527 | 0.80660 | 0.34688 |
| 0.18 | 1.02763 | 0.04753 | 0.04625 | 0.68 | 1.41089 | 0.33029 | 0.23410 | 1.18 | 2.35099 | 0.81897 | 0.34835 |
| 0.19 | 1.03079 | 0.05133 | 0.04979 | 0.69 | 1.42362 | 0.33783 | 0.23730 | 1.19 | 2.37703 | 0.83148 | 0.34980 |
| 0.20 | 1.03413 | 0.05521 | 0.05339 | 0.70 | 1.43657 | 0.34545 | 0.24047 | 1.20 | 2.40340 | 0.84410 | 0.35121 |
| 0.21 | 1.03764 | 0.05919 | 0.05704 | 0.71 | 1.44973 | 0.35315 | 0.24359 | 1.21 | 2.43010 | 0.85686 | 0.35260 |
| 0.22 | 1.04133 | 0.06324 | 0.06073 | 0.72 | 1.46312 | 0.36092 | 0.24668 | 1.22 | 2.45714 | 0.86974 | 0.35397 |
| 0.23 | 1.04520 | 0.06735 | 0.06447 | 0.73 | 1.47674 | 0.36878 | 0.24973 | 1.23 | 2.48452 | 0.88276 | 0.35530 |
| 0.24 | 1.04923 | 0.07161 | 0.06825 | 0.74 | 1.49057 | 0.37672 | 0.25274 | 1.24 | 2.51223 | 0.89591 | 0.35662 |
| 0.25 | 1.05345 | 0.07591 | 0.07206 | 0.75 | 1.50464 | 0.38474 | 0.25570 | 1.25 | 2.54029 | 0.90919 | 0.35791 |
| 0.26 | 1.05784 | 0.08029 | 0.07590 | 0.76 | 1.51893 | 0.39285 | 0.25863 | 1.26 | 2.56870 | 0.92261 | 0.35917 |
| 0.27 | 1.06240 | 0.08475 | 0.07977 | 0.77 | 1.53346 | 0.40104 | 0.26152 | 1.27 | 2.59745 | 0.93616 | 0.36041 |
| 0.28 | 1.06715 | 0.08929 | 0.08367 | 0.78 | 1.54821 | 0.40931 | 0.26438 | 1.28 | 2.62656 | 0.94985 | 0.36163 |
| 0.29 | 1.07207 | 0.09390 | 0.08759 | 0.79 | 1.56320 | 0.41767 | 0.26719 | 1.29 | 2.65603 | 0.96368 | 0.36283 |
| 0.30 | 1.07717 | 0.09859 | 0.09152 | 0.80 | 1.57842 | 0.42611 | 0.26996 | 1.30 | 2.68586 | 0.97765 | 0.36400 |
| 0.31 | 1.08245 | 0.10335 | 0.09548 | 0.81 | 1.59388 | 0.43464 | 0.27269 | 1.31 | 2.71605 | 0.99176 | 0.36515 |
| 0.32 | 1.08790 | 0.10819 | 0.09944 | 0.82 | 1.60958 | 0.44325 | 0.27539 | 1.32 | 2.74661 | 1.00602 | 0.36628 |
| 0.33 | 1.09354 | 0.11310 | 0.10342 | 0.83 | 1.62552 | 0.45196 | 0.27804 | 1.33 | 2.77754 | 1.02042 | 0.36738 |
| 0.34 | 1.09936 | 0.11808 | 0.10741 | 0.84 | 1.64170 | 0.46075 | 0.28065 | 1.34 | 2.80884 | 1.03497 | 0.36847 |
| 0.35 | 1.10536 | 0.12314 | 0.11140 | 0.85 | 1.65812 | 0.46963 | 0.28323 | 1.35 | 2.84052 | 1.04966 | 0.36953 |
| 0.36 | 1.11155 | 0.12826 | 0.11539 | 0.86 | 1.67479 | 0.47860 | 0.28577 | 1.36 | 2.87258 | 1.06451 | 0.37058 |
| 0.37 | 1.11791 | 0.13347 | 0.11939 | 0.87 | 1.69171 | 0.48767 | 0.28827 | 1.37 | 2.90503 | 1.07951 | 0.37160 |
| 0.38 | 1.12446 | 0.13874 | 0.12338 | 0.88 | 1.70888 | 0.49682 | 0.29073 | 1.38 | 2.93786 | 1.09466 | 0.37260 |
| 0.39 | 1.13119 | 0.14408 | 0.12737 | 0.89 | 1.72630 | 0.50607 | 0.29315 | 1.39 | 2.97109 | 1.10997 | 0.37359 |
| 0.40 | 1.13811 | 0.14949 | 0.13135 | 0.90 | 1.74398 | 0.51541 | 0.29554 | 1.40 | 3.00471 | 1.12543 | 0.37455 |
| 0.41 | 1.14522 | 0.15498 | 0.13533 | 0.91 | 1.76191 | 0.52485 | 0.29789 | 1.41 | 3.03874 | 1.14105 | 0.37550 |
| 0.42 | 1.15251 | 0.16054 | 0.13929 | 0.92 | 1.78010 | 0.53438 | 0.30020 | 1.42 | 3.07317 | 1.15683 | 0.37643 |
| 0.43 | 1.15999 | 0.16616 | 0.14325 | 0.93 | 1.79854 | 0.54401 | 0.30247 | 1.43 | 3.10800 | 1.17278 | 0.37734 |
| 0.44 | 1.16766 | 0.17186 | 0.14719 | 0.94 | 1.81725 | 0.55373 | 0.30471 | 1.44 | 3.14325 | 1.18888 | 0.37823 |
| 0.45 | 1.17552 | 0.17763 | 0.15111 | 0.95 | 1.83623 | 0.56355 | 0.30691 | 1.45 | 3.17891 | 1.20515 | 0.37911 |
| 0.46 | 1.18356 | 0.18347 | 0.15502 | 0.96 | 1.85547 | 0.57348 | 0.30907 | 1.46 | 3.21499 | 1.22159 | 0.37997 |
| 0.47 | 1.19180 | 0.18938 | 0.15890 | 0.97 | 1.87498 | 0.58350 | 0.31120 | 1.47 | 3.25150 | 1.23820 | 0.38081 |
| 0.48 | 1.20023 | 0.19536 | 0.16277 | 0.98 | 1.89476 | 0.59362 | 0.31330 | 1.48 | 3.28843 | 1.25498 | 0.38163 |
| 0.49 | 1.20886 | 0.20142 | 0.16662 | 0.99 | 1.91481 | 0.60385 | 0.31536 | 1.49 | 3.32580 | 1.27193 | 0.38244 |
| 0.50 | 1.21768 | 0.20754 | 0.17044 | 1.00 | 1.93514 | 0.61418 | 0.31738 | 1.50 | 3.36360 | 1.28905 | 0.38324 |

TABLE D.XXIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/17$ and x from 0.00 to 1.50.

$\alpha = 5/17$

| $x$ | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ |
|---|---|---|---|
| 1.50 | 3.36360 | 1.28905 | 0.38324 |
| 1.51 | 3.40185 | 1.30636 | 0.38401 |
| 1.52 | 3.44053 | 1.32383 | 0.38478 |
| 1.53 | 3.47967 | 1.34149 | 0.38552 |
| 1.54 | 3.51927 | 1.35933 | 0.38626 |
| 1.55 | 3.55932 | 1.37736 | 0.38697 |
| 1.56 | 3.59983 | 1.39557 | 0.38768 |
| 1.57 | 3.64081 | 1.41397 | 0.38837 |
| 1.58 | 3.68226 | 1.43256 | 0.38904 |
| 1.59 | 3.72419 | 1.45134 | 0.38970 |
| 1.60 | 3.76661 | 1.47031 | 0.39035 |
| 1.61 | 3.80950 | 1.48948 | 0.39099 |
| 1.62 | 3.85289 | 1.50885 | 0.39161 |
| 1.63 | 3.89678 | 1.52842 | 0.39223 |
| 1.64 | 3.94117 | 1.54819 | 0.39283 |
| 1.65 | 3.98606 | 1.56817 | 0.39341 |
| 1.66 | 4.03146 | 1.58835 | 0.39399 |
| 1.67 | 4.07738 | 1.60874 | 0.39455 |
| 1.68 | 4.12383 | 1.62935 | 0.39511 |
| 1.69 | 4.17079 | 1.65016 | 0.39565 |
| 1.70 | 4.21830 | 1.67120 | 0.39618 |
| 1.71 | 4.26634 | 1.69245 | 0.39670 |
| 1.72 | 4.31492 | 1.71392 | 0.39721 |
| 1.73 | 4.36405 | 1.73562 | 0.39771 |
| 1.74 | 4.41373 | 1.75754 | 0.39820 |
| 1.75 | 4.46398 | 1.77969 | 0.39869 |
| 1.76 | 4.51478 | 1.80207 | 0.39915 |
| 1.77 | 4.56616 | 1.82468 | 0.39961 |
| 1.78 | 4.61812 | 1.84753 | 0.40006 |
| 1.79 | 4.67066 | 1.87061 | 0.40050 |
| 1.80 | 4.72379 | 1.89394 | 0.40094 |
| 1.81 | 4.77751 | 1.91751 | 0.40136 |
| 1.82 | 4.83184 | 1.94132 | 0.40178 |
| 1.83 | 4.88677 | 1.96538 | 0.40219 |
| 1.84 | 4.94231 | 1.98970 | 0.40258 |
| 1.85 | 4.99847 | 2.01427 | 0.40298 |
| 1.86 | 5.05526 | 2.03909 | 0.40336 |
| 1.87 | 5.11268 | 2.06417 | 0.40374 |
| 1.88 | 5.17074 | 2.08952 | 0.40410 |
| 1.89 | 5.22944 | 2.11513 | 0.40447 |
| 1.90 | 5.28880 | 2.14100 | 0.40482 |
| 1.91 | 5.34881 | 2.16715 | 0.40517 |
| 1.92 | 5.40949 | 2.19357 | 0.40550 |
| 1.93 | 5.47084 | 2.22027 | 0.40584 |
| 1.94 | 5.53287 | 2.24725 | 0.40616 |
| 1.95 | 5.59558 | 2.27451 | 0.40648 |
| 1.96 | 5.65899 | 2.30205 | 0.40680 |
| 1.97 | 5.72310 | 2.32988 | 0.40710 |
| 1.98 | 5.78791 | 2.35801 | 0.40740 |
| 1.99 | 5.85344 | 2.38643 | 0.40770 |
| 2.00 | 5.91969 | 2.41514 | 0.40798 |

| $x$ | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ |
|---|---|---|---|
| 2.0 | 5.91969 | 2.41514 | 0.40798 |
| 2.1 | 6.62356 | 2.71940 | 0.41057 |
| 2.2 | 7.40830 | 3.05717 | 0.41267 |
| 2.3 | 8.28271 | 3.43221 | 0.41438 |
| 2.4 | 9.25655 | 3.84867 | 0.41578 |
| 2.5 | 10.34068 | 4.31119 | 0.41692 |
| 2.6 | 11.54715 | 4.82490 | 0.41784 |
| 2.7 | 12.88936 | 5.39547 | 0.41860 |
| 2.8 | 14.38218 | 6.02922 | 0.41921 |
| 2.9 | 16.04211 | 6.73315 | 0.41972 |
| 3.0 | 17.88748 | 7.51501 | 0.42013 |
| 3.1 | 19.93859 | 8.38339 | 0.42046 |
| 3.2 | 22.21800 | 9.34765 | 0.42073 |
| 3.3 | 24.75072 | 10.41896 | 0.42096 |
| 3.4 | 27.56449 | 11.60844 | 0.42114 |
| 3.5 | 30.69009 | 12.92930 | 0.42129 |
| 3.6 | 34.16164 | 14.39594 | 0.42141 |
| 3.7 | 38.01697 | 16.02436 | 0.42151 |
| 3.8 | 42.29805 | 17.83227 | 0.42159 |
| 3.9 | 47.05138 | 19.83931 | 0.42165 |
| 4.0 | 52.32855 | 22.06726 | 0.42171 |
| 4.1 | 58.18673 | 24.54025 | 0.42175 |
| 4.2 | 64.68931 | 27.28504 | 0.42179 |
| 4.3 | 71.90656 | 30.33128 | 0.42182 |
| 4.4 | 79.91634 | 33.71185 | 0.42184 |
| 4.5 | 88.90500 | 37.46317 | 0.42186 |
| 4.6 | 98.66820 | 41.62564 | 0.42187 |
| 4.7 | 109.61201 | 46.24399 | 0.42189 |
| 4.8 | 121.75394 | 51.36783 | 0.42190 |
| 4.9 | 135.22424 | 57.05212 | 0.42191 |
| 5.0 | 150.16726 | 63.35776 | 0.42191 |
| 5.1 | 166.74293 | 70.35226 | 0.42192 |
| 5.2 | 185.12852 | 78.11040 | 0.42193 |
| 5.3 | 205.52042 | 86.71506 | 0.42193 |
| 5.4 | 228.13624 | 96.25807 | 0.42193 |
| 5.5 | 253.21712 | 106.84117 | 0.42194 |
| 5.6 | 281.03023 | 118.57709 | 0.42194 |
| 5.7 | 311.87155 | 131.59074 | 0.42194 |
| 5.8 | 346.06903 | 146.02048 | 0.42194 |
| 5.9 | 383.98600 | 162.01963 | 0.42194 |
| 6.0 | 426.02496 | 179.75902 | 0.42194 |

| $x$ | $F_{5/17}(x)$ | $H_{12/17}(x)$ | $T_{5/17}(x)$ |
|---|---|---|---|
| 6.0 | 426.02496 | 179.75802 | 0.42194 |
| 6.1 | 472.63180 | 199.42380 | 0.42194 |
| 6.2 | 524.30050 | 221.22540 | 0.42194 |
| 6.3 | 581.57826 | 245.39370 | 0.42194 |
| 6.4 | 645.07120 | 272.18447 | 0.42194 |
| 6.5 | 715.45078 | 301.88102 | 0.42195 |
| 6.6 | 793.46071 | 334.79717 | 0.42195 |
| 6.7 | 879.92479 | 371.28051 | 0.42195 |
| 6.8 | 975.75548 | 411.71604 | 0.42195 |
| 6.9 | 1081.96340 | 456.53021 | 0.42195 |
| 7.0 | 1199.66792 | 506.19532 | 0.42195 |
| 7.1 | 1330.10877 | 561.23449 | 0.42195 |
| 7.2 | 1474.65900 | 622.22705 | 0.42195 |
| 7.3 | 1634.83927 | 689.81465 | 0.42195 |
| 7.4 | 1812.33367 | 764.70789 | 0.42195 |
| 7.5 | 2009.00727 | 847.69373 | 0.42195 |
| 7.6 | 2226.92552 | 939.64367 | 0.42195 |
| 7.7 | 2468.37571 | 1041.52284 | 0.42195 |
| 7.8 | 2735.89079 | 1154.40000 | 0.42195 |
| 7.9 | 3032.27566 | 1279.45867 | 0.42195 |
| 8.0 | 3360.63634 | 1418.00944 | 0.42195 |
| 8.1 | 3724.41224 | 1571.50353 | 0.42195 |
| 8.2 | 4127.41183 | 1741.54793 | 0.42195 |
| 8.3 | 4573.85221 | 1929.92204 | 0.42195 |
| 8.4 | 5068.40267 | 2138.59612 | 0.42195 |
| 8.5 | 5616.23411 | 2369.75177 | 0.42195 |
| 8.6 | 6223.07069 | 2625.80453 | 0.42195 |
| 8.7 | 6895.25115 | 2909.42892 | 0.42195 |
| 8.8 | 7639.79352 | 3223.58619 | 0.42195 |
| 8.9 | 8464.46804 | 3571.55497 | 0.42195 |
| 9.0 | 9377.87767 | 3956.96523 | 0.42195 |
| 9.1 | 10389.54712 | 4383.83592 | 0.42195 |
| 9.2 | 11510.02154 | 4856.61650 | 0.42195 |
| 9.3 | 12750.97560 | 5380.23309 | 0.42195 |
| 9.4 | 14125.33432 | 5960.13934 | 0.42195 |
| 9.5 | 15647.40677 | 6602.37292 | 0.42195 |
| 9.6 | 17333.03404 | 7313.61793 | 0.42195 |
| 9.7 | 19199.75308 | 8101.27403 | 0.42195 |
| 9.8 | 21266.97790 | 8973.53291 | 0.42195 |
| 9.9 | 23556.20027 | 9939.46293 | 0.42195 |
| 10.0 | 26091.21170 | 11009.10286 | 0.42195 |

TABLE D.XXIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/17$ and $x$ from 1.50 to 10.0.

$\alpha = 12/17$

| $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ | $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ | $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.09018 | 1.57792 | 1.44740 | 1.00 | 1.38093 | 2.72283 | 1.97174 |
| 0.01 | 1.00004 | 0.15064 | 0.15063 | 0.51 | 1.09389 | 1.59946 | 1.46218 | 1.01 | 1.38915 | 2.74866 | 1.97866 |
| 0.02 | 1.00014 | 0.22649 | 0.22645 | 0.52 | 1.09768 | 1.62098 | 1.47674 | 1.02 | 1.39748 | 2.77466 | 1.98547 |
| 0.03 | 1.00032 | 0.28752 | 0.28743 | 0.53 | 1.10155 | 1.64250 | 1.49108 | 1.03 | 1.40592 | 2.80082 | 1.99217 |
| 0.04 | 1.00057 | 0.34058 | 0.34039 | 0.54 | 1.10550 | 1.66402 | 1.50521 | 1.04 | 1.41446 | 2.82717 | 1.99877 |
| 0.05 | 1.00089 | 0.38842 | 0.38808 | 0.55 | 1.10953 | 1.68554 | 1.51914 | 1.05 | 1.42310 | 2.85369 | 2.00526 |
| 0.06 | 1.00128 | 0.43248 | 0.43193 | 0.56 | 1.11364 | 1.70706 | 1.53286 | 1.06 | 1.43186 | 2.88039 | 2.01164 |
| 0.07 | 1.00174 | 0.47365 | 0.47283 | 0.57 | 1.11784 | 1.72860 | 1.54638 | 1.07 | 1.44073 | 2.90728 | 2.01793 |
| 0.08 | 1.00227 | 0.51250 | 0.51135 | 0.58 | 1.12211 | 1.75015 | 1.55970 | 1.08 | 1.44970 | 2.93435 | 2.02411 |
| 0.09 | 1.00287 | 0.54945 | 0.54788 | 0.59 | 1.12646 | 1.77173 | 1.57282 | 1.09 | 1.45879 | 2.96162 | 2.03019 |
| 0.10 | 1.00354 | 0.58480 | 0.58273 | 0.60 | 1.13090 | 1.79333 | 1.58576 | 1.10 | 1.46798 | 2.98907 | 2.03618 |
| 0.11 | 1.00429 | 0.61877 | 0.61613 | 0.61 | 1.13542 | 1.81497 | 1.59850 | 1.11 | 1.47729 | 3.01672 | 2.04206 |
| 0.12 | 1.00511 | 0.65156 | 0.64825 | 0.62 | 1.14002 | 1.83664 | 1.61105 | 1.12 | 1.48671 | 3.04457 | 2.04784 |
| 0.13 | 1.00599 | 0.68330 | 0.67923 | 0.63 | 1.14471 | 1.85835 | 1.62343 | 1.13 | 1.49625 | 3.07263 | 2.05355 |
| 0.14 | 1.00695 | 0.71411 | 0.70918 | 0.64 | 1.14948 | 1.88010 | 1.63561 | 1.14 | 1.50590 | 3.10088 | 2.05916 |
| 0.15 | 1.00798 | 0.74411 | 0.73822 | 0.65 | 1.15433 | 1.90190 | 1.64762 | 1.15 | 1.51567 | 3.12935 | 2.06467 |
| 0.16 | 1.00908 | 0.77336 | 0.76640 | 0.66 | 1.15927 | 1.92375 | 1.65946 | 1.16 | 1.52555 | 3.15802 | 2.07009 |
| 0.17 | 1.01026 | 0.80195 | 0.79381 | 0.67 | 1.16429 | 1.94566 | 1.67112 | 1.17 | 1.53555 | 3.18691 | 2.07542 |
| 0.18 | 1.01150 | 0.82993 | 0.82049 | 0.68 | 1.16940 | 1.96763 | 1.68261 | 1.18 | 1.54567 | 3.21602 | 2.08067 |
| 0.19 | 1.01282 | 0.85736 | 0.84651 | 0.69 | 1.17459 | 1.98966 | 1.69392 | 1.19 | 1.55590 | 3.24534 | 2.08583 |
| 0.20 | 1.01421 | 0.88429 | 0.87190 | 0.70 | 1.17987 | 2.01176 | 1.70507 | 1.20 | 1.56626 | 3.27489 | 2.09090 |
| 0.21 | 1.01567 | 0.91075 | 0.89670 | 0.71 | 1.18523 | 2.03395 | 1.71606 | 1.21 | 1.57674 | 3.30467 | 2.09589 |
| 0.22 | 1.01720 | 0.93680 | 0.92095 | 0.72 | 1.19069 | 2.05618 | 1.72688 | 1.22 | 1.58734 | 3.33467 | 2.10080 |
| 0.23 | 1.01881 | 0.96245 | 0.94468 | 0.73 | 1.19623 | 2.07850 | 1.73755 | 1.23 | 1.59806 | 3.36490 | 2.10562 |
| 0.24 | 1.02049 | 0.98774 | 0.96791 | 0.74 | 1.20186 | 2.10090 | 1.74805 | 1.24 | 1.60890 | 3.39537 | 2.11036 |
| 0.25 | 1.02224 | 1.01270 | 0.99067 | 0.75 | 1.20757 | 2.12339 | 1.75840 | 1.25 | 1.61988 | 3.42608 | 2.11503 |
| 0.26 | 1.02406 | 1.03735 | 1.01298 | 0.76 | 1.21338 | 2.14597 | 1.76859 | 1.26 | 1.63097 | 3.45703 | 2.11961 |
| 0.27 | 1.02596 | 1.06172 | 1.03485 | 0.77 | 1.21928 | 2.16864 | 1.77863 | 1.27 | 1.64219 | 3.48822 | 2.12412 |
| 0.28 | 1.02793 | 1.08582 | 1.05632 | 0.78 | 1.22526 | 2.19141 | 1.78852 | 1.28 | 1.65354 | 3.51966 | 2.12855 |
| 0.29 | 1.02997 | 1.10968 | 1.07739 | 0.79 | 1.23134 | 2.21427 | 1.79826 | 1.29 | 1.66502 | 3.55135 | 2.13291 |
| 0.30 | 1.03209 | 1.13334 | 1.09808 | 0.80 | 1.23751 | 2.23724 | 1.80786 | 1.30 | 1.67663 | 3.58329 | 2.13720 |
| 0.31 | 1.03428 | 1.15673 | 1.11840 | 0.81 | 1.24377 | 2.26031 | 1.81731 | 1.31 | 1.68837 | 3.61549 | 2.14141 |
| 0.32 | 1.03654 | 1.17996 | 1.13837 | 0.82 | 1.25012 | 2.28349 | 1.82662 | 1.32 | 1.70024 | 3.64795 | 2.14555 |
| 0.33 | 1.03888 | 1.20301 | 1.15799 | 0.83 | 1.25657 | 2.30678 | 1.83578 | 1.33 | 1.71225 | 3.68068 | 2.14962 |
| 0.34 | 1.04129 | 1.22593 | 1.17729 | 0.84 | 1.26310 | 2.33019 | 1.84481 | 1.34 | 1.72439 | 3.71367 | 2.15362 |
| 0.35 | 1.04378 | 1.24863 | 1.19626 | 0.85 | 1.26974 | 2.35372 | 1.85370 | 1.35 | 1.73666 | 3.74693 | 2.15755 |
| 0.36 | 1.04634 | 1.27121 | 1.21492 | 0.86 | 1.27646 | 2.37736 | 1.86246 | 1.36 | 1.74907 | 3.78046 | 2.16141 |
| 0.37 | 1.04897 | 1.29367 | 1.23327 | 0.87 | 1.28329 | 2.40113 | 1.87108 | 1.37 | 1.76161 | 3.81427 | 2.16521 |
| 0.38 | 1.05169 | 1.31601 | 1.25133 | 0.88 | 1.29021 | 2.42503 | 1.87957 | 1.38 | 1.77430 | 3.84835 | 2.16895 |
| 0.39 | 1.05447 | 1.33823 | 1.26910 | 0.89 | 1.29722 | 2.44906 | 1.88793 | 1.39 | 1.78712 | 3.88272 | 2.17261 |
| 0.40 | 1.05733 | 1.36036 | 1.28660 | 0.90 | 1.30433 | 2.47322 | 1.89616 | 1.40 | 1.80008 | 3.91738 | 2.17622 |
| 0.41 | 1.06027 | 1.38240 | 1.30381 | 0.91 | 1.31154 | 2.49752 | 1.90426 | 1.41 | 1.81319 | 3.95233 | 2.17976 |
| 0.42 | 1.06329 | 1.40435 | 1.32076 | 0.92 | 1.31885 | 2.52195 | 1.91224 | 1.42 | 1.82644 | 3.98756 | 2.18325 |
| 0.43 | 1.06638 | 1.42623 | 1.33745 | 0.93 | 1.32626 | 2.54653 | 1.92009 | 1.43 | 1.83983 | 4.02310 | 2.18667 |
| 0.44 | 1.06955 | 1.44804 | 1.35389 | 0.94 | 1.33376 | 2.57126 | 1.92782 | 1.44 | 1.85337 | 4.05893 | 2.19003 |
| 0.45 | 1.07279 | 1.46979 | 1.37007 | 0.95 | 1.34137 | 2.59613 | 1.93543 | 1.45 | 1.86705 | 4.09507 | 2.19333 |
| 0.46 | 1.07611 | 1.49150 | 1.38600 | 0.96 | 1.34908 | 2.62116 | 1.94293 | 1.46 | 1.88089 | 4.13152 | 2.19658 |
| 0.47 | 1.07951 | 1.51315 | 1.40170 | 0.97 | 1.35689 | 2.64634 | 1.95030 | 1.47 | 1.89487 | 4.16827 | 2.19977 |
| 0.48 | 1.08299 | 1.53477 | 1.41716 | 0.98 | 1.36480 | 2.67167 | 1.95756 | 1.48 | 1.90900 | 4.20534 | 2.20290 |
| 0.49 | 1.08654 | 1.55636 | 1.43239 | 0.99 | 1.37281 | 2.69717 | 1.96470 | 1.49 | 1.92328 | 4.24272 | 2.20598 |
| 0.50 | 1.09018 | 1.57792 | 1.44740 | 1.00 | 1.38093 | 2.72283 | 1.97174 | 1.50 | 1.93771 | 4.28042 | 2.20901 |

TABLE D.XXIIIA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 12/17$ and $x$ from 0.00 to 1.50.

$\alpha = 12/17$

| $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ |
|---|---|---|---|
| 1.50 | 1.93771 | 4.28042 | 2.20901 |
| 1.51 | 1.95230 | 4.31845 | 2.21198 |
| 1.52 | 1.96705 | 4.35681 | 2.21490 |
| 1.53 | 1.98195 | 4.39550 | 2.21777 |
| 1.54 | 1.99701 | 4.43452 | 2.22058 |
| 1.55 | 2.01222 | 4.47388 | 2.22335 |
| 1.56 | 2.02760 | 4.51359 | 2.22607 |
| 1.57 | 2.04314 | 4.55363 | 2.22874 |
| 1.58 | 2.05885 | 4.59403 | 2.23136 |
| 1.59 | 2.07471 | 4.63479 | 2.23394 |
| 1.60 | 2.09075 | 4.67590 | 2.23647 |
| 1.61 | 2.10695 | 4.71737 | 2.23896 |
| 1.62 | 2.12332 | 4.75921 | 2.24140 |
| 1.63 | 2.13986 | 4.80142 | 2.24379 |
| 1.64 | 2.15658 | 4.84399 | 2.24615 |
| 1.65 | 2.17346 | 4.88695 | 2.24846 |
| 1.66 | 2.19053 | 4.93029 | 2.25073 |
| 1.67 | 2.20777 | 4.97401 | 2.25296 |
| 1.68 | 2.22518 | 5.01812 | 2.25515 |
| 1.69 | 2.24278 | 5.06263 | 2.25730 |
| 1.70 | 2.26056 | 5.10753 | 2.25941 |
| 1.71 | 2.27852 | 5.15283 | 2.26149 |
| 1.72 | 2.29666 | 5.19854 | 2.26352 |
| 1.73 | 2.31499 | 5.24466 | 2.26552 |
| 1.74 | 2.33351 | 5.29120 | 2.26748 |
| 1.75 | 2.35222 | 5.33815 | 2.26941 |
| 1.76 | 2.37112 | 5.38552 | 2.27130 |
| 1.77 | 2.39021 | 5.43333 | 2.27316 |
| 1.78 | 2.40949 | 5.48156 | 2.27499 |
| 1.79 | 2.42897 | 5.53023 | 2.27678 |
| 1.80 | 2.44865 | 5.57934 | 2.27853 |
| 1.81 | 2.46853 | 5.62890 | 2.28026 |
| 1.82 | 2.48861 | 5.67890 | 2.28195 |
| 1.83 | 2.50890 | 5.72936 | 2.28362 |
| 1.84 | 2.52939 | 5.78028 | 2.28525 |
| 1.85 | 2.55008 | 5.83166 | 2.28685 |
| 1.86 | 2.57099 | 5.88351 | 2.28843 |
| 1.87 | 2.59210 | 5.93584 | 2.28997 |
| 1.88 | 2.61343 | 5.98864 | 2.29149 |
| 1.89 | 2.63497 | 6.04192 | 2.29297 |
| 1.90 | 2.65673 | 6.09569 | 2.29443 |
| 1.91 | 2.67871 | 6.14996 | 2.29587 |
| 1.92 | 2.70090 | 6.20472 | 2.29728 |
| 1.93 | 2.72332 | 6.25998 | 2.29866 |
| 1.94 | 2.74597 | 6.31575 | 2.30001 |
| 1.95 | 2.76883 | 6.37204 | 2.30134 |
| 1.96 | 2.79193 | 6.42884 | 2.30265 |
| 1.97 | 2.81526 | 6.48616 | 2.30393 |
| 1.98 | 2.83882 | 6.54402 | 2.30519 |
| 1.99 | 2.86262 | 6.60240 | 2.30642 |
| 2.00 | 2.88665 | 6.66133 | 2.30763 |

| $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ |
|---|---|---|---|
| 2.0 | 2.88665 | 6.66133 | 2.30763 |
| 2.1 | 3.14048 | 7.28143 | 2.31857 |
| 2.2 | 3.42051 | 7.96167 | 2.32762 |
| 2.3 | 3.72927 | 8.70824 | 2.33510 |
| 2.4 | 4.06954 | 9.52794 | 2.34128 |
| 2.5 | 4.44441 | 10.42826 | 2.34638 |
| 2.6 | 4.85728 | 11.41739 | 2.35057 |
| 2.7 | 5.31189 | 12.50435 | 2.35403 |
| 2.8 | 5.81238 | 13.69907 | 2.35688 |
| 2.9 | 6.36332 | 15.01245 | 2.35922 |
| 3.0 | 6.96973 | 16.45652 | 2.36114 |
| 3.1 | 7.63715 | 18.04448 | 2.36272 |
| 3.2 | 8.37170 | 19.79090 | 2.36402 |
| 3.3 | 9.18012 | 21.71181 | 2.36509 |
| 3.4 | 10.06982 | 23.82485 | 2.36597 |
| 3.5 | 11.04898 | 26.14948 | 2.36669 |
| 3.6 | 12.12664 | 28.70711 | 2.36728 |
| 3.7 | 13.31272 | 31.52133 | 2.36776 |
| 3.8 | 14.61817 | 34.61815 | 2.36816 |
| 3.9 | 16.05508 | 38.02620 | 2.36848 |
| 4.0 | 17.63674 | 41.77705 | 2.36875 |
| 4.1 | 19.37781 | 45.90547 | 2.36897 |
| 4.2 | 21.29446 | 50.44979 | 2.36915 |
| 4.3 | 23.40450 | 55.45224 | 2.36930 |
| 4.4 | 25.72756 | 60.95936 | 2.36942 |
| 4.5 | 28.28528 | 67.02246 | 2.36952 |
| 4.6 | 31.10150 | 73.69808 | 2.36960 |
| 4.7 | 34.20252 | 81.04853 | 2.36967 |
| 4.8 | 37.61732 | 89.14249 | 2.36972 |
| 4.9 | 41.37783 | 98.05570 | 2.36976 |
| 5.0 | 45.51926 | 107.87160 | 2.36980 |
| 5.1 | 50.08045 | 118.68220 | 2.36983 |
| 5.2 | 55.10417 | 130.58892 | 2.36986 |
| 5.3 | 60.63760 | 143.70358 | 2.36988 |
| 5.4 | 66.73275 | 158.14943 | 2.36989 |
| 5.5 | 73.44695 | 174.06236 | 2.36991 |
| 5.6 | 80.84343 | 191.59221 | 2.36992 |
| 5.7 | 88.99187 | 210.90415 | 2.36993 |
| 5.8 | 97.96913 | 232.18032 | 2.36993 |
| 5.9 | 107.85993 | 255.62152 | 2.36994 |
| 6.0 | 118.75771 | 281.44917 | 2.36994 |

| $x$ | $F_{12/17}(x)$ | $H_{5/17}(x)$ | $T_{12/17}(x)$ |
|---|---|---|---|
| 6.0 | 118.75771 | 281.44917 | 2.36994 |
| 6.1 | 130.76546 | 309.90742 | 2.36995 |
| 6.2 | 143.99679 | 341.26548 | 2.36995 |
| 6.3 | 158.57695 | 375.82019 | 2.36995 |
| 6.4 | 174.64407 | 413.89892 | 2.36996 |
| 6.5 | 192.35045 | 455.86264 | 2.36996 |
| 6.6 | 211.86407 | 502.10943 | 2.36996 |
| 6.7 | 233.37016 | 553.07830 | 2.36996 |
| 6.8 | 257.07300 | 609.25337 | 2.36996 |
| 6.9 | 283.19787 | 671.16858 | 2.36996 |
| 7.0 | 311.99324 | 735.41276 | 2.36996 |
| 7.1 | 343.73312 | 814.63533 | 2.36996 |
| 7.2 | 378.71974 | 897.55255 | 2.36997 |
| 7.3 | 417.28643 | 988.95440 | 2.36997 |
| 7.4 | 459.80081 | 1089.71216 | 2.36997 |
| 7.5 | 506.66840 | 1200.78687 | 2.36997 |
| 7.6 | 558.33644 | 1323.23849 | 2.36997 |
| 7.7 | 615.29828 | 1458.23620 | 2.36997 |
| 7.8 | 678.09805 | 1607.06964 | 2.36997 |
| 7.9 | 747.33599 | 1771.16132 | 2.36997 |
| 8.0 | 823.67418 | 1952.08037 | 2.36997 |
| 8.1 | 907.84297 | 2151.55767 | 2.36997 |
| 8.2 | 1000.64800 | 2371.50257 | 2.36997 |
| 8.3 | 1102.97798 | 2614.02128 | 2.36997 |
| 8.4 | 1215.81329 | 2881.43[illegible]27 | 2.36997 |
| 8.5 | 1340.23543 | 3176.31367 | 2.36997 |
| 8.6 | 1477.43749 | 3501.47806 | 2.36997 |
| 8.7 | 1628.73567 | 3860.04979 | 2.36997 |
| 8.8 | 1795.58200 | 4255.47013 | 2.36997 |
| 8.9 | 1979.57841 | 4691.53557 | 2.36997 |
| 9.0 | 2182.49218 | 5172.43456 | 2.36997 |
| 9.1 | 2406.27310 | 5702.78798 | 2.36997 |
| 9.2 | 2653.07228 | 6287.69393 | 2.36997 |
| 9.3 | 2925.26302 | 6932.77704 | 2.36997 |
| 9.4 | 3225.46378 | 7644.24298 | 2.36997 |
| 9.5 | 3556.56354 | 8428.93852 | 2.36997 |
| 9.6 | 3921.74978 | 9294.41791 | 2.36997 |
| 9.7 | 4324.53941 | 10249.01609 | 2.36997 |
| 9.8 | 4768.81281 | 11301.92943 | 2.36997 |
| 9.9 | 5258.85149 | 12463.30496 | 2.36997 |
| 10.0 | 5799.37957 | 13744.33875 | 2.36997 |

TABLE D.XXIIIB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 12/17$ and $x$ from 1.50 to 10.0.

$\alpha = 5/21$

| $x$ | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ | $x$ | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ | $x$ | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.26919 | 0.16443 | 0.12956 | 1.00 | 2.16003 | 0.52421 | 0.24268 |
| 0.01 | 1.00011 | 0.00041 | 0.00041 | 0.51 | 1.28035 | 0.16971 | 0.13255 | 1.01 | 2.18569 | 0.53366 | 0.24416 |
| 0.02 | 1.00042 | 0.00118 | 0.00118 | 0.52 | 1.29175 | 0.17506 | 0.13552 | 1.02 | 2.21170 | 0.54322 | 0.24561 |
| 0.03 | 1.00095 | 0.00218 | 0.00213 | 0.53 | 1.30340 | 0.18048 | 0.13847 | 1.03 | 2.23807 | 0.55288 | 0.24703 |
| 0.04 | 1.00168 | 0.00338 | 0.00338 | 0.54 | 1.31529 | 0.18597 | 0.14139 | 1.04 | 2.26480 | 0.56265 | 0.24843 |
| 0.05 | 1.00263 | 0.00475 | 0.00474 | 0.55 | 1.32744 | 0.19153 | 0.14429 | 1.05 | 2.29188 | 0.57252 | 0.24980 |
| 0.06 | 1.00378 | 0.00628 | 0.00625 | 0.56 | 1.33985 | 0.19717 | 0.14716 | 1.06 | 2.31934 | 0.58250 | 0.25115 |
| 0.07 | 1.00515 | 0.00794 | 0.00790 | 0.57 | 1.35247 | 0.20288 | 0.15001 | 1.07 | 2.34716 | 0.59259 | 0.25247 |
| 0.08 | 1.00672 | 0.00973 | 0.00967 | 0.58 | 1.36537 | 0.20867 | 0.15283 | 1.08 | 2.37535 | 0.60278 | 0.25377 |
| 0.09 | 1.00851 | 0.01165 | 0.01155 | 0.59 | 1.37852 | 0.21452 | 0.15562 | 1.09 | 2.40392 | 0.61309 | 0.25504 |
| 0.10 | 1.01051 | 0.01368 | 0.01354 | 0.60 | 1.39192 | 0.22045 | 0.15838 | 1.10 | 2.43287 | 0.62350 | 0.25628 |
| 0.11 | 1.01272 | 0.01583 | 0.01563 | 0.61 | 1.40559 | 0.22646 | 0.16111 | 1.11 | 2.46220 | 0.63403 | 0.25751 |
| 0.12 | 1.01514 | 0.01808 | 0.01781 | 0.62 | 1.41951 | 0.23254 | 0.16381 | 1.12 | 2.49191 | 0.64468 | 0.25871 |
| 0.13 | 1.01778 | 0.02043 | 0.02007 | 0.63 | 1.43369 | 0.23869 | 0.16649 | 1.13 | 2.52201 | 0.65543 | 0.25988 |
| 0.14 | 1.02062 | 0.02288 | 0.02242 | 0.64 | 1.44814 | 0.24492 | 0.16913 | 1.14 | 2.55251 | 0.66630 | 0.26104 |
| 0.15 | 1.02368 | 0.02543 | 0.02484 | 0.65 | 1.46285 | 0.25122 | 0.17174 | 1.15 | 2.58340 | 0.67729 | 0.26217 |
| 0.16 | 1.02695 | 0.02807 | 0.02733 | 0.66 | 1.47782 | 0.25761 | 0.17431 | 1.16 | 2.61470 | 0.68840 | 0.26328 |
| 0.17 | 1.03043 | 0.03080 | 0.02989 | 0.67 | 1.49307 | 0.26406 | 0.17686 | 1.17 | 2.64639 | 0.69963 | 0.26437 |
| 0.18 | 1.03413 | 0.03362 | 0.03251 | 0.68 | 1.50858 | 0.27060 | 0.17937 | 1.18 | 2.67850 | 0.71098 | 0.26544 |
| 0.19 | 1.03804 | 0.03652 | 0.03518 | 0.69 | 1.52436 | 0.27721 | 0.18185 | 1.19 | 2.71101 | 0.72244 | 0.26649 |
| 0.20 | 1.04217 | 0.03951 | 0.03792 | 0.70 | 1.54042 | 0.28390 | 0.18430 | 1.20 | 2.74394 | 0.73404 | 0.26751 |
| 0.21 | 1.04651 | 0.04259 | 0.04070 | 0.71 | 1.55675 | 0.29067 | 0.18672 | 1.21 | 2.77729 | 0.74575 | 0.26852 |
| 0.22 | 1.05107 | 0.04574 | 0.04352 | 0.72 | 1.57336 | 0.29752 | 0.18910 | 1.22 | 2.81106 | 0.75759 | 0.26950 |
| 0.23 | 1.05584 | 0.04898 | 0.04639 | 0.73 | 1.59025 | 0.30445 | 0.19145 | 1.23 | 2.84526 | 0.76956 | 0.27047 |
| 0.24 | 1.06083 | 0.05230 | 0.04930 | 0.74 | 1.60742 | 0.31146 | 0.19376 | 1.24 | 2.87989 | 0.78156 | 0.27142 |
| 0.25 | 1.06604 | 0.05569 | 0.05224 | 0.75 | 1.62498 | 0.31855 | 0.19605 | 1.25 | 2.91496 | 0.79389 | 0.27235 |
| 0.26 | 1.07147 | 0.05917 | 0.05522 | 0.76 | 1.64262 | 0.32572 | 0.19830 | 1.26 | 2.95046 | 0.80624 | 0.27326 |
| 0.27 | 1.07711 | 0.06272 | 0.05823 | 0.77 | 1.66064 | 0.33298 | 0.20051 | 1.27 | 2.98641 | 0.81873 | 0.27415 |
| 0.28 | 1.08297 | 0.06634 | 0.06126 | 0.78 | 1.67896 | 0.34032 | 0.20269 | 1.28 | 3.02281 | 0.83136 | 0.27503 |
| 0.29 | 1.08906 | 0.07004 | 0.06431 | 0.79 | 1.69757 | 0.34774 | 0.20484 | 1.29 | 3.05966 | 0.84411 | 0.27589 |
| 0.30 | 1.09536 | 0.07382 | 0.06739 | 0.80 | 1.71647 | 0.35524 | 0.20696 | 1.30 | 3.09696 | 0.85701 | 0.27673 |
| 0.31 | 1.10189 | 0.07767 | 0.07048 | 0.81 | 1.73567 | 0.36283 | 0.20905 | 1.31 | 3.13473 | 0.87004 | 0.27755 |
| 0.32 | 1.10864 | 0.08159 | 0.07359 | 0.82 | 1.75516 | 0.37051 | 0.21110 | 1.32 | 3.17296 | 0.88321 | 0.27836 |
| 0.33 | 1.11561 | 0.08558 | 0.07671 | 0.83 | 1.77496 | 0.37827 | 0.21311 | 1.33 | 3.21166 | 0.89653 | 0.27915 |
| 0.34 | 1.12280 | 0.08965 | 0.07985 | 0.84 | 1.79506 | 0.38612 | 0.21510 | 1.34 | 3.25083 | 0.90998 | 0.27992 |
| 0.35 | 1.13022 | 0.09379 | 0.08298 | 0.85 | 1.81547 | 0.39406 | 0.21706 | 1.35 | 3.29049 | 0.92358 | 0.28068 |
| 0.36 | 1.13787 | 0.09800 | 0.08613 | 0.86 | 1.83619 | 0.40208 | 0.21898 | 1.36 | 3.33063 | 0.93733 | 0.28143 |
| 0.37 | 1.14574 | 0.10229 | 0.08927 | 0.87 | 1.85721 | 0.41020 | 0.22087 | 1.37 | 3.37126 | 0.95122 | 0.28216 |
| 0.38 | 1.15384 | 0.10664 | 0.09242 | 0.88 | 1.87855 | 0.41840 | 0.22273 | 1.38 | 3.41238 | 0.96526 | 0.28287 |
| 0.39 | 1.16217 | 0.11107 | 0.09557 | 0.89 | 1.90021 | 0.42670 | 0.22455 | 1.39 | 3.45400 | 0.97945 | 0.28357 |
| 0.40 | 1.17073 | 0.11556 | 0.09871 | 0.90 | 1.92218 | 0.43509 | 0.22635 | 1.40 | 3.49612 | 0.99379 | 0.28426 |
| 0.41 | 1.17952 | 0.12013 | 0.10185 | 0.91 | 1.94448 | 0.44357 | 0.22812 | 1.41 | 3.53875 | 1.00829 | 0.28493 |
| 0.42 | 1.18854 | 0.12477 | 0.10498 | 0.92 | 1.96710 | 0.45214 | 0.22985 | 1.42 | 3.58189 | 1.02294 | 0.28559 |
| 0.43 | 1.19779 | 0.12948 | 0.10810 | 0.93 | 1.99004 | 0.46081 | 0.23156 | 1.43 | 3.62555 | 1.03774 | 0.28623 |
| 0.44 | 1.20728 | 0.13426 | 0.11121 | 0.94 | 2.01332 | 0.46958 | 0.23323 | 1.44 | 3.66974 | 1.05271 | 0.28686 |
| 0.45 | 1.21700 | 0.13911 | 0.11431 | 0.95 | 2.03692 | 0.47843 | 0.23488 | 1.45 | 3.71445 | 1.06783 | 0.28748 |
| 0.46 | 1.22696 | 0.14403 | 0.11739 | 0.96 | 2.06086 | 0.48739 | 0.23650 | 1.46 | 3.75970 | 1.08312 | 0.28809 |
| 0.47 | 1.23716 | 0.14903 | 0.12045 | 0.97 | 2.08514 | 0.49645 | 0.23809 | 1.47 | 3.80549 | 1.09857 | 0.28868 |
| 0.48 | 1.24760 | 0.15409 | 0.12351 | 0.98 | 2.10976 | 0.50560 | 0.23965 | 1.48 | 3.85182 | 1.11419 | 0.28926 |
| 0.49 | 1.25827 | 0.15922 | 0.12654 | 0.99 | 2.13472 | 0.51485 | 0.24118 | 1.49 | 3.89870 | 1.12997 | 0.28983 |
| 0.50 | 1.26919 | 0.16443 | 0.12956 | 1.00 | 2.16003 | 0.52421 | 0.24268 | 1.50 | 3.94614 | 1.14592 | 0.29039 |

TABLE D.XXIVA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/21$ and $x$ from 0.00 to 1.50.

$\alpha = 5/21$

| x | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ |
|---|---|---|---|
| 1.50 | 3.94614 | 1.14592 | 0.29039 |
| 1.51 | 3.99414 | 1.16204 | 0.29094 |
| 1.52 | 4.04270 | 1.17834 | 0.29147 |
| 1.53 | 4.09184 | 1.19480 | 0.29200 |
| 1.54 | 4.14156 | 1.21145 | 0.29251 |
| 1.55 | 4.19186 | 1.22827 | 0.29301 |
| 1.56 | 4.24275 | 1.24528 | 0.29351 |
| 1.57 | 4.29424 | 1.26246 | 0.29399 |
| 1.58 | 4.34633 | 1.27983 | 0.29446 |
| 1.59 | 4.39903 | 1.29738 | 0.29493 |
| 1.60 | 4.45234 | 1.31513 | 0.29538 |
| 1.61 | 4.50627 | 1.33306 | 0.29582 |
| 1.62 | 4.56083 | 1.35118 | 0.29626 |
| 1.63 | 4.61603 | 1.36950 | 0.29668 |
| 1.64 | 4.67186 | 1.38801 | 0.29710 |
| 1.65 | 4.72834 | 1.40673 | 0.29751 |
| 1.66 | 4.78548 | 1.42564 | 0.29791 |
| 1.67 | 4.84327 | 1.44476 | 0.29830 |
| 1.68 | 4.90173 | 1.46408 | 0.29869 |
| 1.69 | 4.96086 | 1.48360 | 0.29906 |
| 1.70 | 5.02068 | 1.50334 | 0.29943 |
| 1.71 | 5.08118 | 1.52329 | 0.29979 |
| 1.72 | 5.14238 | 1.54345 | 0.30014 |
| 1.73 | 5.20427 | 1.56383 | 0.30049 |
| 1.74 | 5.26688 | 1.58442 | 0.30083 |
| 1.75 | 5.33020 | 1.60524 | 0.30116 |
| 1.76 | 5.39425 | 1.62628 | 0.30148 |
| 1.77 | 5.45903 | 1.64755 | 0.30180 |
| 1.78 | 5.52454 | 1.66904 | 0.30211 |
| 1.79 | 5.59081 | 1.69077 | 0.30242 |
| 1.80 | 5.65782 | 1.71272 | 0.30272 |
| 1.81 | 5.72560 | 1.73492 | 0.30301 |
| 1.82 | 5.79415 | 1.75735 | 0.30330 |
| 1.83 | 5.86348 | 1.78002 | 0.30358 |
| 1.84 | 5.93359 | 1.80293 | 0.30385 |
| 1.85 | [illegible] | 1.82609 | 0.30412 |
| 1.86 | 6.07620 | 1.84950 | 0.30438 |
| 1.87 | 6.14872 | 1.87316 | 0.30464 |
| 1.88 | 6.22206 | [illegible] | 0.30490 |
| 1.89 | 6.29623 | 1.92125 | 0.30514 |
| 1.90 | 6.37123 | 1.94568 | 0.30539 |
| 1.91 | 6.44708 | 1.97037 | 0.30562 |
| 1.92 | 6.52378 | 1.99533 | 0.30585 |
| 1.93 | 6.60135 | 2.02056 | 0.30608 |
| 1.94 | 6.67978 | 2.04605 | 0.30631 |
| 1.55 | 6.75910 | 2.07182 | 0.30652 |
| 1.96 | 6.83931 | 2.09787 | 0.30674 |
| 1.97 | 6.92041 | 2.12420 | 0.30695 |
| 1.98 | 7.00243 | 2.15081 | 0.30715 |
| 1.99 | 7.08536 | 2.17771 | 0.30735 |
| 2.00 | 7.16923 | 2.20489 | 0.30755 |

| x | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ |
|---|---|---|---|
| 2.0 | 7.16923 | 2.20489 | 0.30755 |
| 2.1 | 8.06109 | 2.49333 | 0.30930 |
| 2.2 | 9.05716 | 2.81633 | 0.31073 |
| 2.3 | 10.16895 | 3.17156 | 0.31189 |
| 2.4 | 11.40929 | 3.56912 | 0.31283 |
| 2.5 | 12.79243 | 4.01157 | 0.31359 |
| 2.6 | 14.33422 | 4.50395 | 0.31421 |
| 2.7 | 16.05230 | 5.05188 | 0.31471 |
| 2.8 | 17.96624 | 5.66160 | 0.31512 |
| 2.9 | 20.09781 | 6.34003 | 0.31546 |
| 3.0 | 22.47120 | 7.09484 | 0.31573 |
| 3.1 | 25.11324 | 7.93457 | 0.31595 |
| 3.2 | 28.05377 | 8.86870 | 0.31613 |
| 3.3 | 31.32590 | 9.90773 | 0.31628 |
| 3.4 | 34.96638 | 11.06332 | 0.31640 |
| 3.5 | 39.01602 | 12.34844 | 0.31650 |
| 3.6 | 43.52012 | 13.77744 | 0.31658 |
| 3.7 | 48.52897 | 15.36627 | 0.31634 |
| 3.8 | 54.09836 | 17.13264 | 0.31669 |
| 3.9 | 60.29020 | 19.09617 | 0.31674 |
| 4.0 | 67.17322 | 21.27865 | 0.31677 |
| 4.1 | 74.82365 | 23.70426 | 0.31680 |
| 4.2 | 83.32607 | 26.39981 | 0.31683 |
| 4.3 | 92.77435 | 29.39504 | 0.31684 |
| 4.4 | 103.27258 | 32.72297 | 0.31686 |
| 4.5 | 114.93627 | 36.42021 | 0.31687 |
| 4.6 | 127.89351 | 40.52735 | 0.31688 |
| 4.7 | 142.28642 | 45.08944 | 0.31689 |
| 4.8 | 158.27260 | 50.15644 | 0.31690 |
| 4.9 | 176.02686 | 55.78376 | 0.31690 |
| 5.0 | 195.74306 | 62.03283 | 0.31691 |
| 5.1 | 217.63620 | 68.97181 | 0.31691 |
| 5.2 | 241.94470 | 76.67625 | 0.31692 |
| 5.3 | 268.93293 | 85.22995 | 0.31692 |
| 5.4 | 298.89406 | 94.72582 | 0.31692 |
| 5.5 | 332.15314 | 105.26690 | 0.31692 |
| 5.6 | 369.07059 | 116.96739 | 0.31692 |
| 5.7 | 410.04600 | 129.95395 | 0.31693 |
| 5.8 | 455.52239 | 144.36700 | 0.31693 |
| 5.9 | 505.99091 | 160.36218 | 0.31693 |
| 6.0 | 561.99606 | 178.11208 | 0.31693 |

| x | $F_{5/21}(x)$ | $H_{16/21}(x)$ | $T_{5/21}(x)$ |
|---|---|---|---|
| 6.0 | 561.99606 | 178.11208 | 0.31693 |
| 6.1 | 624.14143 | 197.80798 | 0.31693 |
| 6.2 | 693.09612 | 219.66195 | 0.31693 |
| 6.3 | 769.60184 | 243.90906 | 0.31693 |
| 6.4 | 854.48071 | 270.80986 | 0.31693 |
| 6.5 | 948.64398 | 300.65317 | 0.31693 |
| 6.6 | 1053.10168 | 333.75908 | 0.31693 |
| 6.7 | 1168.97324 | 370.48239 | 0.31693 |
| 6.8 | 1297.49938 | 411.21630 | 0.31693 |
| 6.9 | 1440.05512 | 456.39662 | 0.31693 |
| 7.0 | 1598.16435 | 506.50631 | 0.31693 |
| 7.1 | 1773.51585 | 562.08059 | 0.31693 |
| 7.2 | 1967.98111 | 623.71260 | 0.31693 |
| 7.3 | 2183.63403 | 692.05941 | 0.31693 |
| 7.4 | 2422.77275 | 767.84999 | 0.31693 |
| 7.5 | 2687.94382 | 851.89061 | 0.31693 |
| 7.6 | 2981.96898 | 945.07636 | 0.31693 |
| 7.7 | 3307.97487 | 1048.39758 | 0.31693 |
| 7.8 | 3669.42379 | 1162.95240 | 0.31693 |
| 7.9 | 4070.16016 | 1289.95734 | 0.31693 |
| 8.0 | 4514.43079 | 1430.76024 | 0.31693 |
| 8.1 | 5006.94950 | 1586.85443 | 0.31693 |
| 8.2 | 5552.93656 | 1759.89436 | 0.31693 |
| 8.3 | 6158.17538 | 1951.71297 | 0.31693 |
| 8.4 | 6829.07318 | 2164.34092 | 0.31693 |
| 8.5 | 7572.72810 | 2400.02780 | 0.31693 |
| 8.6 | 8397.00349 | 2661.26576 | 0.31693 |
| 8.7 | 9310.61029 | 2950.81557 | 0.31693 |
| 8.8 | 10323.19814 | 3271.73548 | 0.31693 |
| 8.9 | 11445.45635 | 3627.41326 | 0.31693 |
| 9.0 | 12689.22565 | 4021.60160 | 0.31693 |
| 9.1 | 14067.62200 | 4458.45735 | 0.31693 |
| 9.2 | 15595.17360 | 4942.58494 | 0.31693 |
| 9.3 | 17287.97268 | 5479.08448 | 0.31693 |
| 9.4 | 19163.84354 | 6073.60503 | 0.31693 |
| 9.5 | 21242.52857 | 6732.40357 | 0.31693 |
| 9.6 | 23545.89436 | 7462.41028 | 0.31693 |
| 9.7 | 26098.15971 | 8271.30082 | 0.31693 |
| 9.8 | 28926.14825 | 9167.57644 | 0.31693 |
| 9.9 | 32059.56811 | 10160.65253 | 0.31693 |
| 10.0 | 35531.32147 | 11260.95681 | 0.31693 |

TABLE D.XXIVB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 5/21$ and $x$ from 1.50 to 10.0.

α = 16/21

| x | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ | x | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ | x | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 0.0 | 0.0 | 0.50 | 1.08350 | 2.28156 | 2.10575 | 1.00 | 1.35212 | 3.66387 | 2.70972 |
| 0.01 | 1.00003 | 0.33692 | 0.33691 | 0.51 | 1.08693 | 2.30779 | 2.12322 | 1.01 | 1.35970 | 3.69503 | 2.71752 |
| 0.02 | 1.00013 | 0.46871 | 0.46865 | 0.52 | 1.09044 | 2.33397 | 2.14039 | 1.02 | 1.36738 | 3.72639 | 2.72520 |
| 0.03 | 1.00030 | 0.56859 | 0.56842 | 0.53 | 1.09402 | 2.36011 | 2.15728 | 1.03 | 1.37516 | 3.75797 | 2.73275 |
| 0.04 | 1.00053 | 0.65216 | 0.65182 | 0.54 | 1.09768 | 2.38624 | 2.17389 | 1.04 | 1.38304 | 3.78976 | 2.74017 |
| 0.05 | 1.00082 | 0.72541 | 0.72481 | 0.55 | 1.10141 | 2.41234 | 2.19024 | 1.05 | 1.39101 | 3.82177 | 2.74747 |
| 0.06 | 1.00118 | 0.79138 | 0.79045 | 0.56 | 1.10521 | 2.43844 | 2.20632 | 1.06 | 1.39909 | 3.85400 | 2.75466 |
| 0.07 | 1.00161 | 0.85168 | 0.85051 | 0.57 | 1.10909 | 2.46455 | 2.22213 | 1.07 | 1.40726 | 3.88646 | 2.76172 |
| 0.08 | 1.00210 | 0.90808 | 0.90618 | 0.58 | 1.11304 | 2.49065 | 2.23770 | 1.08 | 1.41553 | 3.91914 | 2.76867 |
| 0.09 | 1.00266 | 0.96080 | 0.95825 | 0.59 | 1.11707 | 2.51677 | 2.25301 | 1.09 | 1.42391 | 3.95206 | 2.77551 |
| 0.10 | 1.00328 | 1.01062 | 1.00731 | 0.60 | 1.12117 | 2.54291 | 2.26808 | 1.10 | 1.43238 | 3.98522 | 2.78223 |
| 0.11 | 1.00397 | 1.05799 | 1.05380 | 0.61 | 1.12535 | 2.56908 | 2.28291 | 1.11 | 1.44096 | 4.01861 | 2.78884 |
| 0.12 | 1.00473 | 1.10326 | 1.09407 | 0.62 | 1.12961 | 2.59528 | 2.29750 | 1.12 | 1.44965 | 4.05225 | 2.79534 |
| 0.13 | 1.00555 | 1.14670 | 1.14037 | 0.63 | 1.13394 | 2.62151 | 2.31185 | 1.13 | 1.45843 | 4.08614 | 2.80173 |
| 0.14 | 1.00644 | 1.18854 | 1.18093 | 0.64 | 1.13835 | 2.64779 | 2.32598 | 1.14 | 1.46732 | 4.12028 | 2.80802 |
| 0.15 | 1.00739 | 1.22895 | 1.21993 | 0.65 | 1.14284 | 2.67412 | 2.33989 | 1.15 | 1.47632 | 4.15467 | 2.81420 |
| 0.16 | 1.00842 | 1.26810 | 1.25752 | 0.66 | 1.14741 | 2.70051 | 2.35357 | 1.16 | 1.48543 | 4.18932 | 2.82028 |
| 0.17 | 1.00950 | 1.30611 | 1.29382 | 0.67 | 1.15205 | 2.72695 | 2.36704 | 1.17 | 1.49464 | 4.22423 | 2.82626 |
| 0.18 | 1.01066 | 1.34309 | 1.32893 | 0.68 | 1.15677 | 2.75346 | 2.38030 | 1.18 | 1.50395 | 4.25941 | 2.83214 |
| 0.19 | 1.01189 | 1.37915 | 1.36296 | 0.69 | 1.16157 | 2.78004 | 2.39334 | 1.19 | 1.51338 | 4.29486 | 2.83792 |
| 0.20 | 1.01315 | 1.41436 | 1.39599 | 0.70 | 1.16645 | 2.80670 | 2.40618 | 1.20 | 1.52292 | 4.33058 | 2.84360 |
| 0.21 | 1.01452 | 1.44880 | 1.42807 | 0.71 | 1.17141 | 2.83344 | 2.41882 | 1.21 | 1.53257 | 4.36658 | 2.84919 |
| 0.22 | 1.01594 | 1.48253 | 1.45928 | 0.72 | 1.17645 | 2.86026 | 2.43126 | 1.22 | 1.54233 | 4.40285 | 2.85468 |
| 0.23 | 1.01742 | 1.51562 | 1.48967 | 0.73 | 1.18158 | 2.88718 | 2.44350 | 1.23 | 1.55220 | 4.43941 | 2.86008 |
| 0.24 | 1.01896 | 1.54811 | 1.51928 | 0.74 | 1.18678 | 2.91419 | 2.45554 | 1.24 | 1.56218 | 4.47626 | 2.86539 |
| 0.25 | 1.02060 | 1.58005 | 1.54816 | 0.75 | 1.19206 | 2.94130 | 2.46740 | 1.25 | 1.57228 | 4.51340 | 2.87061 |
| 0.26 | 1.02229 | 1.61149 | 1.57636 | 0.76 | 1.19743 | 2.96851 | 2.47907 | 1.26 | 1.58249 | 4.55084 | 2.87574 |
| 0.27 | 1.02404 | 1.64246 | 1.60359 | 0.77 | 1.20288 | 2.99583 | 2.49056 | 1.27 | 1.59282 | 4.58857 | 2.88078 |
| 0.28 | 1.02587 | 1.67300 | 1.63081 | 0.78 | 1.20841 | 3.02327 | 2.50186 | 1.28 | 1.60326 | 4.62660 | 2.88574 |
| 0.29 | 1.02776 | 1.70313 | 1.65713 | 0.79 | 1.21402 | 3.05082 | 2.51299 | 1.29 | 1.61383 | 4.66495 | 2.89061 |
| 0.30 | 1.02972 | 1.73290 | 1.68288 | 0.80 | 1.21972 | 3.07850 | 2.52394 | 1.30 | 1.62451 | 4.70360 | 2.89540 |
| 0.31 | 1.03175 | 1.76232 | 1.70809 | 0.81 | 1.22550 | 3.10630 | 2.53471 | 1.31 | 1.63531 | 4.74257 | 2.90011 |
| 0.32 | 1.03384 | 1.79142 | 1.73278 | 0.82 | 1.23137 | 3.13423 | 2.54532 | 1.32 | 1.64622 | 4.78185 | 2.90474 |
| 0.33 | 1.03601 | 1.82023 | 1.75696 | 0.83 | 1.23733 | 3.16230 | 2.55576 | 1.33 | 1.65726 | 4.82146 | 2.90929 |
| 0.34 | 1.03824 | 1.84876 | 1.78067 | 0.84 | 1.24336 | 3.19051 | 2.56603 | 1.34 | 1.66843 | 4.86139 | 2.91376 |
| 0.35 | 1.04055 | 1.87705 | 1.80391 | 0.85 | 1.24949 | 3.21886 | 2.57614 | 1.35 | 1.67971 | 4.90165 | 2.91815 |
| 0.36 | 1.04292 | 1.90509 | 1.82670 | 0.86 | 1.25570 | 3.24735 | 2.58608 | 1.36 | 1.69112 | 4.94224 | 2.92246 |
| 0.37 | 1.04536 | 1.93293 | 1.84906 | 0.87 | 1.26200 | 3.27600 | 2.59587 | 1.37 | 1.70266 | 4.98318 | 2.92671 |
| 0.38 | 1.04787 | 1.96056 | 1.87100 | 0.88 | 1.26839 | 3.30480 | 2.60550 | 1.38 | 1.71432 | 5.02445 | 2.93088 |
| 0.39 | 1.05045 | 1.98801 | 1.89253 | 0.89 | 1.27487 | 3.33376 | 2.61498 | 1.39 | 1.72610 | 5.06607 | 2.93497 |
| 0.40 | 1.05310 | 2.01529 | 1.91368 | 0.90 | 1.28143 | 3.36288 | 2.62431 | 1.40 | 1.73802 | 5.10803 | 2.93900 |
| 0.41 | 1.05582 | 2.04242 | 1.93444 | 0.91 | 1.28809 | 3.39217 | 2.63349 | 1.41 | 1.75006 | 5.15035 | 2.94295 |
| 0.42 | 1.05861 | 2.06940 | 1.95483 | 0.92 | 1.29483 | 3.42162 | 2.64252 | 1.42 | 1.76224 | 5.19303 | 2.94684 |
| 0.43 | 1.06147 | 2.09626 | 1.97486 | 0.93 | 1.30167 | 3.45126 | 2.65141 | 1.43 | 1.77454 | 5.23607 | 2.95066 |
| 0.44 | 1.06440 | 2.12299 | 1.99454 | 0.94 | 1.30860 | 3.48107 | 2.66015 | 1.44 | 1.78698 | 5.27947 | 2.95441 |
| 0.45 | 1.06741 | 2.14963 | 2.01388 | 0.95 | 1.31562 | 3.51106 | 2.66875 | 1.45 | 1.79955 | 5.32325 | 2.95810 |
| 0.46 | 1.07048 | 2.17617 | 2.03289 | 0.96 | 1.32273 | 3.54123 | 2.67722 | 1.46 | 1.81225 | 5.36739 | 2.96172 |
| 0.47 | 1.07363 | 2.20262 | 2.05157 | 0.97 | 1.32994 | 3.57160 | 2.68554 | 1.47 | 1.82509 | 5.41192 | 2.96528 |
| 0.48 | 1.07684 | 2.22900 | 2.06994 | 0.98 | 1.33724 | 3.60216 | 2.69374 | 1.48 | 1.83807 | 5.45683 | 2.96878 |
| 0.49 | 1.08013 | 2.25532 | 2.08800 | 0.99 | 1.34463 | 3.63291 | 2.70180 | 1.49 | 1.85119 | 5.50212 | 2.97221 |
| 0.50 | 1.08350 | 2.28156 | 2.10575 | 1.00 | 1.35212 | 3.66387 | 2.70972 | 1.50 | 1.86444 | 5.54780 | 2.97559 |

TABLE D.XXVA. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 16/21$ and x from 0.00 to 1.50.

$\alpha = 16/21$

| $x$ | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ |
|---|---|---|---|
| 1.50 | 1.86444 | 5.54780 | 2.97559 |
| 1.51 | 1.87783 | 5.59388 | 2.97890 |
| 1.52 | 1.89137 | 5.64035 | 2.98216 |
| 1.53 | 1.90504 | 5.68723 | 2.98536 |
| 1.54 | 1.91886 | 5.73452 | 2.98850 |
| 1.55 | 1.93283 | 5.78222 | 2.99159 |
| 1.56 | 1.94694 | 5.83033 | 2.99462 |
| 1.57 | 1.96119 | 5.87887 | 2.99760 |
| 1.58 | 1.97560 | 5.92783 | 3.00052 |
| 1.59 | 1.99016 | 5.97722 | 3.00339 |
| 1.60 | 2.00486 | 6.02705 | 3.00622 |
| 1.61 | 2.01972 | 6.07731 | 3.00899 |
| 1.62 | 2.03473 | 6.12802 | 3.01171 |
| 1.63 | 2.04990 | 6.17917 | 3.01438 |
| 1.64 | 2.06522 | 6.23078 | 3.01701 |
| 1.65 | 2.08070 | 6.28285 | 3.01959 |
| 1.66 | 2.09634 | 6.33538 | 3.02212 |
| 1.67 | 2.11214 | 6.38837 | 3.02460 |
| 1.68 | 2.12809 | 6.44184 | 3.02704 |
| 1.69 | 2.14422 | 6.49578 | 3.02944 |
| 1.70 | 2.16050 | 6.55021 | 3.03180 |
| 1.71 | 2.17696 | 6.60512 | 3.03411 |
| 1.72 | 2.19358 | 6.66052 | 3.03638 |
| 1.73 | 2.21036 | 6.71642 | 3.03861 |
| 1.74 | 2.22732 | 6.77283 | 3.04079 |
| 1.75 | 2.24445 | 6.82974 | 3.04294 |
| 1.76 | 2.26175 | 6.88716 | 3.04505 |
| 1.77 | 2.27923 | 6.94509 | 3.04712 |
| 1.78 | 2.29688 | 7.00355 | 3.04916 |
| 1.79 | 2.31471 | 7.06254 | 3.05115 |
| 1.80 | 2.33272 | 7.12206 | 3.05311 |
| 1.81 | 2.35091 | 7.18212 | 3.05504 |
| 1.82 | 2.36928 | 7.24272 | 3.05693 |
| 1.83 | 2.38784 | 7.30387 | 3.05878 |
| 1.84 | 2.40658 | 7.36558 | 3.06060 |
| 1.85 | 2.42551 | 7.42784 | 3.06239 |
| 1.86 | 2.44462 | 7.49068 | 3.06414 |
| 1.87 | 2.46393 | 7.55408 | 3.06587 |
| 1.88 | 2.48343 | 7.61806 | 3.06756 |
| 1.89 | 2.50312 | 7.68262 | 3.06922 |
| 1.90 | 2.52301 | 7.74777 | 3.07084 |
| 1.91 | 2.54310 | 7.81352 | 3.07244 |
| 1.92 | 2.56338 | 7.87986 | 3.07401 |
| 1.93 | 2.58387 | 7.94682 | 3.07555 |
| 1.94 | 2.60455 | 8.01438 | 3.07706 |
| 1.95 | 2.62545 | 8.08256 | 3.07855 |
| 1.96 | 2.64654 | 8.15137 | 3.08000 |
| 1.97 | 2.66785 | 8.22081 | 3.08143 |
| 1.98 | 2.68937 | 8.29068 | 3.08284 |
| 1.99 | 2.71109 | 8.36160 | 3.08421 |
| 2.00 | 2.73304 | 8.43296 | 3.08557 |

| $x$ | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ |
|---|---|---|---|
| 2.0 | 2.73304 | 8.43296 | 3.08557 |
| 2.1 | 2.96463 | 9.18375 | 3.09777 |
| 2.2 | 3.21983 | 10.00683 | 3.10787 |
| 2.3 | 3.50087 | 10.90952 | 3.11623 |
| 2.4 | 3.81023 | 11.89985 | 3.12313 |
| 2.5 | 4.15063 | 12.98661 | 3.12883 |
| 2.6 | 4.52509 | 14.17947 | 3.13352 |
| 2.7 | 4.93692 | 15.48905 | 3.13739 |
| 2.8 | 5.38977 | 16.92700 | 3.14058 |
| 2.9 | 5.88768 | 18.50614 | 3.14320 |
| 3.0 | 6.43507 | 20.24059 | 3.14536 |
| 3.1 | 7.03684 | 22.14584 | 3.14713 |
| 3.2 | 7.69836 | 24.23896 | 3.14859 |
| 3.3 | 8.42557 | 26.53873 | 3.14979 |
| 3.4 | 9.22499 | 29.06581 | 3.15077 |
| 3.5 | 10.10381 | 31.84294 | 3.15158 |
| 3.6 | 11.06995 | 34.89515 | 3.15224 |
| 3.7 | 12.13214 | 38.25000 | 3.15278 |
| 3.8 | 13.29996 | 41.93784 | 3.15323 |
| 3.9 | 14.58400 | 45.99205 | 3.15360 |
| 4.0 | 15.99589 | 50.44942 | 3.15390 |
| 4.1 | 17.54848 | 55.35044 | 3.15414 |
| 4.2 | 19.25568 | 60.73971 | 3.15435 |
| 4.3 | 21.13364 | 66.66633 | 3.15451 |
| 4.4 | 23.19890 | 73.18439 | 3.15465 |
| 4.5 | 25.47054 | 80.35344 | 3.15476 |
| 4.6 | 27.96934 | 88.23910 | 3.15485 |
| 4.7 | 30.71819 | 96.91362 | 3.15493 |
| 4.8 | 33.74231 | 106.45659 | 3.15499 |
| 4.9 | 37.06950 | 116.95567 | 3.15504 |
| 5.0 | 40.73034 | 128.50745 | 3.15508 |
| 5.1 | 44.75855 | 141.21830 | 3.15511 |
| 5.2 | 49.19129 | 155.20544 | 3.15514 |
| 5.3 | 54.06945 | 170.59797 | 3.15516 |
| 5.4 | 59.43814 | 187.53817 | 3.15518 |
| 5.5 | 65.34702 | 206.18274 | 3.15520 |
| 5.6 | 71.85081 | 226.70439 | 3.15521 |
| 5.7 | 79.00982 | 249.29339 | 3.15522 |
| 5.8 | 86.89050 | 274.15939 | 3.15523 |
| 5.9 | 95.56606 | 301.53341 | 3.15524 |
| 6.0 | 105.11717 | 331.67003 | 3.15524 |

| $x$ | $F_{16/21}(x)$ | $H_{5/21}(x)$ | $T_{16/21}(x)$ |
|---|---|---|---|
| 6.0 | 105.11717 | 331.67003 | 3.15524 |
| 6.1 | 115.63276 | 364.84977 | 3.15525 |
| 6.2 | 127.21079 | 401.38180 | 3.15525 |
| 6.3 | 139.95925 | 441.60680 | 3.15525 |
| 6.4 | 153.99713 | 485.90024 | 3.15526 |
| 6.5 | 169.45557 | 534.67593 | 3.15526 |
| 6.6 | 186.47913 | 588.38993 | 3.15526 |
| 6.7 | 205.22710 | 647.54492 | 3.15526 |
| 6.8 | 225.87507 | 712.69492 | 3.15526 |
| 6.9 | 248.61659 | 784.45058 | 3.15526 |
| 7.0 | 273.66498 | 863.48502 | 3.15526 |
| 7.1 | 301.25537 | 950.54017 | 3.15526 |
| 7.2 | 331.64698 | 1046.43388 | 3.15526 |
| 7.3 | 365.12552 | 1152.06769 | 3.15526 |
| 7.4 | 402.0059[illegible] | 1268.43541 | 3.15527 |
| 7.5 | 442.63554 | 1396.63260 | 3.15527 |
| 7.6 | 487.39701 | 1537.86701 | 3.15527 |
| 7.7 | 536.71235 | 1693.47011 | 3.15527 |
| 7.8 | 591.04679 | 1864.90980 | 3.15527 |
| 7.9 | 650.91321 | 2053.80437 | 3.15527 |
| 8.0 | 716.87709 | 2261.93803 | 3.15527 |
| 8.1 | 789.56187 | 2491.27791 | 3.15527 |
| 8.2 | 869.65491 | 2743.99286 | 3.15527 |
| 8.3 | 957.91412 | 3022.47422 | 3.15527 |
| 8.4 | 1055.17512 | 3329.35863 | 3.15527 |
| 8.5 | 1162.35930 | 3667.55333 | 3.15527 |
| 8.6 | 1280.48263 | 4040.26395 | 3.15527 |
| 8.7 | 1410.66536 | 4451.02519 | 3.15527 |
| 8.8 | 1554.14278 | 4903.73471 | 3.15527 |
| 8.9 | 1712.27703 | 5402.69045 | 3.15527 |
| 9.0 | 1886.57019 | 5952.63186 | 3.15527 |
| 9.1 | 2078.67866 | 6558.78532 | 3.15527 |
| 9.2 | 2290.42904 | 7226.91425 | 3.15527 |
| 9.3 | 2523.83568 | 7963.37445 | 3.15527 |
| 9.4 | 2781.12001 | 8775.17510 | 3.15527 |
| 9.5 | 3064.73183 | 9670.04603 | 3.15527 |
| 9.6 | 3377.37290 | 10656.51197 | 3.15527 |
| 9.7 | 3722.02284 | 11743.97446 | 3.15527 |
| 9.8 | 4101.96782 | 12942.80217 | 3.15527 |
| 9.9 | 4520.83209 | 14264.43063 | 3.15527 |
| 10.0 | 4982.61287 | 15721.47212 | 3.15527 |

TABLE D.XXVB. LANCHESTER-CLIFFORD-SCHLÄFLI Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 16/21$ and x from 1.50 to 10.0.

## Chapter 7. MODELLING TACTICAL ENGAGEMENTS

### 7.1. Introduction

The fundamental role of ground-combat troops (in the U.S. Army's own words, e.g. see [164, p. iv]) is to "shoot, move, and communicate." Consequently models of tactical engagements must in some manner represent the attendant processes of attrition, movement, and $C^3$ (i.e. command, control, and communications). In this chapter we will focus on the modelling of force-on-force attrition in tactical engagements, although some consideration does have to be given to the other two processes of movement and $C^3$, especially as they influence the attrition process. The two attrition-modelling approaches that are principally used in the United States for assessing casualties in simulated combat engagements and that we will examine in detail are as follows:

(A1) detailed LANCHESTER-type models of attrition in tactical engagements,

(A2) aggregated-force casualty-assessment models based on the use of index numbers to quantify military capabilities.

We will try to be fairly comprehensive in our examination of these two approaches for assessing casualties in tactical engagements, and when details must be omitted, references to further details in the literature will be given. Moreover, there is a third approach that also merits mention:

(A3) Coordinated use of a detailed combat model with a less detailed casualty-assessment model.

Although it has been rather widely used for defense-planning purposes in both England and West Germany, this third approach (i.e. the hierarchical-modelling approach) has not been as widely used in the United States as the first two. Consequently, we will only briefly discuss the hierarchical-modelling approach and not examine it in nearly as much detail as the other two.

Combat (especially that between company-sized units and larger) is a fantastically complex random process. Nevertheless, deterministic models of combat attrition are commonly used in studies for computational reasons, since many people believe that they give essentially the same results for the average course of combat as do corresponding stochastic models[1] and these stochastic attrition models are considerably less convenient to handle (<u>see</u> Chapter 4 for further details). Hence, in the chapter at hand we will consider only deterministic models of force-on-force attrition for assessing casualties in tactical engagements. Even so, the inherent complexity of the combat process leads to great complexity in operational models of combat attrition. However, for purposes of understanding the modelling approaches and concepts that may be used to build such operational models, it is convenient to abstract much simpler auxiliary models and to study them[2]. Thus, we will examine some simplified versions of tactical-engagement models, with the understanding that a more complicated model would be desirable for investigation of actual planning or operational problems.

As we indicated in Chapter 1, two divergent (but yet complementary) trends in the use of combat models are the following[3]:

(T1) their simplification in order to more easily obtain insights into the dynamics of combat,

(T2) their enrichment in details in order to better duplicate real-world combat activities.

In previous chapters we have concentrated primarily on obtaining insights into the dynamics of combat from relatively simple models rather than enriching such models in details. Thus, we have emphasized studying relatively simple combat models in order to better understand their basic nature and to hopefully perceive some significant interrelationships that are difficult to discern in more complicated models. However, such simple models may also be the point of departure for building complex operational models.

In other words, one approach for understanding the reasons why a large-scale complex operational model produces certain output results for particular numerical input data is to abstract a simpler model (e.g. one with fewer variables or simpler functional relations between them) from the complex one. This simple auxiliary model is then used to investigate the system dynamics of the more complex model by considering alternative assumptions and data estimates. The simplified auxiliary model should be intuitively plausible and transparent but yet it should capture the basic essence of the complex operational model. This idea of using relatively simple auxiliary models in conjunction with a complex operational model is, of course, not new[4], but the author knows of no clear articulation of this approach for understanding large-scale combat models. Thus, the simple models that we will consider in this chapter should not be taken literally but should be considered as a point of departure in the building of more

complex models enriched and elaborated upon in numerous details. In order that our simple models not be taken literally by the inexperienced modeller, we will explicitly discuss a few general ideas about modelling, the process of building a model. Our remakrs should provide some insight into how complex models like, for example, ATLAS, BONDER/IUA, and VECTOR-2 have evolved[5].

Many people (e.g. see MORRIS [114] or BONDER [12]) have come to realize that models and modelling are two completely different subjects. Thus, an individual can be quite knowledgeable about models (i.e. he may understand the assumptions on which they are based and also their characteristics and properties), but he may still be quite incapable of building his own model to fit given requirements of, for example, military analysis. It is not an easy task to adapt (i.e. to "bend and twist") a model to fit specific scenario and analysis requirements. Modelling (i.e. model building) is an art, which is probably best learned by active experiences (see BONDER [12] and MORRIS [114] for further discussions). Thus, the simple models presented in the rest of this chapter should not be considered as final products but rather should be considered as points of departure in the building of operational models.

W. T. MORRIS [114] has hypothesized that the process of model building may be considered to consist of the following three aspects:

(A1) the process of enriching or elaborating upon a basic logical structure,

(A2) the use of analogy or association with previously developed logical sturctures to determine the starting point for this enrichment process,

and (A3) the interactive (i.e. "looping") nature of the model-building process.

The enrichment process itself may be considered to consist of the following elements: (1) making constants into variables, (2) adding more variables, (3) using more complicated (i.e. nonlinear) functional relations between variables, (4) using weaker assumptions and restrictions and (5) not suppressing randomness. These general ideas about modelling should be kept in mind as we subsequently review models of combat attrition. Combat-modelling theories only provide the "skeleton," and the military operations research (OR) worker must add the "meat" to the body of the attrition model.

Let us finally make a few observations about the impact of the modern digital computers on modelling. The computer has essentially freed the military OR analyst from having to worry about mathematical tractability and allows him to focus on model formulation (i.e. model building). For example, with respect to attrition modelling, the military analyst's efforts should be focused on analyzing the combat process and formulating the appropriate casualty-assessment equations, since numerical results can always be generated with the help of a digital computer using standard numerical integration techniques. However, before the age of digital computers one had to worry about building "useful" models that could be conveniently "solved." Of course, the mathematical aspects of models are still important, since many times in the process of model building it is useful (even essential) to understand the mathematical properties of the logical structures being enriched in details.

## 7.2. Additional Operational Factors to be Considered in LANCHESTER-type Models.

In adapting LANCHESTER-type models to represent the dynamics of combat in actual tactical engagements, one should consider a number of additional operational factors that were omitted by the relatively simple models considered previously in this book. In particular LANCHESTER's classic combat formulations essentially considered only the fire effectiveness (assumed constant) and the numbers of opposing combatants. We can enrich such simple attrition models by considering additional operational factors such as those shown in Table 7.I in order to reflect more of the inherent complexity of combat (see also Sections 2.6 and 2.7 above).

The LANCHESTER-type models that we consider here and in Sections 7.4 and 7.8 are all deterministic in the sense that each of them will always yield the same output for a given set of input data. Even though combat between two military forces is a complex random process, such deterministic combat models are commonly used for computational reasons in defense-planning studies, for example, to assess the relative importance of various weapon-system and force-level parameters, since many people believe that they give essentially the same results for the mean course of combat as do corresponding stochastic attrition models[6].

Let us now briefly discuss the operational factors shown in Table 7.I. Some of them have been considered in previous portions of this book, and many will be further discussed in this chapter. To begin with, we have already discussed (see Section 5.5 above) how for "aimed" fire the corresponding LANCHESTER attrition-rate coefficients depend directly

TABLE 7.I. Additional Operational Factors to be Considered in LANCHESTER-Type Models.

(1) Range-dependent weapon-system capabilities

(2) Other temporal variations in fire effectiveness

(3) Unit breakpoints

(4) Unit deterioration due to attrition

(5) Target-acquisition considerations

(6) Diversity of weapon-system types

(7) Command, control, and communications

(8) Effects of terrain

(9) Suppressive effects of weapon systems

(10) Effects of logistics constraints

on factors such as firing rate, rate of target acquisition, hit probabilities, etc. and indirectly on factors such as range between firer and target, tactical postures of firers and targets, relative motion of firers and targets, etc. Many people (e.g. BONDER and FARRELL [15]) feel that for many tactical situations the principal factor is the range between firer and target, and we have examined the consequences of such *range dependence for attrition-rate coefficients* in BONDER's constant-speed-attack model (*see* Section 6.2 above). In other cases, however, one may want to have the attrition-rate coefficients also depend on *other operational factors* (e.g. firing rate, target posture, etc.) *that may change over time*.

We have already considered modelling battle termination through *unit breakpoints* and *unit deterioration due to attrition* in Chapter 3 (in particular, *see* Section 3.10; *see* also Section 2.8). Additionally, for combat between two homogeneous forces *target acquisition* is explicitly considered through $t_a$, which appears in (5.4.1) through (5.4.2), in BONDER's expression for the LANCHESTER attrition-rate coefficient in the case of MARKOV-dependent fire. In Section 5.10 we examined an important limiting case for such a coefficient when the constraining factor for killing targets is acquiring them (after ideas of H. BRACKNEY [20]). We found that under such conditions the rate of "aimed"-fire attrition took the form

$$\frac{dx}{dt} = -\tilde{a}xy \ , \qquad (7.2.1)$$

where $\hat{a}$ is a constant of proportionality related to the reciprocal of the time required to acquire a target by visually searching a region (see Section 5.10 for further details). Moreover, Vector Research, Inc. (see [154, pp. 103-108] or [117, pp. 43-45]) has developed a more refined (i.e. enriched in operational details) model for the target-acquisition process in engagements between heterogeneous forces and its consequent impact on the attrition process. Since we have not discussed heterogeneous forces yet, let us do so (and also command, control, and communications) before returning to a brief general discussion of target-acquisition effects (including terrain effects and target selection).

Actual combat (especially large-scale operations) consists of many different weapon-system types (e.g. infantry, tanks, artillery, mortars, etc.) operating together as "combined-arms teams," and such diversity of weapon-system types may be modelled by explicitly considering the attrition of each different type. In other words, attention is given to differences in weapon-system capability, and each side's forces are disaggregated by explicitly considering many different weapon-system types that can be individually attrited. We will consider in greater detail the modelling of attrition in combat between such heterogeneous forces in Section 7.7 below. Essentially one keeps track of the losses from all opposing weapon-sytem types for each target type. The extension of the attrition-modelling ideas of, for example, Chapter 2 is straightforward and is primarily a problem of bookkeeping and notation in the simplest case.

One may consider command, control, and communications ($C^3$) as influencing the efficiency of fire directed at enemy targets. Let us briefly examine

a simple model that was developed by T. S. SCHREIBER [127] and provides some insight into the contribution of $C^3$ systems to combat effectiveness[7]. SCHREIBER considered a battle between two homogeneous forces in which each unit remains in its original position and fires on enemy units until it is destroyed by enemy fire or the battle ends. At the beginning of battle, each force has complete information about enemy unit locations. SCHREIBER argued that an <u>intelligence system</u> provides information on the effects of fire on enemy units and also the status of friendly units, and a <u>command and control system</u> redirects fire (using information from the intelligence system) uniformly over surviving enemy units[8]. He hypothesized that the effectiveness of the intelligence and command and control systems in this type of battle could be represented by the fraction of the enemy's destroyed units from which fire has been redirected. If this fraction is one, fire is being directed at only "live" enemy units with no "overkill;" but if it is zero, fire is being directed at the original enemy positions with attendant "overkill." Consequently, SCHREIBER postulated that the following LANCHESTER-type equations (for $x$ and $y > 0$) would model such a combat situation.

$$\begin{cases} \dfrac{dx}{dt} = -a\left\{\dfrac{xy}{x_0 - e_Y(x_0 - x)}\right\} & \text{with } x(0) = x_0 , \\[2ex] \dfrac{dy}{dt} = -b\left\{\dfrac{xy}{y_0 - e_X(y_0 - y)}\right\} & \text{with } y(0) = y_0 , \end{cases} \tag{7.2.2}$$

where $x(t)$ and $y(t)$ denote the X and Y force levels, $a$ denotes the usual LANCHESTER attrition-rate coefficient for "aimed" fire [i.e. it is given by (5.3.1) and (5.3.2)], $e_Y$ denotes the "command efficiency" of the Y force, and $b$ and $e_X$ denote corresponding quantitites for the X force. The above equations (7.2.2) have the same functional form as those for BRACKNEY's model with target-acquisition times inversely proportional to target density (5.10.11). Also, $0 \leq e_X, e_Y \leq 1$ in (7.2.2).

It is instructive to examine the extreme cases for the above attrition process as postulated by SCHREIBER. The maximum combat efficiency for the Y force occurs when $e_Y = 1$, and then

$$\frac{dx}{dt} = -ay , \tag{7.2.3}$$

which is the usual attrition rate for "aimed" fire when target-acquisition times do not depend on the number of enemy targets. The maximum "overkill" by the Y force (i.e. the least combat efficiency) occurs when $e_Y = 0$, and then

$$\frac{dx}{dt} = -a\left(\frac{y}{y_0}\right)x , \tag{7.2.4}$$

which is the same functional form for the attrition rate for "area" fire against a constant-density defense. HELMBOLD [78] has also noted that attrition rates take the form (7.2.3) when fire is concentrated on the surviving targets, and (7.2.4) when it is directed at the original positions with no redistribution.

SCHREIBER [127] assumed that the "command efficiencies" $e_X$ and $e_Y$ were constant in (7.2.2) and used this simple model to show that an increase in the efficiency of intelligence and command and control systems can be equivalent to a substantial increase in numerical strength (up to 41.4 percent). His analysis used the following analytical results. Since the instantaneous casualty-exchange ratio for SCHREIBER's model (7.2.2) is given by

$$\frac{dx}{dy} = \frac{a}{b}\frac{\left\{y_0 - e_X(y_0 - y)\right\}}{\left\{x_0 - e_Y(x_0 - x)\right\}} \tag{7.2.5}$$

and the "command efficiencies" $e_X$ and $e_Y$ are assumed to be constant, one readily obtains the state equation for SCHREIBER's model.

$$b\left\{x_0 - x(t)\right\}\left\{\frac{x_0}{2}(2 - e_Y) + e_Y\frac{x(t)}{2}\right\}$$

$$= a\left\{y_0 - y(t)\right\}\left\{\frac{y_0}{2}(2 - e_X) + e_X\frac{y(t)}{2}\right\}, \tag{7.2.6}$$

which readily yields (cf. Section 3.5) the following condition for a draw in a fight-to-the-finish ("parity" condition)

$$\frac{bx_0^{\,2}}{(2 - e_X)} = \frac{ay_0^{\,2}}{(2 - e_Y)} \tag{7.2.7}$$

Although a state equation is thus readily obtained, for example, the X force-level equation is not equivalent to any standard differential-equation form, and consequently the X force level X(t) is apparently not expressible in terms of "elementary" functions. Considering the left-hand side of the parity condition (7.2.7), we can easily show that an increase in the value of the command efficiency from $e_X^0$ to $e_X$ increases the

combat power by the same amount as an increase in numerical strength by a fraction $f$ given by

$$f = \sqrt{\frac{2 - e_X^0}{2 - e_X}} - 1 \,, \tag{7.2.8}$$

when follows SCHREIBER's conclusion about the tradeoff of numerical strength and the efficiency of $C^3$ systems.

Let us finally note the following two significant shortcomings of SCHREIBER's above tradeoff analysis: (S1) in the case of mobile units they would not remain in their original positions, and (S2) "command efficiency" would decline during battle due to damage to the intelligence and command and control systems. Nevertheless, SCHREIBER's simple model (7.2.2) with constant "command efficiencies" $e_X$ and $e_Y$ has provided some important insights into the influence of $C^3$ systems on combat power.

We now return to target-acquisition considerations with a brief general discussion of target-acquisition modelling for combat between heterogeneous forces. We continue our discussion of Vector Research's refined model of the target-acquisition process and its influence on the attrition rate. Vector Research, Inc. (see [154, pp. 103-108] or [117, pp. 43-45]) considers that the two major factors determining the value of an attrition-rate coefficient are (1) the acquisition and selection of targets, and (2) the conditional kill rate (i.e. the rate at which acquired targets are destroyed). Concerning target acquisition and selection, the proportion of time that a weapon is actively engaging an enemy target depends on the interaction of three processes:

(P1) the line-of-sight process (which determines when a given target is visible or invisible to a potential firer),

(P2) the target-acquisition process (which determines the time required for a firer to acquire a particular target),

and (P3) the target selection process (which specifies a scheme by which a weapon crew chooses to engage a particular target from among those that have been acquired).

In other words, the effects of terrain are considered by computing intervisibility (i.e. existence of line of sight) for each target-firer pair based on their map locations. Therefore the complex operational models developed by Vector Research must keep track of all firer and target positions during the evolution of battle[9]. The exact way in which the above three processes interact depends in an essential way on which of two kinds of acquisition and target-selection modes the weapon systems employ--serial or parallel acquisition (see Section 5.16 for further details; see also [39], [154], or [117]). Suppressive effects of weapon systems may be accommodated in Vector Research's models (e.g. see [72]), but the phenomenological basis of such suppressive effects is poorly understood at this time (see the "Report of the Army Scientific Advisory Panel Ad Hoc Group on Suppression" [45]).

Although the process of suppression is poorly understood, most military analysts feel that the suppressive effects of weapon systems should be included in any model of combat operations. In general, two ways to model suppressive effects within the context of detailed LANCHESTER-type formulations are (see TAYLOR [141, pp. A-56 - A-60] or BARR [8] for further details):

(a) modify LANCHESTER attrition-rate coefficients to reflect degraded fire effectiveness of the firing units due to firers being suppressed[10],

(b) consider combatants of a given class to be in different states (in the simplest model there are two states: unsuppressed and suppressed) with different fire effectiveness and vulnerability to enemy fire in each state; this approach requires some model of state transitions.

The reader can see from the above that there is no problem in modelling suppressive effects. However, there unfortunately is no supportable data on troop behavior when under fire to use in such models. Thus, the major problems are to scientifically determine functional relations and to estimate the parameters in any hypothesized model of suppressive effects. Although the U.S. Army Combat Developments Experimentation Command (CDEC) has conducted many suppression experiments and the U.S. Army has reviewed the entire topic of fire suppression (see [45]), the representation of suppressive effects in casualty-assessment models (in particular, LANCHESTER-type models) remains a major problem area.

The effects of logistics constraints may be modelled in various ways. The main approach is to represent the consumption and distribution of various types of supplies (e.g. ammunition, fuel, etc.). When supplies of a particular type are depleted to some given critical level, the combat effectiveness of a unit is appropriately modified (see [117], BONDER and FARRELL [15], KERLIN and COLE [98], and CHASE [28] for further details).

## 7.3. Modelling Small-Scale Engagements versus Modelling Large-Scale Ones.

There is a fundamental difference between modelling (with differential equations) small-scale engagements and modelling large-scale ones: for small-scale operations it may be possible to reasonably represent force interactions and attendant attrition rates with a few differential equations, but for large-scale operations of conventional armed forces the same approach might well involve hundreds (and possibly even thousands) of differential equations tied together through battlefield operations. In other words, large-scale warfare involves a seemingly overwhelming amount of detail because of the very scale of operations. Small-scale operations are usually considered as fire fights between at most a few different weapon-system types on each side, but in large-scale warfare one must consider many different weapon-system types (both combat and combat-support systems) operating as combined-arms teams in sustained operations that involve not only fire fights but also maneuver, reconnaissance, logistics, committing of reserves, allocation of tactical aircraft to missions, etc. Thus, in large-scale warfare there are not only many more military units and types of systems, but these systems and units engage in a much wider variety of activities than do the few types in small-scale engagements.

Moreover, the scale of combat operations actually dictates what is a feasible approach for modelling a particular type of engagement (see Figure 7.1). As we saw in Chapter 1, there are three main approaches used for assessing outcomes (in particular, casualties) of simulated tactical engagements:

(A1) firepower-score approach[11],

(A2) Monte-Carlo-simulation approach,

(A3) LANCHESTER-type-model approach.

FEASIBLE MODELLING APPROACH

| SCALE OF OPERATIONS | MONTE CARLO SIMULATION MODEL | DETAILED LANCHESTER-TYPE MODEL | AGGREGATED-FORCE (i.e. FIREPOWER-SCORE) MODEL |
|---|---|---|---|
| INDIVIDUAL-FIRER ENGAGEMENT | X | | |
| SMALL-UNIT ENGAGEMENT (BATTALION-SIZED UNITS AND SMALLER) | X | X | |
| LARGE-UNIT ENGAGEMENT (DIVISION-SIZED UNITS AND LARGER) | | X | X |

Figure 7.1. Feasible modelling approach related to scale of combat operations.

Each of these approaches involves a different level and amount of detail, and each provides a different degree of resolution to battlefield operations. The higher the degree of resolution, the higher (of course) is the amount of details that the model considers. Furthermore, the total amount of details that is feasible to handle depends on current computer technology.

As we saw in Chapter 1 (recall Table 1.III relating combat-assessment approach to the scale of combat operations), Monte Carlo simulations have been used to assess casualties in small-unit combat (i.e. combat between battalion-sized units and smaller), while the firepower-score approach applies primarily to large-scale (i.e. corps-level and theater-level) combat. However, LANCHESTER-type models[12] have been developed in the United States for the full spectrum of combat operations, from small-unit combat to large-scale operations. Thus, if one wants to assess casualties for simulated tactical engagements between battalion-sized units or larger, there are essentially only two types of models that have been widely used in the United States for assessing casualties in such tactical engagements:

(T1) detailed LANCHESTER-type models,

and (T2) aggregrated-force models based on quantifying military capabilities with index numbers (i.e. firepower-score models).

Although one could also consider a third approach of employing a hierarchy of models, such an approach has not been widely used in the United States, and we will consequently not consider in detail in this monograph (see Section 7.20, however, for a brief conceptual discussion).

For very simple small-scale engagements it has always been possible to model in detail attrition in fire fights (provided that forces and operations are not too complicated). Here, we mean not just to formulate a combat model but to develop an operational model from which numerical results may be obtained. However, for large-scale warfare it has been possible only relatively recently to model attrition in detail (i.e. to attrite each different type of weapon system individually). The modern large-scale digital computer has provided the computational capability for detailed modelling of large-scale military operations. In fact, without the modern digital computer operational models of virtually any degree of complexity would be impossible. In particular, the advent of the modern high-speed large-scale digital computer has made feasible not only high-resolution Monte Carlo combat simulations such as DYNTACS and CARMONETTE, automated "quick games" such as ATLAS, and other theater-level firepower-score-based combat models such as CEM and TBM-68, but it has also made possible differential combat models such as BONDER/IUA and its many derivatives[13]. Furthermore, the relation between feasible modelling approach and the scale of combat operations (as portrayed in Figure 7.1) depends in an essential way on the state of the art of computer technology.

All the above complex operational models that are conceptually based on LANCHESTER-type equations (e.g. BONDER/IUA, DIVOPS, or VECTOR-2), however, model combat attrition in detail and explicitly consider the many different weapon-system types that can be individually attrited. These weapon-system types include different types of weapon systems in maneuver units and different types of fixed-wing aircraft, as well as separately represented field artillery, air defense artillery, and helicopter weapon systems. Such LANCHESTER-type models represent attrition in a way that reflects the internal dynamics of combat activities and relates these dynamics to specific

weapon-system parameters and tactics considered important in small-unit engagements. The effects of individual weapon-system types on the outcome of a theater-level campaign are clearly observable and bear a clear relationship to the input performance assumed (see [117] for further details).

A different approach for modelling attrition in large-scale (i.e. theater-level) combat operations is to represent attrition in a macroscopic fashion. The many different weapon systems on one side are all combined together by using firepower scores into a single scalar quantity, the "combat capability" (or firepower index) of the force, and combat causes attrition of this index number. The attrition of combat capability is determined with the help of casualty-rate curves that relate the relative combat capabilities of the forces (expressed in terms of the two firepower indices) and other tactical factors to their casualty rates (expressed in an aggregated fashion). Losses of individual weapon-system types are then determined by some means of disaggregation. Such aggregated loss-rate relations are apparently largely judgmentally determined (although having some alleged basis in empirical combat data), and the author knows of no conceptual approach or mathematical models for relating weapon-system-performance parameters and other operational variables to the numberical determination of these aggregated-force loss rates.

In the rest of this chapter we will discuss various aspects of modelling tactical engagements. We will first consider a number of examples from guerrilla-warfare applications because the engagements are of small enough scale to yield simple (but yet detailed) LANCHESTER-type models and also because such modelling information is readily available in the open literature. We will then progress to more complicated LANCHESTER-type models, including models of combat between heterogeneous forces. The firepower-score approach and aggregated-force models are then discussed. Finally, we briefly discuss current operational models of large-scale conventional warfare.

## 7.4. Applications to Guerrilla Warfare.

The literature on applications of LANCHESTER-type models to the study of guerrilla warfare (see DEITCHMAN [44] and SCHAFFER [125] is small but of particular interest because it contains the only examples of tactical engagements (particularly ambushes) to appear in the open literature. These two papers contain many interesting modelling ideas as well as several detailed models of small-scale engagements. Moreover, the ambush models considered by these authors have much wider applicability than just to guerrilla warfare, since (for example) the "force-oriented defense" (see HOLDSWORTH [88]), which has been proposed for NATO operations, is based on a tactical doctrine of rather wide-spread use of ambushes.

DEITCHMAN [44] in 1962 introduced the idea of modelling an ambush with "aimed" fire for the ambushers and "area" fire for the ambushees, e.g. F/FT attrition. He used such a simple model to argue that the attacking guerrillas, heavily outnumbered overall, can win if both sides are divided into small groups, and the guerrillas always attack in ambushes. Such a result is in consonnance with recent history, which shows that defending regulars must have overall force ratios above ten to one to meet such local guerrilla attacks at all successfully. SCHAFFER [125] subsequently in 1965 studied guerrilla-warfare engagements in more detail and under a variety of operational conditions (i.e. skirmish, ambush, and siege). He developed several LANCHESTER-type models for small-force guerrilla engagements that are typical of the early stages of insurgency. These models included the effects of supporting weapons and the discipline or morale of the troops involved, and they allowed for temporal variations in weapon-system effectiveness (i.e. firepower). His paper is an excellent source of modelling ideas. SCHAFFER used these models to develop insights into the important attack parameters in guerrilla warfare and also to quantitatively justify some new military hardware. We will now examine the ideas of these two important papers in more detail.

## 7.5. DEITCHMAN's Basic Ambush Model.

The goal of DEITCHMAN's investigation [44] was to develop a quantitative explanation of why high counterguerrilla/guerrilla force ratios have been required for regulars (i.e. counterguerrillas or counterinsurgents) to defeat insurgents in guerrilla warfare (see Figure 7.2). He sought to explain this empirical fact with a simple model. DEITCHMAN's simplified conceptualization of guerrilla warfare was as follows:[14] the defending regular army (counter-insurgents) must fragment itself to defined the many possible points that are vulnerable to guerrilla attack and to hunt down the many guerrilla bands; guerrilla warfare itself occurs as a sequence of engagements between small groups drawn from the overall forces. Thus, the overall forces do not engage each other directly in combat, but small groups drawn from them sequentially fight battles. As Figure 7.2 shows us, history indicates that the defending regulars must have overall force ratios above ten to one to defeat the guerrillas under such circumstances.

DEITCHMAN thus considered guerrilla warfare as a sequence of engage-ments between small groups drawn from overall forces. H. K. WEISS [158] had developed the following LANCHESTER-type equations to approximately represent such combat between two homogeneous forces in which both sides use "aimed" fire (with constant target-acquisition times)

$$\begin{cases} \dfrac{dx}{dt} = -a\,\dfrac{xy}{m} & \text{with } x(0) = x_0\,, \\[2ex] \dfrac{dy}{dt} = -b\,\dfrac{xy}{n} & \text{with } y(0) = y_0\,, \end{cases} \qquad (7.5.1)$$

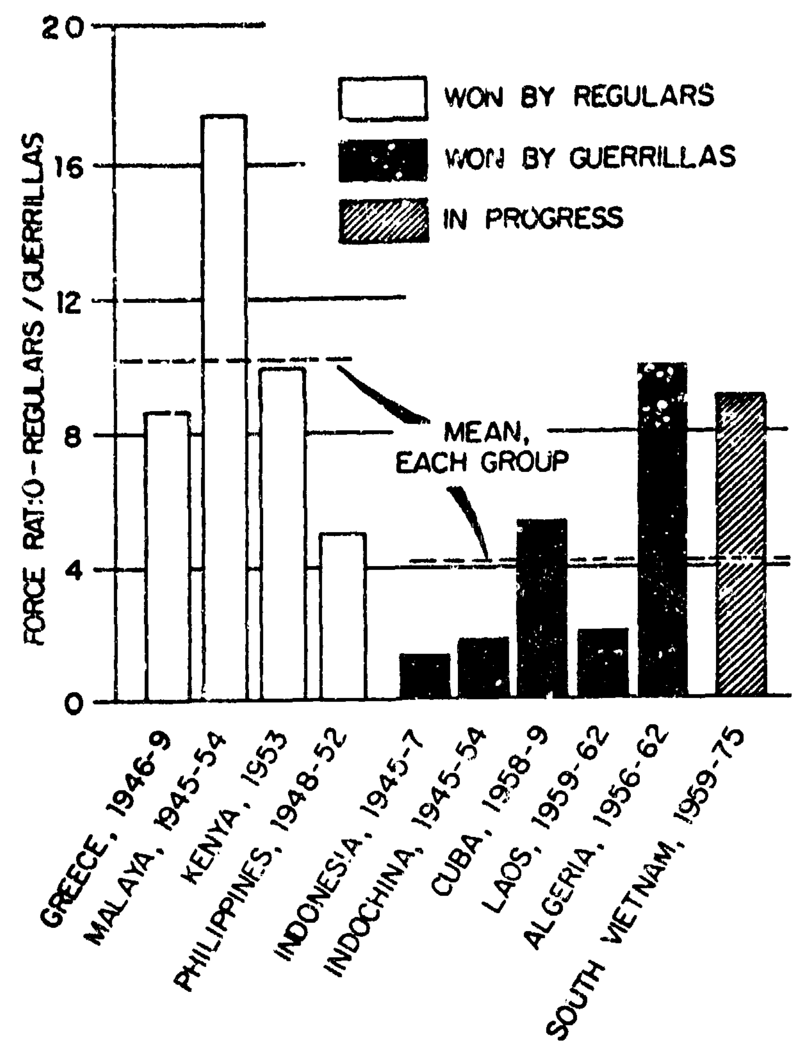


Figure 7.2. Estimated force ratios in guerrilla wars between the end of World War II and 1962 (from DEITCHMAN [44]). Although the end of the Vietnam War has been indicated, the data upon which this figure is based dates from no later than 1962.

where $x(t)$ denotes the overall X force level, $m$ denotes the (initial) size of X's combat groups, $b$ denotes a constant LANCHESTER attrition-rate coefficient representing the fire effectiveness of a single X combatant, and $y(t)$, $n$, and $a$ denote corresponding quantities for the Y force. We will sketch the derivation of these equations at the end of this section.

The condition for a <u>draw in a fight to the finish</u> (i.e. "parity" condition) is readily obtained from (7.5.1) as (<u>cf</u>. Section 3.5)

$$\frac{x_0}{y_0} = \left(\frac{a}{b}\right)\left(\frac{n}{m}\right) . \tag{7.5.2}$$

For larger values of the initial force ratio, i.e. $x_0/y_0 > (a/b)(n/m)$, X will win such a fight to the finish; and for smaller ones, the X force will lose. Thus, engagement outcome depends on three relative parameters (<u>cf</u>. Section 2.2. and 6.6 above): (1) the initial overall force ratio $(x_0/y_0)$, (2) the relative fire effectiveness $(b/a)$, and (3) the relative (initial) size of the small groups $(m/n)$. The break-even (or parity) point expressed in terms of the initial force ratio as a function of relative group size is shown in Figure 7.3 for various values of relative fire effectiveness $b/a$. This figure shows that a side that is heavily outnumbered overall can still win if in all the individual engagements its groups are larger than the enemy's or if the relative fire effectiveness is sufficiently in its favor.

DEITCHMAN [44] argued that for all "reasonable" values of the above relative parameters (i.e. $x_0/y_0$, $b/a$, and $m/n$), the parity condition (7.5.2) implies that an excessively large (initial) local force ratio is required for the guerrillas to win. For example, let X be the counterinsurgents and

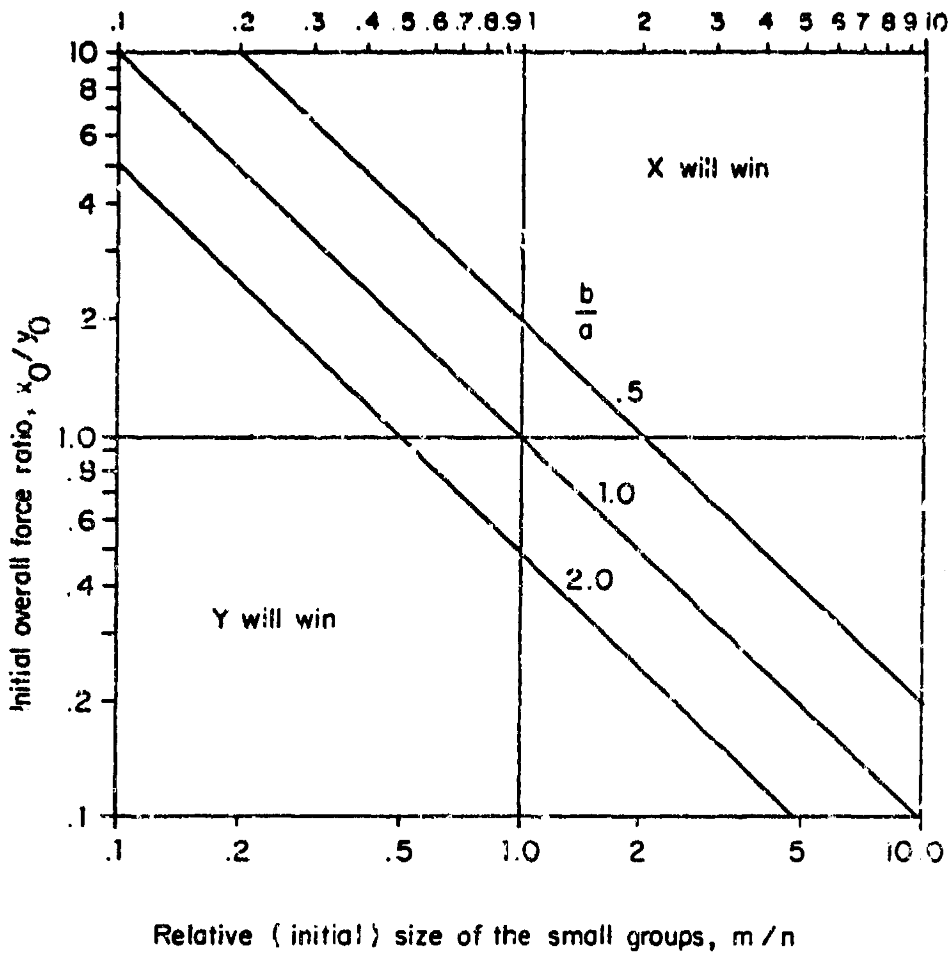


Figure 7.3. Break-even (or parity) point in the initial overall force ratio as a function of relative group size for combat between small groups drawn from overall larger forces (after DEITCHMAN [44]). This figure shows us that, for example, for $(b/a) = 2.0$ a value of $(m/n) = 0.125$ is required for parity when $(x_0/y_0) = 4.0$. If we let X denote the counterinsurgents (counterguerrillas) and Y denote the guerrillas, then the counterinsurgents X will win such a sequence of engagements with $(b/a) = 2.0$ for all combinations of $(m/n)$ and $(x_0/y_0)$ lying above the straight line labelled $(b/a) = 2.0$.

Y be the guerrillas. Then (7.5.2) (or, equivalently, Figure 7.3) says that, for example, a local (initial) force ratio of $(m/n) = 0.125$ is required for parity when $(x_0/y_0) = 4.0$ and $(b/a) = 2.0$. The latter two values are taken to represent the guerrillas being less numerous overall and possessing relatively less effective firepower than the counterinsurgents. [DEITCHMAN argued that relative fire effectiveness (b/a) should favor the counterinsurgents, since one would expect the guerrillas to use crude weapons or a limited number of captured ones.] Thus, for such "typical" values WEISS's [158] model (7.5.1) requires that the guerrillas must heavily outnumber the counterinsurgents in all the local engagements in order to be able to win. Hence, WEISS's model is in this case at variance with empirical evidence that guerrillas can win (and, indeed, many times have [recall Figure 7.2]) with equal or inferior numbers in the local engagements. DEITCHMAN then sought to find tactics that would allow the guerrillas to win with equal or inferior numbers in the local engagements: he consequently postulated that ambush tactics by the guerrillas could achieve this end.

Thus, DEITCHMAN conceptualized that a counterinsurgent force, say X, would move through an area searching for guerrillas or intending to attack a guerrilla base. The guerrillas, denoted as the Y force, should counter such a tactic by preparing an ambush for the approaching counterinsurgents. In this ambush engagement, the force being ambushed (i.e. the ambushees X) are in plain sight (i.e. full view) of the ambushers Y, who use "aimed" fire, so that X's casualty rate is proportional to only the number of Y ambushers, with target-acquisition times negligible. On the other hand, the ambushers are hidden, and the ambushees (who have been "caught by surpirse") fire blindly into the general area occupied by the ambushers (i.e. they return

"area" fire) so that Y's casualty rate is proportional to the product of the numbers of both X ambushees and Y ambushers. Thus, DEITCHMAN [44] hypothesized that attrition in such a homogeneous-force ambush could be modelled by[15] (see Figure 7.4)

$$\begin{cases} \frac{dx}{dt} = -ay \quad \text{(AMBUSHEE ATTRITION)} \quad \text{with} \quad x(0) = x_0 , \\ \frac{dy}{dt} = -bxy \quad \text{(AMBUSHER ATTRITION)} \quad \text{with} \quad y(0) = y_0 , \end{cases} \tag{7.5.3}$$

where for the simplest case considered by DEITCHMAN the attrition-rate coefficients $a$ and $b$ would be given by (see Chapter 5 for further details about more sophisticated models for them)

$$a = \nu_Y P_{SSK_{XY}} , \qquad \text{and} \qquad b = \nu_X \frac{a_{V_X}}{A_Y} . \tag{7.5.4}$$

with $\nu_X$ and $\nu_Y$ denoting the firing rates of X and Y, $P_{SSK_{XY}}$ denoting the single-shot kill probability of Y against X, $a_{V_X}$ denoting the vulnerable area of a single X target, and $A_Y$ denoting the "presented" area occupied by the Y force. Here, we assume that the ambushees return "small arms" fire (see Section 5.13 for other types of "area" fire, e.g. "artillery" fire). We also assume that the X force fires into the actual region occupied by the Y force, with modification of (7.5.4) being required if this does not coincide with the region in which X believes the ambushers to occupy and into which he consequently directs his fire.

The state equation for DEITCHMAN's ambush model (7.5.3) is given by

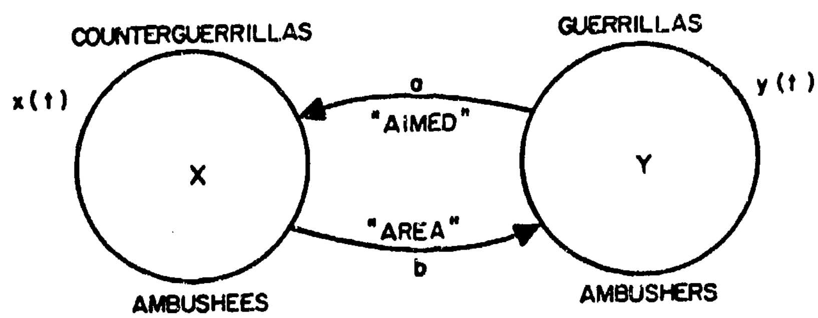


Figure 7.4. Schematic of ambush situation considered by DEITCHMAN [44].

$$\frac{b}{2}(x_0^2 - x^2) = a(y_0 - y) \,, \qquad (7.5.5)$$

so that (see Section 3.5) the ambushing Y force will win an engagement with fixed force-level breakpoints $x_{BP} = f_{BP}^X x_0$ and $y_{BP} = f_{BP} y_0$ if and only if

$$\frac{(x_0)^2}{y_0} < \frac{2a}{b} \frac{\{1 - f_{BP}^Y\}}{\{1 - (f_{BP}^X)^2\}} \,. \qquad (7.5.6)$$

Thus, parity exists between the forces in a fight to the finish for

$$\frac{(x_0)^2}{y_0} = \frac{2a}{b} = \frac{2\nu_Y}{\nu_X} \frac{P_{SSK_{XY}}}{(a_{V_X}/A_Y)} \,. \qquad (7.5.7)$$

Let us finally note that these results all hold for a single engagement.

DEITCHMAN [44] used the above simple ambush model [and, in particular, the parity condition (7.5.7)] to conclude that:

(C1) attacking guerrillas, heavily outnumbered overall, can win if both sides are subdivided into small groups and the guerrillas attack with local numerical superiority, but the local superiority required on the part of the guerrillas is greatly reduced if ambush tactics are used,

(C2) all things being equal, the ambushee cannot win in such an ambush engagement,

and (C3) the counterinsurgents' use of ambush tactics is a powerful tool against guerrillas.

DEITCHMAN added that the overall high defender/guerrilla force ratios that have historically been required for counterguerrillas to win against guerrilla attacks are very difficult to reduce significantly. These conclusions were based on the following type of analysis. Consider an ambush by guerrillas such as we have examined above. Then parity between the guerrillas and the ambushees is given by (7.5.7), and $(2a/b)$ in (7.5.7) can easily be on the order of several hundred so that a few ambushers can annihilate many ambushees. For example, $P_{SSK_{XY}}$ may be on the order of 0.1, $a_{V_X}$ may be about 1 square foot for a man taking available cover in the terrain, and 2 men can easily be hidden in a region of uncertainty of 1600 square feet; then for equal firing rates, $(2a/b) = 320$ so that according to (7.5.7), for example, 2 ambushers can annihilate a force of 25 ambushees.

The <u>force levels as functions of time</u>, i.e. $x(t)$ and $y(t)$, are rather complicated for the simple model (7.5.3). To develop them, for example, we may solve (7.5.5) for $y$ and substitute the result into the first equation of (7.5.3) to obtain

$$\frac{dx}{\{x^2 + [(2b/a)y_0^2 - x_0^2]\}} = -\frac{a}{2}\,dt ,$$

whence integration (e.g. <u>see</u> the "C.R.C. tables" [87]) yields the results shown in Table 7.II. The complexity of these results for the simple model with constant coefficients (7.5.3) provides some insight into why numerical integration techniques must usually be used for LANCHESTER-type models of any degree of complexity. DEITCHMAN gave numerical results for $x(t)$ and $y(t)$ for a number of illustrative battles. He observed that the victor can reduce his fractional loss (i.e. casualties expressed as a fraction of the unit's

TABLE 7.II. Analytical Expressions for Force Levels $x(t)$ and $y(t)$ in DEITCHMAN's Ambush Model (7.5.3).

(a) <u>Whem ambusher Y wins a fight to the finish</u> (i.e. $\frac{b}{2} x_0^2 < ay_0$):

for $0 \leq t \leq B/A$

$$x(t) = \sqrt{\frac{2a}{b} y_0 - x_0^2} \quad \tan(-At + B)$$

$$y(t) = \{y_0 - \frac{b}{2a} x_0^2\} \{\sec(-At + B)\}^2$$

for $B/A \leq t$

$$x(t) = 0 \quad \text{and} \quad y(t) = y_0 - \frac{b}{2a} x_0^2$$

where

$$A = \frac{b}{2} \sqrt{\frac{2a}{b} y_0 - x_0^2}$$

$$B = \tan^{-1}\left(\frac{x_0}{\sqrt{\frac{2a}{b} y_0 - x_0^2}}\right)$$

(b) <u>When ambushee X wins a fight to the finish</u> (i.e. $\frac{b}{2} x_0^2 > ay_0$):

for $0 \leq t$

$$x(t) = \sqrt{x_0^2 - \frac{2a}{b} y_0} \coth(A't + B')$$

$$y(t) = \frac{\{\frac{b}{2a} x_0^2 - y_0\}}{\{\sinh(A't + B')\}^2}$$

where

$$A' = \frac{b}{2} \sqrt{x_0^2 - \frac{2a}{b} y_0}$$

$$B' = \coth^{-1}\left(\frac{x_0}{\sqrt{x_0^2 - \frac{2a}{b} y_0}}\right)$$

initial force level) by initially committing more men to battle (see Section 8.9 for more general results of this nature).

Let us finally sketch the development of WEISS's model for combat among small groups (7.5.1). WEISS [158] observed that warfare was tending in the mid 1950's towards employment of small combat groups operating independently. He consequently sought to develop a simple model for aggregating a large number of such engagements between small groups. Let us therefore consider an $X$ force of overall numerical strength $x_0$ and assume that it is divided into "combat groups," each of which initially contains $m_0$ combatants. There will be $N_X = x_0/m_0$ such groups. Similar quantities for the $Y$ force are analogously defined, with $n_0$ denoting the initial strength of their combat groups. We will consider "aimed-fire" combat between two such small groups; it may be modelled by

$$\begin{cases} \dfrac{dm}{dt} = -an & \text{with } m(0) = m_s , \\ \\ \dfrac{dn}{dt} = -bm & \text{with } n(0) = n_s , \end{cases} \tag{7.5.8}$$

where $m(t)$ and $n(t)$ now denote the force levels of the two small groups at time t in the engagement, and $m_s$ and $n_s$ denote their initial (or starting) values (equal to $m_0$ and $n_0$ when two "fresh" units fight). For one engagement, we then have

$$a(n_s^2 - n_f^2) = b(m_s^2 - m_f^2) , \tag{7.5.9}$$

where the subscript $f$ denotes a final value (i.e. a value at the end of such an engagement).

Consider now a <u>sequence of engagements</u> between pairs of two such combat groups drawn from the overall forces (which have initial strengths $x_0$ and $y_0$) such that (1) each engagement is a fight to the finish, (2) the survivors of one engagement subsequently take on a fresh enemy combat group (a full initial strength) in another fight to the finish, and (3) the sequence ultimately leads to a draw (i.e. all the initial overall forces $x_0$ and $y_0$ are annihilated). By repeatedly applying (7.5.9) to engagements of such a sequence and adding, we find that all terms not involving the initial strengths cancel out, and the condition for a draw consequently is

$$\left(\frac{y_0}{n_0}\right) an_0^2 = \left(\frac{x_0}{m_0}\right) bm_0^2 ,$$

or

$$an_0 y_0 = bm_0 x_0 , \qquad (7.5.10)$$

since there were, for example, $(x_0/m_0)$ engagements in which the X group started at full strength. Notice that when all men on each side are in a single unit, (7.5.10) reduces to LANCHESTER's square law; but when we have a sequence of engagements between two individuals, (7.5.10) reduces to LANCHESTER's linear law. Can we devise LANCHESTER-type equations that yield (7.5.10) as a parity condition for a fight to the finish? Denoting $m_0$ and $n_0$ simply as $m$ and $n$, we observe that the equations (7.5.1) yield the desired parity condition, and this is how WEISS [158] developed his equations for combat among small groups.

## 7.6. SCHAFFER's Models of Guerrilla Engagements.

SCHAFFER's [125] goal was to develop LANCHESTER-type models for studying (e.g. for evaluating casualty claims for) small-force guerrilla engagements that are typical of Phase II insurgency.[16] His models included the effects of supporting weapon systems and the discipline or morale of the troops involved. SCHAFFER also allowed the fire effectivenesses of the different weapon-system types to vary with time and model temporal variations in firepower due to, for example, changes in the tactical postures of combatants during a fire fight. A number of his models explicitly considered such time-dependent attrition-rate coefficients. As is the case for operational models with almost any degree of complexity, analytical results were not expressible in terms of "elementary" functions, and numerical results had to be generated by numerical integration.

SCHAFFER's article [125] is particularly important because it is apparently the first reported use of LANCHESTER-type models to study actual combat situations and because of the many interesting models that it contains. He apparently used these models in studies at RAND to provide insights into the important attack parameters in guerrilla warfare and to quantitatively justify new hardware concepts (e.g. fast-response from supporting weapons).

SCHAFFER [125] first considered the overall military manpower flow in Phase II insurgency (see Figure 7.5) and examined small (typically 100-man) engagements classified as (1) skirmishes, (2) ambushes, or (3) sieges. Thus, each side has a large manpower pool from which small fighting groups are drawn for guerrilla-type operations. He assumed that for such operations food, weapons, and ammunition were inexhausbile. Traditional LANCHESTER combat theory had previously considered only battlefield casualties, but SCHAFFER added operational losses and captures to his models.

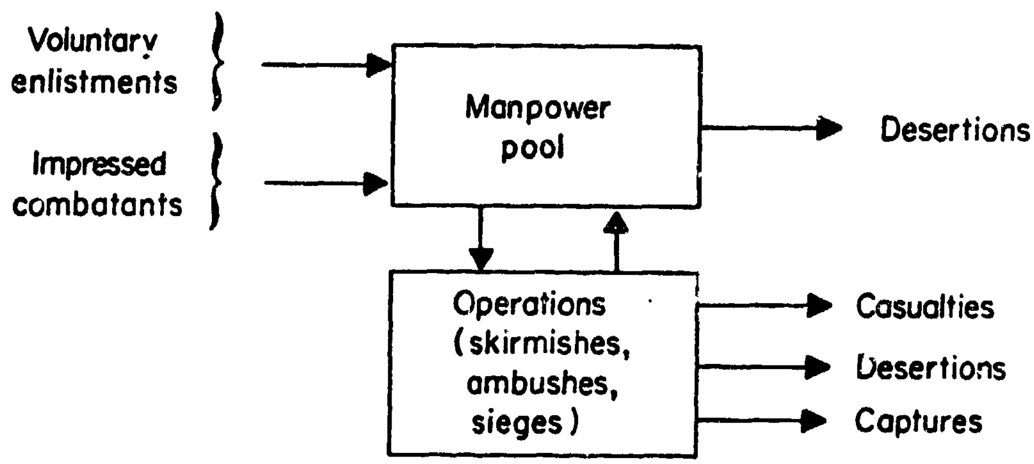


Figure 7.5. SCHAFFER's conceptualization of the military manpower flow in Phase II insurgency (from SCHAFFER [125]).

SCHAFFER developed a generalized LANCHESTER theory for force depletion in such small engagements. He considered losses due to the following sources:

(S1) battlefield casualties,

and (S2) surrenders and desertions.

Let X denote the counterinsurgents and Y denote the guerrillas (insurgents). SCHAFFER considered combat between small groups of infantry with supporting weapons and took the <u>rates of battlefield casualties</u> to be given by

$$\begin{cases} \left(\dfrac{dx}{dt}\right)_c = -a(t,x)y - \sum\limits_i E_i(t,x)\, W_i(t) \quad \text{with} \quad x(0) = x_0 , \\[2ex] \left(\dfrac{dy}{dt}\right)_c = -b(t,y)x - \sum\limits_j E_j(t,y)\, W_j(t) \quad \text{with} \quad y(0) = y_0 , \end{cases} \qquad (7.6.1)$$

where $(dx/dt)_c$ denotes the casualty rate for the X force, $b = b(t,y)$ denotes the fire effectiveness of a single X combatant (i.e. LANCHESTER attrition-rate coefficient), $E_j(t,y)$ denotes the effectiveness of X's <u>jth</u> supporting weapon-system type, $W_j(t)$ denotes the number of X's <u>jth</u> supporting weapon-system type that is firing at time t, and $(dy/dt)_c$, $a = a(t,x)$, $E_i$, and $W_i$ denote similar quantities for the Y force. Here the subscript i refers to the Y force and j to the X force. $W_i$ and $W_j$ have been taken to be functions of time, since the supporting weapons are taken to be employed for only portions of the battle.

The rate of surrenders and desertions were hypothesized by SCHAFFER to depend on (1) the friendly casualty rate, and (2) the difference between the friendly/enemy force ratio and unity (i.e. an unfriendly force ratio causes the friendly forces to "fade away"). Assuming that the surrender and desertion rates were expressible as sums of separate power series, SCHAFFER wrote

$$\left\{\begin{aligned} \left(\frac{dx}{dt}\right)_{s+d} &= r_X - \left[ p_{X_1}\left(\frac{dx}{dt}\right)_c + p_{X_2}\left(\frac{dx}{dt}\right)_c^2 + \cdots \right] \\ &\quad - \left[ q_{X_1}\left(\frac{y}{x} - 1\right) + q_{X_2}\left(\frac{y}{x} - 1\right)^2 + \cdots \right], \\ \left(\frac{dy}{dt}\right)_{s+d} &= r_Y - \left[ p_{Y_1}\left(\frac{dy}{dt}\right)_c + p_{Y_2}\left(\frac{dy}{dt}\right)_c^3 + \cdots \right] \\ &\quad - \left[ q_{Y_1}\left(\frac{x}{y} - 1\right) + q_{Y_2}\left(\frac{x}{y} - 1\right)^2 + \cdots \right], \end{aligned}\right. \tag{7.6.2}$$

where $q_{X_k} = 0$ for $y/x < 1$, and $q_{Y_k} = 0$ for $x/y < 1$. SCHAFFER [125, pp. 461-462] went on to discuss what restrictions should be placed on the signs of the coefficients $p$, $q$, and $r$ in (7.6.2). He pointed out that for the types of engagements between small units considered by him (i.e. both $dx/dt$ and $dy/dt$ must always be $\leq 0$), one must always have both $(dx/dt)_{s+d}$ and $(dy/dt)_{s+d} \leq 0$ (i.e. a net rate of loss due to surrenders and desertions), and hence he assumed that both $r_X$ and $r_Y \leq 0$. [SCHAFFER observed that "in a self-policing military group" it can be assumed that $r = 0$.] Thus, on

physical/operational grounds we must always have both $q_{X_k}$ and $q_{Y_k} \geq 0$ for all integers $k \geq 1$; we must analogously have both $p_{X_1}$ and $p_{Y_1} \leq 0$. The coefficients $p_{X_k}$, $p_{Y_k}$, $q_{X_k}$, and $q_{Y_k}$ reflect the motivation and discipline of the troops involved in the engagement, and the greater the magnitude of the absolute value of such a coefficient, the poorer the motivation and discipline of the troops involved.[17]

For computational purposes SCHAFFER only retained the first few terms in (7.6.2). Thus, his equations for the total rate of force depletion, e.g. $dx/dt = (dx/dt)_c + (dx/dt)_{s+d}$, were (see Figure 7.5)

$$\begin{cases} \dfrac{dx}{dt} = -(1-p_X)\, a(t,x)y - q_{X_1}\left(\dfrac{y}{x} - 1\right) - q_{X_2}\left(\dfrac{y}{x} - 1\right)^2 \\ \qquad\qquad - (1-p_X) \sum_i E_i(t,x)\, W_i(t) \quad \text{with} \quad x(0) = x_0 , \\ \dfrac{dy}{dt} = -(1-p_Y)\, b(t,y)x - q_{Y_1}\left(\dfrac{x}{y} - 1\right) - q_{Y_2}\left(\dfrac{x}{y} - 1\right)^2 \\ \qquad\qquad - (1-p_Y) \sum_j E_j(t,y)\, W_j(t) \quad \text{with} \quad y(0) = y_0 , \end{cases} \qquad (7.6.3)$$

where both $p_X$ and $p_Y \leq 0$, $q_{X_k} \geq 0$ with $q_{X_k} = 0$ when $y/x < 1$, and $q_{Y_k} \geq 0$ with $q_{Y_k} = 0$ when $x/y < 1$. The larger that $|p_X|$, $|p_Y|$, $|q_{X_k}|$, or $|q_{Y_k}|$ is, the poorer is the motivation and discipline of the soliders involved (i.e. as discussed above, these coefficients model the morale and discipline of the troops involved). Also, for the appropriate choices of values for $q_{X_k}$ and $q_{Y_k}$, the terms that contain $(y/x - 1)$ and $(x/y - 1)$ can simulate the act of breaking off an engagement, which is in keeping with the guerrilla tactic of fading off into the jungle (i.e. when guerrilla forces are outnumbered

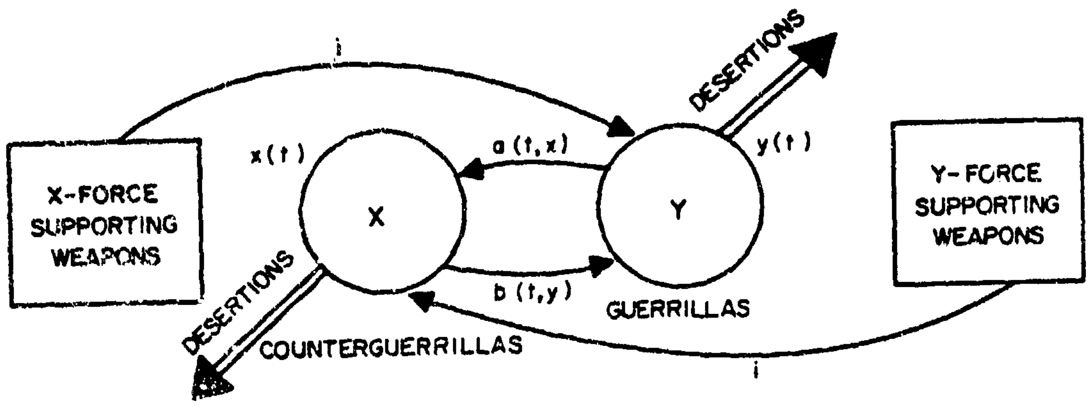


Figure 7.6. Diagram of guerrilla-warfare engagement to which SCHAFFER's general model applies.

or at some other disadvantage, they will gradually disengage, with the remaining troops fighting a rear-guard action). SCHAFFER then applied his above generalized attrition equations (7.6.3) to the following three special types of guerrilla-warfare engagements: (I) skirmish, (II) ambush, and (III) siege. As noted above, the solution to a LANCHESTER-type model as complex as (7.6.3) is most likely not expressible in terms of "elementary" functions,[18] and consequently one must use numerical-integration techniques to generate numerical results for specific battles.

SCHAFFER [125, p. 463] used the word skirmish to denote an engagement with a relatively limited commitment of resources. He assumed that the primary force[19] on each side is composed of riflemen and that every rifleman on each side uses "aimed" fire (see Sections 2.2 and 6.5 for further discussions of "aimed" fire) with an associated constant attrition-rate coefficient modelling their fire effectiveness. In this case equations (7.6.3) become

$$\begin{cases} \dfrac{dx}{dt} = -(1-p_X)ay - q_{X_1}\left(\dfrac{y}{x} - 1\right) - q_{X_2}\left(\dfrac{y}{x} - 1\right)^2 \\ \qquad\qquad - (1-p_X)\sum\limits_i E_i(t,x)\, W_i(t) \quad \text{with} \quad x(0) = x_0 , \\ \dfrac{dy}{dt} = -(1-p_Y)bx - q_{Y_1}\left(\dfrac{x}{y} - 1\right) - q_{Y_2}\left(\dfrac{x}{y} - 1\right)^2 \\ \qquad\qquad - (1-p_Y)\sum\limits_j E_j(t,y)\, W_j(t) \quad \text{with} \quad y(0) = y_0 , \end{cases} \tag{7.6.4}$$

where $a$ and $b$ denote constant attrition-rate coefficients.

SCHAFFER examined numerical results (generated by numerical-integration techniques) for a variety of specific battles modelled by (7.6.4). He concluded

that morale and discipline (in addition to weapon-system effectivenesses and the initial force ratio) can have a significant effect on the outcome of battle. He showed that the numerically weaker side can win when discipline/morale factors outweigh firepower disparities. In his calculations SCHAFFER took numerical values of 0, -0.5, and -1.0 for both $p_X$ and $p_Y$ [see (7.6.4)], where (for example) $p_X = -1.0$ means that one X combatant deserts his fighting group for each casualty that the group sustains. Values of 0.04 were assigned to both a and also b. SCHAFFER modelled these attrition-rate coefficients (see Chapter 5 for more sophisticated models) by, for example,

$$a = \nu_Y P_{SSK_{XY}} = \nu_Y P(K|H)_{XY} \cdot P_{SSH_{XY}} , \tag{7.6.5}$$

where

$$P_{SSH_{XY}} = \frac{A_{T_X}}{2\pi\sigma_Y^2} . \tag{7.6.6}$$

Here $\nu_Y$ denotes Y's firing rate, $A_{T_X}$ denotes the presented area of a prone X infantryman to rifle fire over average terrain, $P(K|H)_{XY}$ denotes the probability that an X target is killed when he is hit by a round of Y's fire, and $P_{SSH}$ denotes single-shot hit probability. SCHAFFER actually gave sample numerical values for these parameters to the above model (7.6.4). An illustrative average rate of fire of $\nu = 5$ pounds/minute would lead to expenditure of 10 lbs of .22-cal rifle ammunition in about 80 minutes. SCHAFFER considered the following values to be typical: $A_T = 0\ 1\ ft^2$ at a range of 100 feet, $P(K|H) = 0.5$, and $\sigma = 1$ ft (corresponding to 10 mils at 100-ft range). The single-shot hit probability $P_{SSH_{XY}}$ given by (7.6.6) is computed according to the "small-target" approximation (see MORSE and KIMBALL [115, p. 112]), which

applies when the single-shot dispersions are "much larger" than the target. Some skirmish results for the case in which there are no supporting weapons on either side are shown in Figure 7.7.

SCHAFFER also considered skirmishes in which a single type of supporting weapon backed up the weaker side. For example, when the counterguerrillas bring up supporting weapons, he modelled combat by (see Figure 7.8)

$$\begin{cases} \dfrac{dx}{dt} = -(1-p_X)\, a(S_c)y - q_X\left(\dfrac{y}{x} - 1\right)^2 & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -(1-p_Y)\,\{bx + S_c(t,y)\} - q_Y\left(\dfrac{x}{y} - 1\right)^2 & \text{with } y(0) = y_0. \end{cases} \tag{7.6.7}$$

where $S_c(t,y) = \sum_j E_j(t,y)\, W_j(t)$ and the integer index $j$ takes on a single value. In other words, $S_c = S_c(t,y)$ denotes the effectiveness of the single type of supporting weapon. Suppressive effects of the supporting weapons are considered by having the fire effectiveness of enemy infantry decreased by the supporting fire, i.e. $a = a(S_c)$ with $a(S_c)$ being a decreasing function of $S_c$. The effectiveness of the supporting fires is modelled by the simplified formula given in Section 5.13, namely[20]

$$S_c = \nu_U \frac{a_{L_U}}{A_Y}\, y, \tag{7.6.8}$$

where $\nu_U$ denotes the firing rate of X's supporting weapons, $a_{L_U}$ denotes the lethal area of a single round of these supporting weapons, and $A_Y$ denotes the area of the region in which the Y force is considered to be randomly dispersed (and into which the supporting weapons are assumed to deliver "area" fire). SCHAFFER conceptualized that such a skirmish would begin without any supporting weapons for the counter insurgents, supporting fires would be called for at some time after engagement initiation, and after some additional

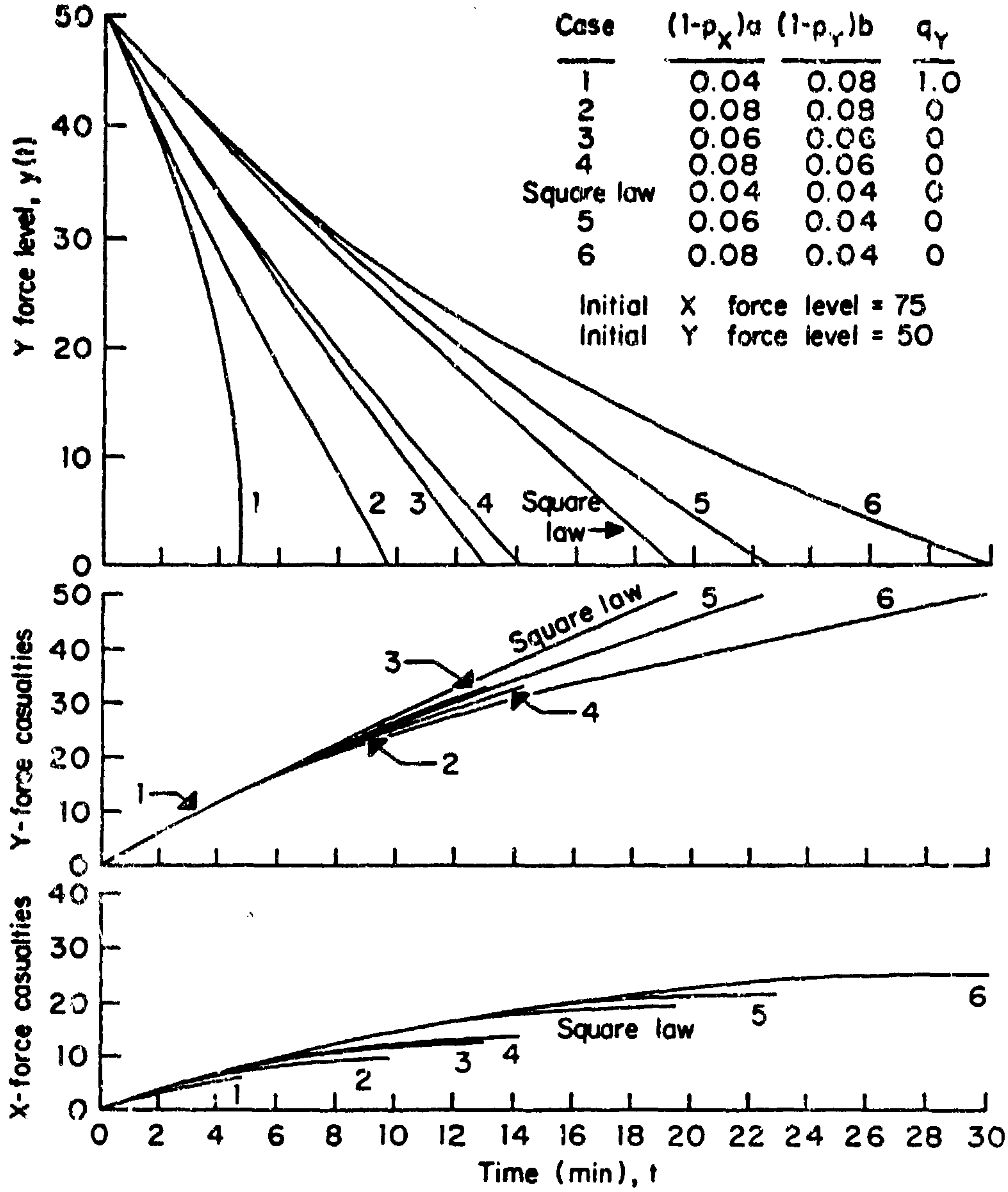


Figure 7.7. Results for model of skirmish with no supporting weapons on either side (after SCHAFFER [125]). In this case, the battle dynamics are given by

$$\begin{cases} \dfrac{dx}{dt} = -(1 - p_X)ay - q_X\left(\dfrac{y}{x} - 1\right)^2 , \\ \dfrac{dy}{dt} = -(1 - p_Y)bx - q_Y\left(\dfrac{x}{y} - 1\right)^2 . \end{cases}$$

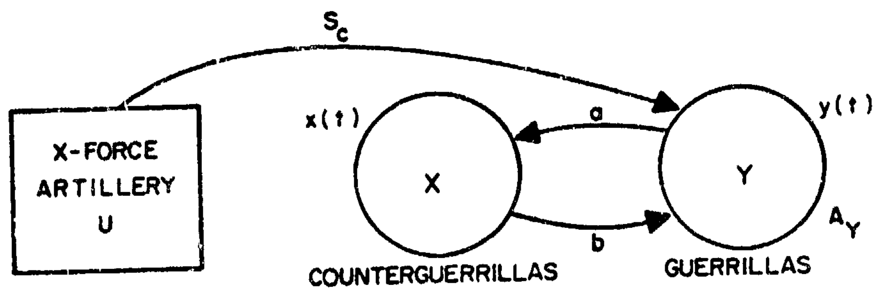


Figure 7.8. Skirmish in which the counterguerrillas bring up supporting weapons. Here $S_c$ denotes the effectiveness of supporting fires and is modelled by (7.6.8). If we ignore surrenders and desertions, then the combat dynamics are given by

$$\begin{cases} \dfrac{dx}{dt} = -a(S_c)y, \\ \\ \dfrac{dy}{dt} = -bx - S_c(t,y). \end{cases}$$

Suppressive effects are modelled by taking $a = a(S_c)$, i.e. the fire effectiveness of a $Y$ combatant (guerrilla) is degraded by the effectiveness of the X-force artillery fire.

delay the supporting fires would arrive. He modelled this process by

$$S_c(t,y) = w_{YU}(t)y\ H(t - t_d) , \tag{7.6.9}$$

where $w_{YU}(t) = \nu_U a_{L_U}/A_Y$ [cf. equation (7.6.8) above], $t_d$ denotes the delay time for the supporting fires to be added to the battle, and $H(t-t_d)$ denotes the "unit step function"

$$H(t-t_d) = \begin{cases} 0 & \text{for } 0 \leq t < t_d , \\ 1 & \text{for } t_d \leq t . \end{cases} \tag{7.6.10}$$

Such a step function allows us to "turn on" the supporting fires after a given amount of delay.

SCHAFFER took DEITCHMAN's ambush model (see the previous section) as a point of departure and added temporal variations to fire effectiveness modelled by attrition-rate coefficients. SCHAFFER emphasized that it was important to use time-dependent attrition-rate coefficients (cf. Section 6.2 above) and that such time dependence was the dominant factor in an ambush. He argued that temporal variations in fire effectiveness are the result of changes in cover (i.e. shielding) available to the ambushees and their gradual transition from area to aimed fire over the course of the ambush. Because of the element of surprise in the ambush, the ambushees' cover is initially minimal but improves as they "take cover." On the other hand, the ambushers' position is relatively secure and it does not change until they choose to break off the engagement. The ambushees initially return area fire because they have been "caught by surprise," and this fire transitions (i.e. changes) to aimed fire as they

recover their tactical discipline from the initial shock of the ambush. On the other hand, the ambushers always use aimed fire, although its quality deteriorates over time. During the early stages of the ambush, the ambushers have little motivation to desert or surrender, but after a time $t_c$, they may decide to withdraw. SCHAFFER quantified the effects of these potential acts through the quantity $q_T(t)$ defined as

$$q_Y(t) = |q_Y| \quad H(t-t_c) \quad H(x/y - 1) \ . \tag{7.6.11}$$

In other words, $q_Y(t) > 0$ for $t > t_c \geq 0$ or $x/y > 1$, and it is zero otherwise.

Based on the above considerations, SCHAFFER modelled such an ambush with the following LANCHESTER-type equations (<u>see</u> Figure 7.9)

$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = -(1-p_X) \ a(t)y - q_X\left(\dfrac{y}{x} - 1\right)^2 - (1-p_X) \displaystyle\sum_i E_i(t,x) \ W_i(t) \\ \qquad\qquad \text{(AMBUSHEE ATTRITION)} \quad \text{with} \quad x(0) = x_0 \ , \\ \\ \dfrac{dy}{dt} = -b(t,y)x - q_Y(t) \left( \dfrac{x}{y} - 1\right)^2 - \displaystyle\sum_j E_j(t,y) \ W_j(t) \\ \qquad\qquad \text{(AMBUSHER ATTRITION)} \quad \text{with} \quad y(0) = y_0 \ , \end{array} \right. \tag{7.6.12}$$

where $q_Y(t)$ is given by (7.6.11), and the attrition-rate coefficient $a(t)$ representing a Y-firer's fire effectiveness is given by

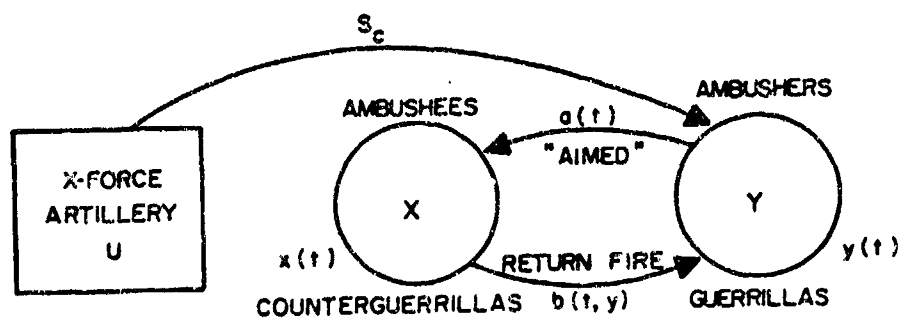


Figure 7.9. Schematic diagram of battlefield situation corresponding to SCHAFFER's model (7.6.12) of ambush in which counter-insurgents have a single type of fire support (here, artillery) with fire effectiveness denoted as $S_c = S_c(t,y)$. For this guerrilla-warfare engagement, (7.6.12) reduces to

$$\begin{cases} \dfrac{dx}{dt} = -(1 - p_X)\ a(t)y - q_X\left(\dfrac{y}{x} - 1\right)^2 , \\[2ex] \dfrac{dy}{dt} = -b(t,y)x - S_c(t,y) - q_Y(t)\left(\dfrac{x}{y} - 1\right)^2 , \end{cases}$$

where the attrition-rate coefficient for the ambusher "aimed" fire $a(t)$ is modelled by (7.6.13) and the ambushee return-fire effectiveness $b(t,y)$ is modelled by (7.6.14). Here the ambushee return fire, as modelled by (7.6.14), transitions from pure "area" fire to pure "aimed" fire.

$$a(t) = \frac{\nu_Y A_{T_X}(t)\ P(K|H)_{XY}}{2\pi\sigma_Y^2} \tag{7.6.13}$$

with the presented area of a single X ambushee being modelled by

$$A_{T_X}(t) = \frac{A_{T_\infty}}{1 - e^{-\alpha t-\beta}} .$$

Here, $A_{T_\infty}$ denotes the "steady-state" value for the vulnerable area of a single X ambushee, and $\alpha$ and $\beta$ reflect the speed with which an ambushee can approach this level of maximum cover. A typical value for $A_{T_\infty}$ for prone troops against rifle fire is 0.1 $ft^2$. SCHAFFER modelled the ambushee's return fire against the ambushers with

$$b(t,y) = \underbrace{b_1(1 - e^{-\gamma t})}_{\substack{\text{"aimed-fire"} \\ \text{contribution}}} + \underbrace{b_2 y\ e^{-\gamma t}}_{\substack{\text{"area-fire"} \\ \text{contribution}}} \tag{7.6.14}$$

where $b_1$ and $b_2$ denote attrition-rate coefficients for "aimed" and "area" fire respectively, and $\gamma$ denotes the transition rate from "area" to "aimed" fire. The parameter $\gamma$ is used to model how fast the ambushees recover from being "caught by surprise" in the ambush. SCHAFFER, however, expressed in terms of two other parameters: a factor of increase in the effectiveness of the ambushees' return fire, F, and a time for this increase to occur, $\tau$. We then have

$$Fb_2y_0 = b_1(1 - e^{-\gamma\tau}) + b_2ye^{-\gamma\tau}, \tag{7.6.15}$$

whence

$$\gamma = \left(\frac{1}{\tau}\right) \ln\left[\frac{1 - b_2 y/b_1}{1 - F b_2 y_0/b_1}\right] . \tag{7.6.16}$$

SCHAFFER observed that typical values for the battle parameters yield $b_2 y/b_1 < b_2 y_0/b_1 << 1$, whence we have the approximation for (7.6.16)

$$\gamma \approx \left(\frac{1}{\tau}\right) \ln\left[\frac{1}{1 - F b_2 y_0/b_1}\right] . \tag{7.6.17}$$

Some "typical" results for ambushes modelled by (7.6.12) are shown in Figure 7.10. SCHAFFER concluded from his study of ambushes modelled by (7.6.12) that "in the absence of supporting weapons, ambushes can be successful against forces that are numerically twice as large as the ambusher's force, provided the ambushee has less than perfect discipline and/or is sluggish in attaining aiming parity with his opponent." His analysis showed that a properly conducted ambush should be an excellent tactic (see SCHAFFER [125, pp. 483-484] for further details).

Finally, SCHAFFER considered sieges, which he divided into two stages: (i) a "softening-up" phase with supporting weapons, and (II) an assault stage during which the artillery fire must be lifted. He modelled an assault with the following LANCHESTER-type equations (after work by BRACKNEY [20] on tactical posture and the functional form for an attrition rate; see also Section 7.2 above)

$$\begin{cases} \dfrac{dx}{dt} = -(1-p_X)\,\dfrac{P_{SSK_{XY}}}{t_{XY}}\,y \quad \text{(ATTACKER ATTRITION) with } x(0) = x_0 , \\[2ex] \dfrac{dy}{dt} = -(1-p_Y)\,\dfrac{xy}{k_X A_Y} \quad \text{(DEFENDER ATTRITION) with } y(0) = y_0 , \end{cases} \tag{7.6.18}$$

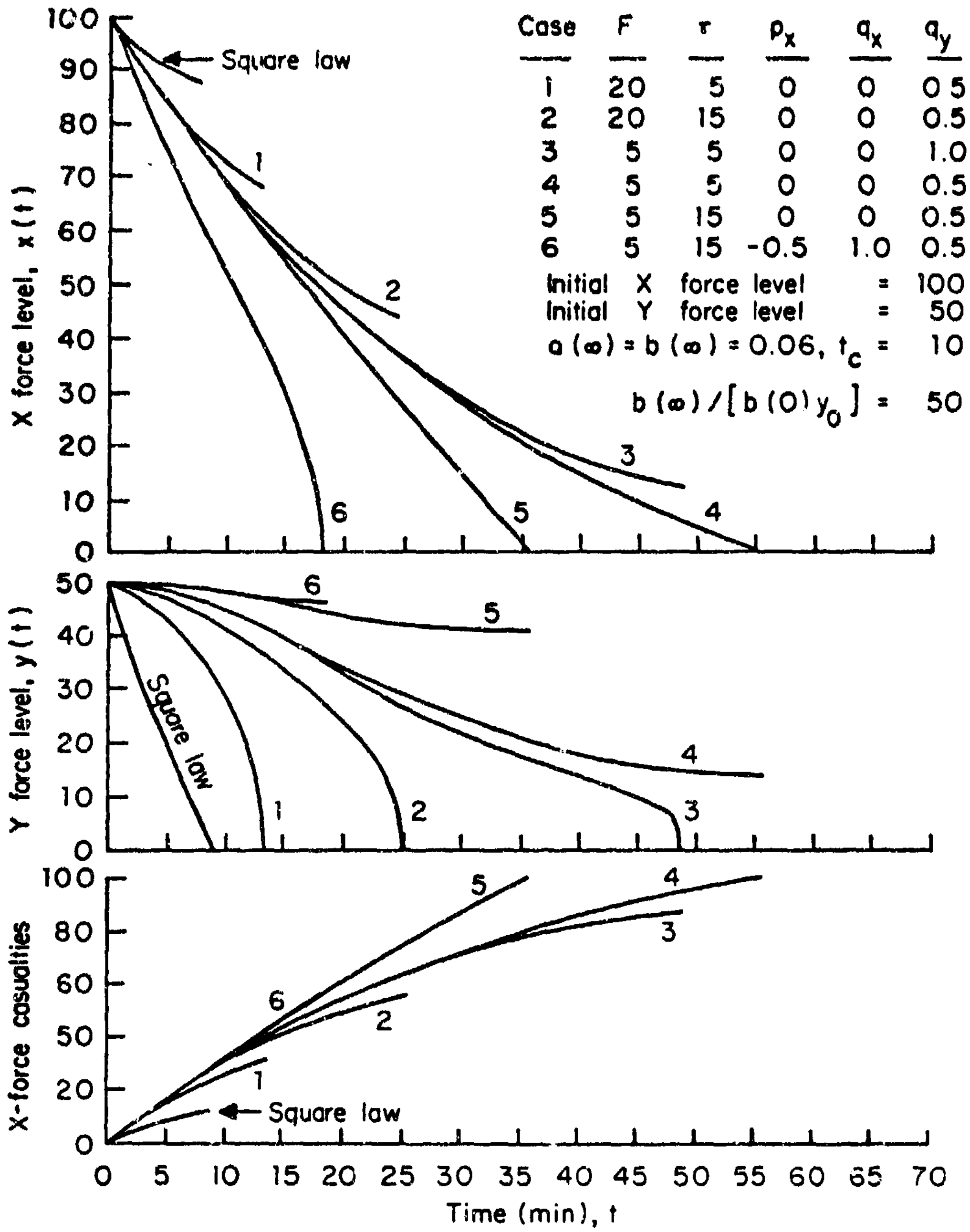


Figure 7.10. Results for model of ambush with no supporting weapons on either side (after SCHAFFER [125]). In this case, the battle dynamics are given by

$$\begin{cases} \dfrac{dx}{dt} = -(1 - p_X)\, a(t)y - q_X\left(\dfrac{y}{x} - 1\right)^2 , \\[2ex] \dfrac{dy}{dt} = -\{b_1(1-e^{-\gamma t}) + b_2 y e^{-\gamma t}\}x - q_Y(t)\left(\dfrac{x}{y} - 1\right)^2 , \end{cases}$$

where $a(t)$ is modelled by (7.6.13).

where $t_{XY}$ denotes the average time between the firing of two successive rounds by a single defender (with target-acquisition times being assumed negligible), and the average time for an attacker to acquire a target by visual search of the defender's position (with "presented area" $A_Y$) is assumed to be inversely proportional to target density (with constant of proportionality $k_X$) and is assumed to be the dominant (i.e. constraining) factor in the target-attrition process for the defenders. In the assault modelled by (7.6.18), X is the attacker and Y is the defender. Thus, the time for a single assaulting firer to destroy an emeny defensive target is approximately equal to the time for him to acquire one, and the average time for an assault troop to acquire such a defensive target is given by $k_X A_Y/y$. The model (7.6.18), of course, only applies to the assault situation up until the time the defensive perimeter is overrun or until a counterattack is launched.

Thus, SCHAFFER [125] developed a number of detailed LANCHESTER-type models of small-scale guerrilla-warfare engagements. These were apparently the first detailed LANCHESTER-type models of tactical engagements to be developed and applied to military-analysis problems in the United States. His models contained a number of significant operational enrichments (e.g. time-dependent attrition-rate coefficients reflecting changes in tactical posture, fire discipline, calling in of supporting fires, etc.) over previously considered simplistic LANCHESTER-type models (e.g. the classic constant-coefficient models (2.2.1) and (2.4.1) of LANCHESTER [104]). SCHAFFER developed a number of important quantitative insights into the dynamics of guerrilla-warfare operations from exercising these models (see SCHAFFER [125] for further details).

## 7.7. Modelling Attrition for Combat Between Heterogeneous Forces.

So far in this book we have considered various aspects of attrition modelling for combat between two homogeneous forces, but actual combat consists of many different weapon-system types operating together as "combined-arms teams." For example, there may be infantry (armed with several types of weapons), tanks, artillery, mortars, etc. on each side. Let us therefore consider combat between such heterogeneous forces and briefly indicate how the above basic ideas on modelling combat attrition are extended and adapted to such cases.

For illustrative purposes, we consider an engagement with $m$ different types of weapon systems on the $X$ side and $n$ for $Y$ (see Figure 7.11). Although more complicated types of force interactions may be postulated, we will consider the "natural" extension of (2.2.1) to this combat situation. We accordingly assume that

(A1) the attrition effects of various different enemy weapon-system types against a particular friendly target type are additive (no mutual support, i.e. no synergistic effects),

and (A2) the loss rate to each enemy weapon-system type is proportional to the number of enemy firers of that type.

Let $Y_{ij}$ denote those $Y_j$ who engage $X_i$, and let $y_{ij}$ denote the corresponding number of $Y_{ij}$ and similarly for $y_j$. Similar quantities are analogously defined for the $X$ force. We observe that we then have

$$y_j = \sum_{i=1}^{m} y_{ij} . \qquad (7.7.1)$$

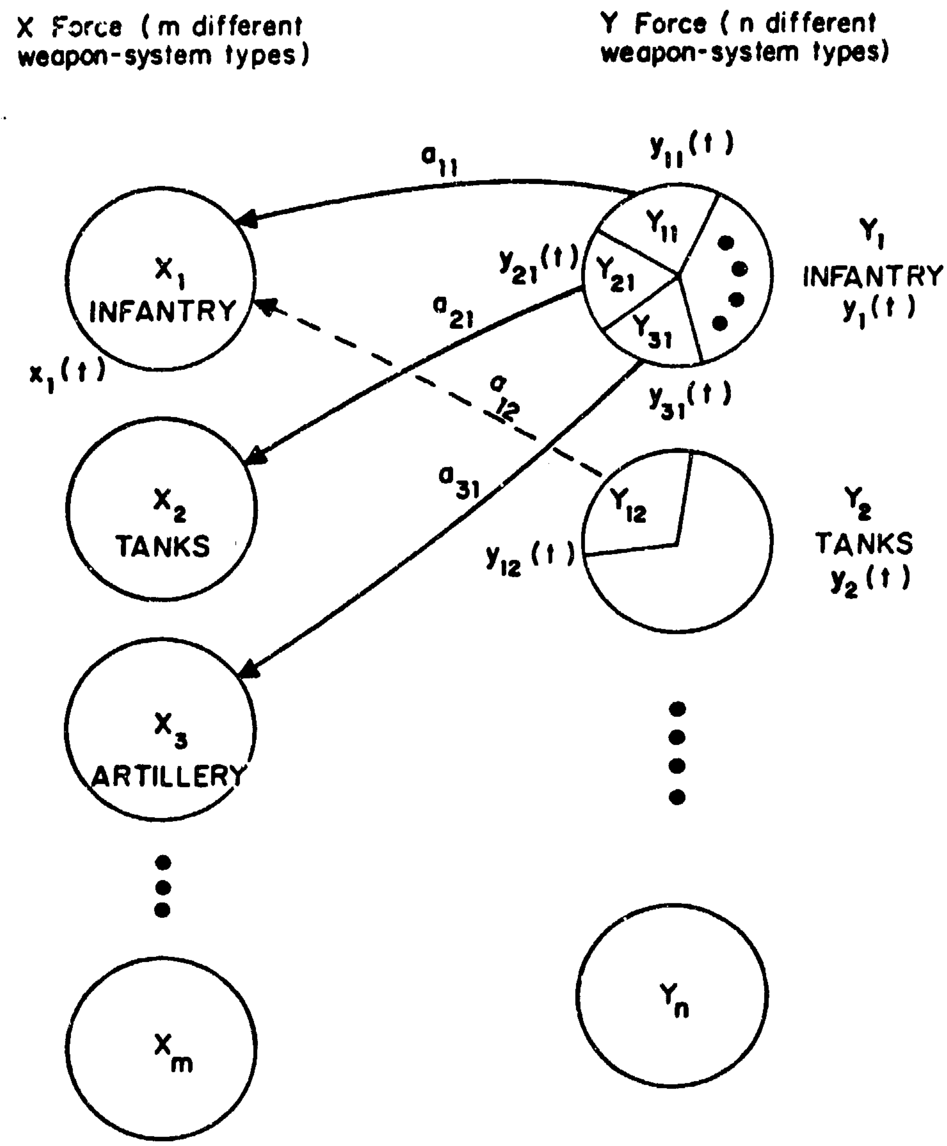


Figure 7.11. Schematic of combat between heterogeneous forces. In this figure $Y_{ij}$ denotes those $Y_j$ who are engaging $X_i$, and $y_{ij}$ denotes the corresponding number of $Y_{ij}$ and similarly for $y_j$. Also, $a_{ij}$ denotes the "inherent" weapon-system kill rate of one $Y_j$ against live $X_i$ targets, i.e. the rate at which one $Y_j$ can kill $X_i$ targets.

For notational convenience we will always let the subscript $i$ refer to the $X$ force and the subscript $j$ refer to the $Y$ force. Thus (recall Figure 7.11), the index $i$ will always take on the integer values 1 through $m$ and the index $j$ will always take on the integer values 1 through $n$. In other words, $X_{ji}$ denotes those $X_i$ who engage $Y_j$ with $i = 1,2,\ldots, m$ and $j = 1,2,\ldots, n$. Hence, without further specification if we say $x_i > 0$, it will be understood that the inequality holds for $i = 1,2,\ldots,m$.

For modelling combat between heterogeneous forces, one must take into account that a particular firer type can try to engage various different enemy target types. Hence, we must represent how fire is distributed over enemy target types. Accordingly, we will now introduce the allocation factor $\psi_{ij} = y_{ij}/y_j$ = fraction of $Y_j$ who engage $X_i$. It follows that

$$y_{ij} = \psi_{ij} y_j . \tag{7.7.2}$$

To complete our notational preliminaries, we let $a_{ij}$ denote the "inherent" weapon-system kill rate of $Y_j$ against live $X_i$ targets, i.e. the rate at which one $Y_j$ can kill $X_i$ targets.

Let us now examine how (A1) and (A2) lead to the following linear model (no synergistic effects for weapon systems in joint operations) for $x_i$ and $y_j > 0$

$$\begin{cases} \dfrac{dx_i}{dt} = -\displaystyle\sum_{j=1}^{n} \psi_{ij} a_{ij} y_j & \text{with } x_i(0) = x_i^0 , \\[2ex] \dfrac{dy_j}{dt} = -\displaystyle\sum_{i=1}^{m} \phi_{ji} b_{ji} x_i & \text{with } y_j(0) = y_j^0 , \end{cases} \tag{7.7.3}$$

where $0 \le \phi_{ji}, \psi_{ij} \le 1$, and on physical grounds $a_{ij}$ and $b_{ji} \ge 0$. Let us now develop (7.7.3) from assumptions (A1) and (A2) above. Assumption (A1) may be stated in mathematical terms as, for example,

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} \begin{pmatrix} X_i \text{ loss rate} \\ \text{due to } Y_j \end{pmatrix}, \tag{7.7.4}$$

while assumption (A2) means that

$$\begin{pmatrix} X_i \text{ loss rate} \\ \text{due to } Y_j \end{pmatrix} = a_{ij}y_{ij} = a_{ij}\psi_{ij}y_j , \tag{7.7.5}$$

whence follows (7.7.3) from combination with (7.7.4). If we "absorb" the allocation factors into the attrition-rate coefficients, e.g. let $A_{ij} = \psi_{ij}a_{ij}$, then our linear combat model (7.7.3) may be written as (for $x_i$ and $y_j > 0$)

$$\begin{cases} \dfrac{dx_i}{dt} = -\displaystyle\sum_{j=1}^{n} A_{ij}y_j & \text{with } x_i(0) = x_i^0 , \\[2ex] \dfrac{dy_j}{dt} = -\displaystyle\sum_{i=1}^{m} B_{ji}x_i & \text{with } y_j(0) = y_j^0 . \end{cases} \tag{7.7.6}$$

If we add operational losses [or attrition from enemy supporting weapons not subject to attrition (<u>see</u> Sections 6.12 and 6.13 for further details)], then our combat model becomes (again, for $x_i$ and $y_j > 0$)

$$\begin{cases} \dfrac{dx_i}{dt} = -\displaystyle\sum_{j=1}^{n} A_{ij}y_j - \beta_i x_i & \text{with } x_i(0) = x_i^0 , \\[2ex] \dfrac{dy_j}{dt} = -\displaystyle\sum_{i=1}^{m} B_{ji}x_i - \alpha_j y_j & \text{with } y_j(0) = y_j^0 , \end{cases} \tag{7.7.7}$$

where $\alpha_j$ denotes an attrition-rate coefficient modelling the operational losses of $Y_j$ and similarly for $\beta_i$. On physical grounds, we must have $\alpha_j$ and $\beta_i \geq 0$.

In complex operational LANCHESTER-type combat models like BONDER/IUA and its may derivatives,[21] attrition-rate coefficients corresponding to $A_{ij}$ and $B_{ji}$ in (7.7.6) above are (as they are in the real world) complex functions of the weapon-system capabilities, target characteristics, distribution of the targets, allocation procedures for assigning weapons to targets, etc. These models then attempt to reflect these complexities by partitioning the attrition process into four distinct subprocesses:

(1) the fire effectiveness of weapon-system types firing on live targets,

(2) the allocation process of assigning weapons to targets,

(3) the inefficiency of fire when weapon-system types engage other than live targets,

and (4) the effects of terrain on limiting firing activities of weapon-system types and on mobility of the systems.

BONDER and FARRELL [15, pp. 16-17] have included the effects of the first three subprocesses above on an attrition-rate coefficient, for example, as

$$A_{ij}(r) = \psi_{ij} I_{ij}^{Y} a_{ij}(r) , \tag{7.7.8}$$

where $\psi_{ij}$ denotes the allocation factor (the fraction of $Y_j$ who are assigned to engage $X_i$), $I_{ij}^{Y}$ denotes the intelligence factor (the fraction of $Y_{ij}$ who are actually engaging live $X_i$ targets), and $a_{ij}(r)$ denotes the "inherent" weapon-system kill rate (the rate at which one $Y_j$ kills live $X_i$ targets when it is engaging only them). Here, for simplicity, we have assumed that the inherent weapon-system-kill capability (as quantified by $a_{ij}$) depends on only the range between firer and target (see BONDER and FARRELL [15] for further details). Similar to the case of homogeneous forces, the "inherent" weapon-system kill rate $a_{ij}$ is computed as

$$a_{ij} = \frac{1}{E[T_{X_iY_j}]} , \tag{7.7.9}$$

where $T_{X_iY_j}$ ( a r.v.) denotes the time for a single $Y_j$ firer to kill an $X_i$ target.

Thus, BONDER and FARRELL's [15] approach (see also CHERRY [30] and [117; 154]) basically decomposes the battlefield into unit engagements, and there are further decomposed into a series of one-on-one duels between opposing weapon-system types. For each firer-target pair one must perform a detailed analysis of a single firer engaging a passive target. Force interactions are then tied together with attrition equations similar to (7.7.6), and these assessment equations are made to respond to the evolution of combat (e.g. changing firer positions) through the operational factors influencing kill rates. Terrain effects are incorporated into such models by computing intervisibility (i.e. existence of line-of-sight) for each target-firer pair based on their map

locations. Consideration is given to cover, concealment, terrain roughness, etc. but time does not allow us to go into further details here (see Chapter 5, especially Section 5.16, for further developments, however).

Let us finally consider the determination of numerical values for the allocation factors $\phi_{ji}$ and $\psi_{ij}$ in the heterogeneous-force model (7.7.3). We first observe that (in some sense) X controls (i.e. influences or can affect) $\phi_{ji}$ but such an allocation factor is not directly affected by Y. Similarly, Y controls $\psi_{ij}$. There are then two basically different approaches for determining numerical values[22] for such allocation factors in a tactical engagement:

(1) the <u>descriptive approach</u> (based on asking the question, "How would fire be allocated?"),

and (2) the <u>normative approach</u> (based on asking the question, "How should fire be allocated?").

Both these approaches involve building a model of the allocation process. The descriptive approach is based on observing how people make such decisions in real-world situations, while the normative approach is based on modelling human behavior as a "rational process" with an optimization problem. This latter normative approach may also be thought of as being based on asking the question, "What is the 'best' choice for the allocation factors?" Further discussion of this important topic of determining values for such allocation factors would take us too far afield from our main subject of modelling tactical engagements, but we will return to it in Chapter 8 (see also Section 5.16).

## 7.8. Analytical Results for Heterogeneous-Force Models.

Let us now briefly discuss what analytical results have been obtained for the heterogeneous-force model (7.7.7). We will find out that, except for some special cases, only a few analytical results of limited usefulness have been developed. In fact, it is essentially impossible to analytically solve systems of differential equations like (7.7.7) for combat interactions with any degree of complexity (recall Figure 6.11). Consequently, numerical-integration methods (see Appendix E) must be generally used to generate numerical results for particular battles of any degree of complexity. Thus, such numerical-integration methods are essentially always used to numerically determine the force levels as functions of time, i.e. $x_i(t)$ and $y_j(t)$, in complex operational models like BONDER/IUA.

In general an attrition-rate coefficient such as $A_{ij}$ in (7.7.7) varies with time $t$ and the force levels of the combatants. When the attrition-rate coefficients $A_{ij}$ and $B_{ji}$ depend on the force levels $x_i$ and $y_j$, the system of differential equations (7.7.7) is nonlinear. We will not consider this case, however, since no useful analytical results are apparently available for such systems of nonlinear ordinary differential equations. When the attrition-rate coefficients do not depend on the force levels, we may take them to depend on time,[23] and we will therefore consider (again for $x_i$ and $y_j > 0$) the following linear combat model with time-dependent attrition-rate coefficients

$$\begin{cases} \dfrac{dx_i}{dt} = -\sum\limits_{j=1}^{n} A_{ij}(t)y_j - \beta_i(t)x_i & \text{with } x_i(0) = x_i^0 , \\ \dfrac{dy_j}{dt} = -\sum\limits_{i=1}^{m} B_{ji}(t)x_i - \alpha_j(t)y_j & \text{with } y_j(0) = y_j^0 , \end{cases} \tag{7.8.1}$$

where (as above) the subscript $i$ runs over the integer values 1 through $m$ and $j$ over 1 through $n$ when such ranges are not explicitly given. However, the substitution

$$\begin{cases} p_i(t) = x_i(t)\exp\{\int_0^t \beta_i(s)ds\}, \\ q_j(t) = y_j(t)\exp\{\int_0^t \alpha_j(s)ds\}, \end{cases} \tag{7.8.2}$$

transforms (7.8.1) into

$$\begin{cases} \dfrac{dp_i}{dt} = -\sum_{j=1}^{m} \tilde{A}_{ij}(t)q_j & \text{with } p_i(0) = x_i^0, \\ \dfrac{dq_j}{dt} = -\sum_{i=1}^{m} \tilde{B}_{ji}(t)p_i & \text{with } q_j(0) = y_j^0, \end{cases} \tag{7.8.3}$$

where

$$\tilde{A}_{ij}(t) = A_{ij}(t)\exp\{\int_0^t [\beta_i(s) - \alpha_j(s)]ds\},$$

and

$$\tilde{B}_{ji}(t) = B_{ji}(t)\exp\{-\int_0^t [\beta_i(s) - \alpha_j(s)]ds\}. \tag{7.8.4}$$

Thus, in discussing the development of analytical solutions, we may without loss of generality consider (7.8.1) with $\beta_i(t)$ and $\alpha_j(t)$ identically equal to zero, i.e. for $x_i$ and $y_j > 0$

$$\begin{cases} \dfrac{dx_i}{dt} = -\sum_{j=1}^{n} A_{ij}(t)y_j & \text{with } x_i(0) = x_i^0, \\ \dfrac{dy_j}{dt} = -\sum_{i=1}^{m} B_{ji}(t)x_i & \text{with } y_j(0) = y_j^0. \end{cases} \tag{7.8.5}$$

Although equations (7.8.5) are a linear differential-equation combat model and consequently all the results from the theory of linear ordinary differential equations may be invoked, essentially no explicit analytical results for $x_i(t)$ and $y_j(t)$ of practical significance for military OR are known to this author. We can, of course, in theory use the method of successive approximations (cf. Section 6.5 above) to determine $x_i(t)$ and $y_j(t)$, but the details are prohibitively complex. Let us proceed just far enough to indicate such difficulties to the reader.

It is, moreover, convenient to express such computations in a more compact notation. Therefore, let us write (7.8.5) in vector/matrix notation as

$$\begin{cases} \dot{\underset{\sim}{x}} = -A(t)\underset{\sim}{y} & \text{with } \underset{\sim}{x}(0) = \underset{\sim}{x}_0, \\ \dot{\underset{\sim}{y}} = -B(t)\underset{\sim}{x} & \text{with } \underset{\sim}{y}(0) = \underset{\sim}{y}_0, \end{cases} \tag{7.8.6}$$

where $\dot{\underset{\sim}{x}}$ denotes $d\underset{\sim}{x}/dt$, $\underset{\sim}{x}$ denotes a column vector of the $m$ force levels of the heterogeneous $X$ force [i.e. $\underset{\sim}{x}^T = (x_1, x_2, \ldots, x_m)$], $B(t)$ denotes an $n \times m$ matrix of attrition-rate coefficients (i.e. $B(t) = [B_{ji}(t)]$, where $[B_{ji}(t)]$ denotes the matrix with element $B_{ji}(t)$ in the $j$th row and $i$th column for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$), and similar quantities for the $Y$ force are analogously defined, with $\underset{\sim}{y}$ being an n-vector and $A(t)$ an $m \times n$ matrix. We may write (7.8.6) in even more compact notation by introducing $\underset{\sim}{w}^T = (\underset{\sim}{x}^T, \underset{\sim}{y}^T)$ so that it becomes (for $\underset{\sim}{w} > 0$)

$$\dot{\underset{\sim}{w}} = -C(t)\underset{\sim}{w} \qquad \text{with } \underset{\sim}{w}^T(0) = \underset{\sim}{w}_0^T = (\underset{\sim}{x}_0^T, \underset{\sim}{y}_0^T), \tag{7.8.7}$$

where $C(t)$ denotes the following $(m + n) \times (m + n)$ matrix

$$C(t) = \begin{bmatrix} 0 & A(t) \\ B(t) & 0 \end{bmatrix} \quad .$$

Assuming the appropriate integrability of the coefficients [i.e. $C(t) \in L(0,T)$ for any finite $T$], and apply the method of successive approximations (cf. Section 6.5), one may show (e.g. see REID [122, pp. 62-63]) that the solution to (7.8.7) is given by

$$\underset{\sim}{w}(t) = \Omega_0^t(C)\underset{\sim}{w}_0 \quad , \tag{7.8.8}$$

where $\Omega_0^t(C)$ denotes the following infinite series of matrices

$$\Omega_0^t(C) = I - \int_0^t C(s_1)ds_1 + \int_0^t C(s_1)\ \{\int_0^{s_1} C(s_2)ds_2\}ds_1$$
$$- \int_0^t C(s_1)\left\{\int_0^{s_1} C(s_2)\ \{\int_0^{s_2} C(s_3)ds_3\ \}ds_2\right\}ds_1 + \cdots , \tag{7.8.9}$$

$I$ denotes the $(m + n) \times (m + n)$ identity matrix, and the integrals are matrix integrals. The matrix quantity $\Omega_0^t(C)$ is sometimes called the matrizant [122, p. 63]. It is the $(m + n) \times (m + n)$ matrix of fundamental solutions to (7.8.7) and satisfies the matrix differential equation (see REID [122] for further details)

$$\dot{W} = - C(t)W \qquad \text{with } W(0) = I , \tag{7.8.10}$$

where $W(t) = \Omega_0^t(C)$.

Example 7.8.1. We may obtain the representation (6.5.16) for the solution $C_X(t)$ to (6.5.13) as a special case of (7.8.8). To see this, we let $\underset{\sim}{w}^T = (C_X, S_Y)$ and then (6.5.13) may be written in the form (7.8.7) with $\underset{\sim}{w}_0^T = (1,0)$ and

$$C(t) = \begin{bmatrix} 0 & a(t)/\sqrt{\lambda_R} \\ b(t)\sqrt{\lambda_R} & 0 \end{bmatrix} .$$

If we substitute the above into (7.8.8) and (7.8.9), we find that $C_X(t)$ is given by (6.5.16). Thus, the successive-approximation results of Chapter 6 for the hyperbolic-like GLF may be viewed as special cases of the matrizant (7.8.9).

The reader should note that (7.8.8) also applies to the more general model

$$\begin{cases} \dot{\underset{\sim}{x}} = -A(t)\underset{\sim}{y} - G(t)\underset{\sim}{x} & \text{with } \underset{\sim}{x}(0) = \underset{\sim}{x}_0 , \\ \dot{\underset{\sim}{y}} = -B(t)\underset{\sim}{x} - H(t)\underset{\sim}{y} & \text{with } \underset{\sim}{y}(0) = \underset{\sim}{y}_0 , \end{cases} \tag{7.8.11}$$

in which case $C(t)$ is given by

$$C(t) = \begin{bmatrix} G(t) & A(t) \\ B(t) & H(t) \end{bmatrix} . \tag{7.8.12}$$

The above results are also readily extended to the case in which replacements are continuously added to the battle (7.8.6) [or, equivalently, (7.8.11)]. Accordingly, we let $\underset{\sim}{q}(t)$ denote an $m \times n$ column vector of replacement rates. Our model (7.8.7) then becomes

$$\dot{\underset{\sim}{w}} = -C(t)\underset{\sim}{w} + \underset{\sim}{g}(t) \qquad \text{with} \quad \underset{\sim}{w}^T(0) = \underset{\sim}{w}_0^T = (\underset{\sim}{x}_0^T, \underset{\sim}{y}_0^T) \tag{7.8.13}$$

The solution to (7.8.13) may be written as (e.g. see REID [122] again)

$$\underset{\sim}{w}(t) = \Omega_0^t(C)\underset{\sim}{w}_0 + \Omega_0^t(C) \int_0^t [\Omega_0^s(C)]^{-1} \underset{\sim}{g}(s)ds \ , \tag{7.8.14}$$

where $\Omega_0^t(C)$ is given by (7.8.9) and $[\Omega_0^t(C)]^{-1}$ denotes the inverse operator $\Omega^{-1}(C)$ of $\Omega(C)$. Thus, the force levels as functions of time are even more complicated when replacements are continuously committed to LANCHESTER-type combat [cf. (6.12.8) and Figure 6.11]. As we noted in Chapter 6 (recall Figure 6.11), it is impossible to "solve" the differential-equation combat model (7.8.13) when both $m$ and $n > 1$, although a formal solution such as (7.8.14) may, of course, be written down.

The solutions (7.8.8) and (7.8.14) are formal symbolic solutions to (7.8.7) and (7.8.13) for the vector of force levels $\underset{\sim}{w}^T(t) = (\underset{\sim}{x}^T(t), \underset{\sim}{y}^T(t))$. Unfortunately, they are of no computational use when both $m$ and $n > 1$. Thus, although they symbolically represent the force levels, the "solutions" (7.8.8) and (7.8.14) have been put to no practical use.

Let us now consider the model (7.8.6) in the special case of constant attrition-rate coefficients, i.e. for $\underset{\sim}{x} > 0$ and $\underset{\sim}{y} > 0$

$$\begin{cases} \dot{\underset{\sim}{x}} = -A\underset{\sim}{y} & \text{with} \quad \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \ , \\ \dot{\underset{\sim}{y}} = -B\underset{\sim}{x} & \text{with} \quad \underset{\sim}{y}(0) = \underset{\sim}{y}_0 \ . \end{cases} \tag{7.8.15}$$

where $A$ denotes an $m \times n$ matrix of constant attrition-rate coefficients modelling the fire effectiveness of the heterogeneous $Y$ force and $B$ denotes an $n \times m$ matrix of constant attrition-rate coefficients for the $X$ force. Similar to what we saw in Chapter 2, the two basic vehicles for answering questions concerning the outcome of combat modelled by the constant-coefficient differential equations (7.8.15) are: (1) the state equation, and (2) the $X$ and $Y$ force levels as a function of time $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$. Unlike the case of combat between two homogeneous forces, though, we now deal with vectors and matrices, not scalars, and far fewer explicit analytical results have been developed.

To obtain information concerning parity (i.e. equal military strength) between the two opposing heterogeneous forces we consider the <u>state equation</u>. By parity we mean that neither force ever "wins," and of course we must specify battle-termination conditions for such a determination. We will limit our discussion to a fight to the finish, since results have only appeared for this special case. Since negative force levels make no physical sense (<u>cf</u>. our discussion in Section 2.2), we must accordingly extend the model (7.8.15), which holds for $x$ and $y > 0$, to cases in which one or more of the component forces of either heterogeneous force become annihilated. If we are to retain constant coefficients, we must essentially assume that there is no redistribution of fire by friendly forces after an enemy target type has been annihilated. In this case, the natural extension of (7.8.15) is

$$\begin{cases} \dot{\underset{\sim}{x}} = -E_X(\underset{\sim}{x})A\underset{\sim}{y} & \text{with } \underset{\sim}{x}(0) = \underset{\sim}{x}_0 , \\ \dot{\underset{\sim}{y}} = -E_Y(\underset{\sim}{y})B\underset{\sim}{x} & \text{with } \underset{\sim}{y}(0) = \underset{\sim}{y}_0 , \end{cases} \tag{7.8.16}$$

where $E_X(\underset{\sim}{x})$ is an $m \times m$ diagonal matrix with diagonal element

$$e_{ii}^X(\underset{\sim}{x}) = \begin{cases} 1 & \text{for } x_i > 0 , \\ 0 & \text{otherwise} \end{cases} \tag{7.8.17}$$

and similarly for $E_Y(\underset{\sim}{y})$.

Equations (7.8.16) and (7.8.17) are nothing more than the generalization to heterogeneous-force combat of LANCHESTER's equations written in a form to avoid the physical absurdity of negative force levels. In other words, $X_i$ only suffers attrition according to the appropriate component of (7.8.15) as long as $x_i > 0$ (i.e. $dx_i/dt = -\sum_{j=1}^{n} a_{ij}y_j$ for $x_i > 0$), and such an attrition equation is "turned off" once $x_i = 0$ (i.e. $dx_i/dt = 0$ for $x_i \leq 0$) [cf. (2.2.2)]. By parity between the forces, we simply mean that $x_i(t)$ and $y_j(t) > 0$ for all $i$, $j$, and finite $t \geq 0$. Unfortunately, there is generally no extension of LANCHESTER's square law of parity between two homogeneous forces (2.1.6) to combat between such heterogeneous forces. However, SNOW [133] has shown[24] that in one and only one special case does the square law (2.1.6) generalize to combat between heterogeneous forces: namely, the condition for parity between heterogeneous X and Y forces is given by the following quadratic expression for the force levels

$$a_{IJ} \sum_{i=1}^{m} \left(\frac{b_{Ji}}{a_{iJ}}\right) x_i^2 = b_{JI} \sum_{j=1}^{n} \left(\frac{a_{Ij}}{b_{jI}}\right) y_j^2 , \tag{7.8.18}$$

if and only if for any two fixed indices $I$ and $J$

$$\frac{a_{IJ}a_{ij}}{a_{iJ}a_{Ij}} = \frac{b_{JI}b_{ji}}{b_{Ji}b_{jI}} , \tag{7.8.19}$$

where $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. The condition (7.8.19) was called Condition M by SNOW [133].

For developing, for example, the X force level as a function of time $\underset{\sim}{x}(t)$, there are two different (but equivalent) methods for constant attrition-rate coefficients and heterogeneous forces:

(M1) a matrix-theory approach that involves evaluation of a <u>matrix exponential function</u>,

and (M2) algebraic elimination to obtain the $\underline{X_i}$ <u>force-level equation</u> (which contains only $x_i$).

Although (in both cases) one finds that $x_i(t)$ is simply a sum of certain exponential functions of time weighted by coefficients that are functions of only the attrition-rate coefficients and initial force levels, explicit results (even for the simplest $2 \times 2$ case) have not been generally obtained for (7.8.15) (<u>recall</u> Figure 6.11 of Chapter 6). Thus, although the general form of the solution is well known, it is so complex that explicit analytical results have not been obtained except in special cases. We will now briefly illustrate each of the above solution methods. In both examinations we will only consider the case in which $x_i$ and $y_j > 0$, and then (7.8.15) applies.

The <u>matrix-theory approach</u> consists of considering the vector differential equation (7.8.7) for $\underset{\sim}{w}^T = (\underset{\sim}{x}^T,\underset{\sim}{y}^T)$, namely

$$\dot{\underset{\sim}{w}} = -C\underset{\sim}{w} \qquad \text{with } \underset{\sim}{w}^T(0) = \underset{\sim}{w}_0^T = (\underset{\sim}{x}_0^T,\underset{\sim}{y}_0^T) , \qquad (7.8.20)$$

where $C$ denotes the $(m + n) \times (m + n)$ matrix of constant attrition-rate coefficients given by

$$C = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} . \qquad (7.8.21)$$

In this case the matrizant (7.8.9) reduces to the matrix exponential

$$e^{-Ct} = \sum_{k=0}^{\infty} (-1)^k C^k \left(\frac{t^k}{k!}\right) , \qquad (7.8.22)$$

and the solution to (7.8.20) may be written in terms of this matrix exponential as

$$w(t) = e^{-Ct} w_0 . \qquad (7.8.23)$$

Thus, we are left with the task of evaluating the matrix exponential $e^{-Ct}$ with $C$ given by (7.8.21).

The complexity of evaluating the matrix exponential depends essentially on whether or not the matrix $C$ has distinct eigenvalues. Let $|C|$ denote the determinant of $C$ and

$$\Delta(\lambda) = |C - \lambda I| . \qquad (7.8.24)$$

The eigenvalues of $C$ (as the reader will recall) are the roots of the $(m + n)$ degree polynomial equation

$$\Delta(\lambda) = 0 = |C - \lambda I| . \qquad (7.8.25)$$

Consider now the $(m + n)$ roots of (7.8.25) and assume that there are $q$ distinct values. Let $N_k$ denote the multiplicity minus one of the kth eigenvalue. It

follows that $q + \sum_{k=1}^{q} N_k = m + n$. By the confluent form of SYLVESTER's theorem (see FRAZER, DUNCAN, and COLLAR [59, pp. 78-85]) the matrix exponential $e^{-Ct}$ is given by

$$e^{-Ct} = \sum_{k=1}^{q} \left\{ \sum_{r=0}^{N_k} \left(\frac{t^r}{r!}\right) Z_{N_k - r}(\lambda_k) \right\} e^{-\lambda_k t} , \tag{7.8.26}$$

where

$$Z_{N_k}(\lambda_k) = \frac{1}{(N_k)!} \left\{ \frac{d^{N_k}}{d\lambda^{N_k}} \left(\frac{F(\lambda)}{\Delta_{N_k}(\lambda)}\right)\right\}_{\lambda=\lambda_k} ,$$

$$\Delta_{N_k}(\lambda) = \prod_{r \neq k} (\lambda - \lambda_r)^{(N_r+1)} ,$$

and $F(\lambda)$ denotes the transposed matrix of the cofactors of $\lambda I - C$. In the English mathematical literature $F(\lambda)$ is called the adjoint of $\lambda I - C$ (see [59, p. 21]). The result (7.8.26) may be equivalently developed by considering the JORDAN canonical form for the matrix $C$ (see CODDINGTON and LEVINSON [38, Chapter 3]). In the case of distinct eigenvalues for $C$, the above expression for $e^{-Ct}$ simplifies considerably: namely (cf. HILDEBRAND [82, pp. 64-66])

$$e^{-Ct} = \sum_{k=1}^{m+n} Z_0(\lambda_k)\, e^{-\lambda_k t} , \tag{7.8.27}$$

where

$$Z_0(\lambda_k) = \frac{\prod_{r \neq k} (C - \lambda_r I)}{\prod_{r \neq k} (\lambda_k - \lambda_r)} .$$

As the reader may have already guessed, no really useful analytical results have so far been obtained for (7.8.26) except in special cases when other methods are more convenient (see below). Thus, matrix-theory methods show us the form of the solution to (7.8.20) for the X and Y force levels $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$, but these results are generally of little computational use (recall Figure 6.11 of Chapter 6).

Example 7.8.2. For the $(F + T)|(F + T)$ attrition process, we have $m = n = 1$, and (7.8.20) holds with [see equation (2.12.2)]

$$C = \begin{bmatrix} \beta & a \\ b & \alpha \end{bmatrix} .$$

Invoking (7.8.23) with $e^{-Ct}$ given by (7.8.27), we find that, for example,

$$x(t) = e^{-\frac{1}{2}(\alpha+\beta)t} \{ x_0 \cosh \theta t - \frac{1}{\theta} [ay_0 + \frac{1}{2} (\beta-\alpha)] \sinh \theta t\} ,$$

where $\theta = \sqrt{ab + \{(\beta-\alpha)/2\}^2}$ .

The algebraic-elimination approach relies on the differential-equation combat model's special structure to use differentiation and algebraic elimination to develop a Nth order (where $N < m + n$) linear differential equation for each of the force levels. When there is a simple solution for the force levels to the linear combat model (7.8.15), this approach is the simplest one for obtaining it. Let us now illustrate the algebraic-elimination approach with a simple example. Consider a homogeneous Y force in combat against two enemy weapon-system types. Then, for $x_i$ and $y > 0$, we have

$$\begin{cases} \dfrac{dx_1}{dt} = -a_1 y & \text{with} \quad x_1(0) = x_1^0 , \\ \dfrac{dx_2}{dt} = -a_2 y & \text{with} \quad x_2(0) = x_2^0 , \\ \dfrac{dy}{dt} = -b_1 x_1 - b_2 x_2 & \text{with} \quad y(0) = y_0 . \end{cases} \tag{7.8.28}$$

The $Y$ force level equation is readily obtained by differentiating the last equation of (7.8.28) with respect to time and combining the result with the previous two equations. We find that

$$\frac{d^2y}{dt^2} - (a_1 b_1 + a_2 b_2) y = 0 , \tag{7.8.29}$$

with initial conditions $y(0) = y_0$ and $dy/dt(0) = - b_1 x_1^0 - b_2 x_2^0$. It follows that

$$y(t) = y_0 \cosh \theta t - \left(\frac{z_0}{\theta}\right) \sinh \theta t , \tag{7.8.30}$$

where $z_0 = b_1 x_1^0 + b_2 x_2^0$ and $\theta = \sqrt{a_1 b_1 + a_2 b_2}$ . Also,

$$x_i(t) = x_i^0 + a_i \left\{ \left(\frac{z_0}{\theta^2}\right)(\cosh \theta t - 1) - \left(\frac{y_0}{\theta}\right) \sinh \theta t \right\} . \tag{7.8.31}$$

We may also use algebraic elimination and elementary integration to obtain the following state equation from (7.8.28)

$$z_0^2 - z^2 = \theta^2 (y_0^2 - y^2) , \tag{7.8.32}$$

where $z = z(t) = b_1 x_1 + b_2 x_2$. When the $X$ force is composed of $m$ different weapon system types, the state equation is still given by (7.8.32) and the force

levels by (7.8.30) and (7.8.31), only with $z, z_0$, and $\theta$ now given by $z = z(t) = \sum_{i=1}^{m} b_i x_i(t)$, $z_0 = z(0)$, and $\theta = \sqrt{\sum_{i=1}^{m} a_i b_i}$. The above results for (7.8.28) are the only simple ones known to the author for combat between heterogeneous forces (<u>recall</u> Figure 6.11).

7.9. <u>Current Detailed LANCHESTER-Type Operational Models of Tactical Engagements</u>.

The following are current operational[25] models (used in the United States) that employ detailed LANCHESTER-type equations to assess casualties in tactical engagements:[26]

battalion-level combat: BONDER/IUA and its many derivatives such as BONDER AIRCAV (or IHA), BLDM, AMSWAG, FAST,

division-level combat: DIVOPS

theater-level combat: VECTOR-2

As we have pointed out in Section 1.3, in these models attrition is modelled analytically, but movement is modelled in a simulatory manner. Consequently, these models are not exactly analytical ones, but they are more precisely called <u>hybrid analytical-simulation models</u>. Since all the above detailed differential-equation combat models have been developed by the principals of Vector Research, Inc. (VRI) (<u>see</u> also Footnote 21 above), it seems appropriate to briefly discuss the combat-modelling approach of VRI.

The basic idea[27] behind the modelling approach of VRI is to develop analytical structures that can be used to forecast the evolution of combat over time in terms of battlefield geometry (i.e. troop positions), force levels, and supplies. It is also hypothesized that there exists a functional relation between the results of battle and the initial numbers of forces, types and capabilities of their weapon systems, their doctrine of employment, and the environment, i.e.

$$\begin{pmatrix} \text{Results} \\ \text{of} \\ \text{Battle} \end{pmatrix} \quad \text{are a function of} \quad \left\{ \begin{array}{l} \text{Number of Forces} \\ \text{Types of Weapon Systems} \\ \text{Weapon Capabilities} \\ \text{Doctrine of Employment} \\ \quad \text{(tactics, organization)} \\ \text{Environment} \end{array} \right.$$

Unfortunately, because of the large number of variables involved, such a functional relation is not known for the overall evolution of battle, nor is there sufficient data to develop it empirically. It is therefore assumed that subprocesses can be quantified and modelled for at least short periods of time and extrapolated.

Thus, the VRI approach is to examine the battle for short periods of time and to hypothesize that for each side during such a short period of time:

(1) locations change due to tactical movement,

(2) weapon systems are attrited by enemy activity,

(3) resources are expended,

and

(4) personnel become casualties due to enemy activity.

Heterogeneous-force LANCHESTER-type equations (cf. Section 7.7) are used to represent the loss of weapon systems and personnel. Implicit in such use is the assumption that if the state of the battle is known at the beginning of a small time interval and the actions that take place during this interval are also known,

then the rate at which losses occur can be predicted for this small time interval. Because of this rate focus, differential equations (i.e. LANCHESTER-type equations) are the appropriate modelling tool. Conceptually these models are based on the following two components:

(1) the concept of the state space,

(2) the concept of process models.

As we mentioned in Section 1.6, the state space consists of those variables that allow one to predict the future course of combat, e.g. numbers and locations of different weapon systems, target lists, plans and intentions, etc.

The VRI approach (BONDER and FARRELL [15]; see also [39; 117; 154] and CHERRY [30]) in essence conceptually decomposes the battlefield into unit engagements, which are further decomposed into a series of one-on-one duels between opposing weapon-system types. For each firer-target pair one must perform a detailed analysis of a single firer engaging a passive target (e.g. recall Section 5.3). Force interactions are then tied together with LANCHESTER-type heterogeneous-force attrition equations similar to (7.7.6), and these assessment equations are made to respond to the evolution of combat (e.g. changing firer positions) through the operational factors influencing the kill rates. The evolution of other state variables (e.g. ammunition supplies or battlefield information) are similarly modelled with differential equations. Terrain effects are incorporated into the combat model by computing intervisibility (i.e existence of line of sight) for each target-firer pair based on their map locations.

Consideration is given to cover, concealment, terrain roughness, etc., but time does not allow us to go into further details here (see Section 5.16 or [39; 117; 154] for further details). In such a complex system model, the LANCHESTER-type equations are numerically integrated.

The modern large-scale digital computer has made such detailed models possible, especially those of large-scale combat. Because of the detailed weapon-system-performance information used in their combat assessments, i.e. to compute LANCHESTER attrition-rate coefficients (see Chapter 5, especially Section 5.16), the data and data-base problems associated with such models are, however, formidable although no less so than those for detailed Monte Carlo combat simulations. For example, VECTOR-2 may require between 200,000 and 300,000 pieces of input data for a "typical" run (see BONDER [14] for further details). The interested reader can find further information about the time and resource requirements for actually using these models in [9] (e.g. the time required to acquire input data, the time required to structure this data in the model's input format, the time required to run the model, and the time required to analyze and evaluate the model's results). Such models consider heterogeneous forces, battle plans (ground order of battle and air order of battle), target acquisition, allocation of fire, fire support by ground weapons, movement, intelligence, command and control, logistics, etc. The full extent of combat systems and processes that have been incorporated into the VRI models is indicated in Tables 7.III and 7.IV (see CHERRY [30] and [39; 117; 154] for further details). These very complicated operational models, however, have been developed from the basic analytical structure discussed above by the process of enrichment, which we have also considered above (e.g. see Section 7.1).

TABLE 7.III. Weapon Systems Included in the Differential Combat Models Developed by Vector Research, Inc. (from CHERRY [30]).

Tanks, including secondary armament
APC's, including multiple armament systems
Anti-Tank Guns and Missiles
Assault Guns
Heavy Machine Guns
Mortars
Rifle-Squad Weapons, including
 light and medium machine guns
 grenade launchers
 mixed-mode weapons
 rifles
Convention, ICM, and Laser-Guided Artillery
Attack Helicopters with
 automatic weapons
 rockets
 command-guided missiles
 self-guided missiles
 laser-guided missiles
Rocket or Missile Artillery
Fixed-Wing Tactical Aircraft with Conventional or Advanced Ordnance
Air Defense Guns and Missiles
Land Mines, including scatterable mines
Jeep and Truck Mounted Weapons
Laser Designators
Target-Acquisition Systems, whether ground or air based,
 including optical and other electromagnetic systems and
 seismic, audio, and other systems
Smoke or Other Obscurant Aerosal, however delivered.

TABLE 7.IV. Processes Modelled in the Differential Combat Models Developed by Vector Research, Inc. (from CHERRY [30]).

Acquisition, "serial" or "parallel," including false acquisitions, acquisitions of dead targets, and mis-identification (and loss of acquisition)

Target Selection, including criteria for the acceptance of low-priority targets (an approximate minimax target-selection process is available in addition to descriptive models)

Aiming, Round Selection, and Mode-of-Fire Selection, including fire adjustment process

Firing, direct and indirect: single rounds, volley, and burst; adjusted and unadjusted; ballistic ordnance, command-guided ordnance, self-guided ordnance, illumination-guided ordnance; etc.

Ordnance Lethality, immediate or delayed, against weapons-system hardware or crew, including multiple damage states (which may involve damage to only one component or sub-system of the weapons system, such as a mobility kill or a partial firepower kill)

Maneuver

Deliberate Deterministic or Stochastic Use of Local Terrain or Vegetation for Cover and Concealment, including (but not limited to) suppression by artillery or direct fires

Communication of Target-Acquisition Information Between Weapon Systems

Damage Recovery, including re-manning of a weapon system which has suffered a crew kill

Minefield Encounter, including initial encounter attrition, attrition during reorganization (if any), clearing- or passage-tactics decision, maneuver alterations for clearing, passage, or attempted bypassing, and attrition by mines during passage, clearing, etc.

Aerosol Generation and Consequent Acquisition and Illimunation Environmental Degradation

## 7.10. Overview of Aggregated-Force Models of Attrition in Tactical Engagements.

In stark contrast to the detailed LANCHESTER-type models of attrition in tactical engagements are the aggregated-force attrition models that combine all the various different weapon-system types on a side in some particular geographical combat area (or "sector") into a single equivalent homogeneous force. This force's combat capability is quantified by a single scalar quantity called the unit's firepower index. As we discussed above in Section 7.3, the firepower-index approach is only used for modelling large-scale combat (i.e. division-level operations and larger). The quotient of the firepower indices of the two opposing forces is called the force ratio and is the principal measure of relative combat capability used in analyses of simulated conventional ground combat. It is a major factor considered in the assessment of casualties and the movement of forces against enemy resistance.

Moreover, the daily loss in combat power as quantified by the unit's firepower index is assessed on the basis of several operational factors, principal of which is the force ratio (actually the ratio of the attacker's firepower index to that of the defender). Current theater-level combat models typically use curves of daily fractional (or percentage) casualties versus the force ratio (for both the attacker and also the defender for each of several engagement types such as meeting engagement, attack of prepared position, etc.) for assessing such losses. These curves supposedly have an empirical basis (see [164,pp. 23-28] or ANDERSON et al. [6, p. 53]; however, COCKRELL and BALL [37, especially p. 1-2] have a different opinion). Unfortunately, there is no explicit

relationship between weapon-system parameters, operational factors, and attrition as there is for detailed LANCHESTER-type models (e.g. recall (5.2.1), (5.2.3), and (5.4.1) above in Chapter 5; see also [117, pp. 3-4] or [154]).

Although such aggregated-force models are much simpler than the detailed differential combat models and therefore more computationally convenient, a large-scale digital computer is still required for their implementation. Such aggregated-force models have been fairly widely criticized (see, for example, BONDER [13], HONIG et al. [90], or STOCKFISCH [135]), but large-scale conventional-force ground-combat models that use such aggregation techniques have been and continue to be essentially the only analysis tools used for large-scale conventional-force military analyses in the United States (see [ 9]) and also NATO countries [94]. The simple fact is that some type of aggregation must be done in order to model theater-level combat in a computationally convenient manner.

## 7.11. Aggregation of Forces in Combat Analyses.

The modern battlefield contains many diverse weapon-system types that complement each other and operate as "combined-arms teams." For example, there can be both mounted and dismounted infantry, tanks, various types of anti-tank weapon systems, artillery, mortars, infantry with rifles, infantry with machine guns, etc. One must then either model such operations in great detail or find some means for aggregating forces. Military planners[28] and military operations analysts have consequently developed various index-number approaches for aggregating the diverse combat capabilities of such a heterogeneous military force into a single scalar measure of combat power. Although there are many such indices[29] of the relative combat capabilities of military units, all[30] are essentially variations on the same theme, and consequently we will generically refer to any such index-number approach as a firepower-score approach.

The firepower-score approach develops one single number (referred to as the firepower index) to represent the "combat potential" of a military unit. A linear model is used to develop this index number, i.e. the firepower index, from the scores of individual weapon systems as Table 1.II of Chapter 1 shows. As STOCKFISCH [135] has emphasized, however, the words score and index should not be regarded as being synonymous. We should use the term firepower score to refer to the military capability or value of a specific weapon system and use the term firepower index-- which is obtained by summing scores--to refer to the military capability or value of some aggregation of diverse weapons. In other words, the firepower index of the $X$ force, denoted as $I_X$, is given by

$$I_X = \sum_{i=1}^{n} s_i x_i \quad ,$$

where $s_i$ denotes the firepower score of the ith X system and $x_i$ denotes the number effective in the unit (see Table 1.II again).

Although many firepower-score methods claim that the firepower score of a weapon system is determined as the product of a measure of single-round lethality and the expected expenditure of ammunition during a fixed period of time, in actuality varying amounts of subjectivity are involved in the development of such a firepower score. For this and other reasons (e.g. see HONIG et al. [90]), the firepower-score approach has received a fair amount of criticism. Nevertheless, it is essentially the only approach that has been used to model large-scale combat in currently operational ground-combat models (e.g. see [9]). In other words, unless one duplicates large-scale combat in detail, one must use some type of index-number approach to aggregate the many different types of forces involved in modern large-scale military operations (see last paragraph of Section 7.9). Thus, although it has received varying amounts of criticism from different sources, the firepower-score approach is used by essentially all currently operational large-scale ground-combat models.

In large-scale (i.e. division-level and above) ground-combat models,[31] firepower indices are used as a surrogate for unit strength to[32]:

(1) determine engagement outcomes,

(2) assess casualties,

and (3) determine FEBA movement.

The force ratio is a major factor (but not the only one) used to make such assessments. Here, however, the term <u>force ratio</u> means the ratio of the attacker's firepower index to that of the defender. Consider, for example, the 7th Infantry Division of the U. S. Army and assume that the firepower scores and other data shown in Table 1.II apply. Then the 7th Infantry Division would have a firepower index of 32,640. If an attacking enemy army group were to have a firepower index of 146,880, then we would have a force ratio of 4.5 (A/D), where A refers to the attacker and D to the defender.

## 7.12. General Mathematical Structure of Attrition Calculations in Aggregated-Force Models.

The usual approach (e.g. see [64]) for assessing casualties in firepower-score-based combat models is to have daily casualties (i.e. the casualty rates) depend directly on the following two factors:

(F1) the force ratio,

and (F2) the engagement type.

It will be instructive for us to hold the last factor constant and further examine how casualty assessment depends on the firepower scores and indices.

The basic mathematical structure of the attrition calculation in aggregated-force models may be thought of as being done in two steps and may be explained as follows:

STEP (I)
(Aggregation of Forces)

$$\begin{cases} x_0 = \sum_{i=1}^{n_X} s_i^X x_i^0 \\ \\ y_0 = \sum_{i=1}^{n_Y} s_i^Y y_i^0 , \end{cases} \tag{7.12.1}$$

STEP (II)
(Mutual Attrition of the Aggregated Forces)

$$\begin{cases} \left(-\frac{1}{x}\frac{dx}{dt}\right) = A\left(\frac{x}{y}\right) \quad \text{with} \quad x(0) = x_0, \\ \\ \left(-\frac{1}{y}\frac{dy}{dy}\right) = B\left(\frac{y}{x}\right) \quad \text{with} \quad y(0) = x_0 , \end{cases} \tag{7.12.2}$$

where $s_i^X$ denotes the firepower score of the ith X weapon-system type, $x_i^0$ denotes the initial number of the ith X system, $x_0$ denotes the

initial value of the firepower index for the X force, $x(t)$ denotes its value at time t, $A(x/y)$ denotes a given function of the force ratio, $t = 0$ denotes the start of the attrition calculation, and similarly for the corresponding Y quantities. This calculation is then repeated for each "sector" on the battlefield (see Figure 7.15 in Section 7.15 below). Thus, casualties in terms of a loss in the force's combat power are computed from some expression like (7.12.2). In other words, we only know how much the force's combat power was reduced by a day of combat action, and losses of individual component weapon-system types must be obtained by some means of disaggregation.

ATLAS basically computes casualties in the above manner, with the firepower scores (i.e. $s_i^X$ and $s_i^Y$) being held constant over time. However, IDAGAM dynamically recomputes weapons' values which correspond to the firepower scores $s_i^X$ and $s_i^Y$ above, according to the antipotential-potential (or eigenvector) method (see Section 7.18 below; also HOWES and THRALL [92] or ANDERSON [3; 5]). The latter calculation involves the numbers of enemy targets, allocations of friendly fire, and kill probabilities against enemy targets.

We have given the basic structure for attrition calculations in aggregated-force models above. In actual application such models give attention to a multitude of details on combat operations, e.g. positioning of units, logistics considerations, allocation of fire (especially supporting fires), air defense, air operations including allocation of aircraft to tactical missions, unit breakpoints, terrain factors, intelligence, command and control, order of battle, etc. (e.g. see documentation on

on CEM [25; 106] or IDAGAM [6] for further details). Such operational and tactical factors influence exactly how (7.12.1) is computed.

Finally, let us briefly discuss how the engagement type, the second factor (F2) considered in casualty assessment, is determined. In CEM [15, p. 21; 56, p. 35], for example, the type of engagement is determined by the missions (which are, in turn, determined from an estimate of the situation at various echelons of command) of the opposing forces and, where appropriate, the type of defensive position (see Table 7.V). In the "mission matrix" shown in Table 7.V, the entries are the engagement types, while the rows and columns denote the missions and types of defensive positions of the two opposing forces. Thus, we see that in CEM there are three possible missions (for each side), two types of defensive position, and eight possible types of engagement. Similar methods of engagement-type determination are used in all such large-scale combat models.

TABLE 7.V. Engagement-Type Determination According to Mission and Type of Defensive Position of Each of the Two Opposing Forces (from CEM [25; 106]).

| Red Mission / Blue mission | Red position type / Blue position type | Attack | Defend | | Delay |
|---|---|---|---|---|---|
| | | -- | Prepared | Hasty | -- |
| Attack | -- | Meeting engagement | Blue attack of prepared position | Blue attack of hasty position | Blue advance |
| Defend | Prepared | Red attack of prepared position | Static | Static | Static |
| | Hasty | Red attack of hasty position | Static | Static | Static |
| Delay | -- | Red advance | Static | Static | Static |

## 7.13. Fitting a Differential-Equation Model to Loss-Rate Curves Typically Used to Represent Large-Scale Ground-Combat Attrition.

In this section[33] we will develop a general attrition model, whose general form fits the shape of most loss-rate curves typically used to model large-scale ground combat[34]. All currently operational large-scale combat models in one way or another assess casualties for each side by using such a loss-rate curve consisting of casualty rate (expressed as a fraction or percentage of current strength lost per unit time) plotted against the force ratio. Here, as above, the term force ratio means the ratio of the firepower index of the attacker to that of the defender, denoted as A/D. Also, loss here means loss of value for the side's firepower index, which can then be disaggregated into losses in numbers of different weapon-system types.

In other words, the firepower-score approach takes each side's heterogeneous forces and converts them into an equivalent homogeneous force quantified in terms of a firepower index, daily reduction in each side's capability (expressed as a reduction in firepower index) is then determined from the ratio of the two such firepower indices, and finally casualties (i.e. losses in numbers of the different weapon-system types) are assessed by some means of disaggregation. We will now discuss how a relatively simple pair of differential equations may be used to model this process and fit these loss-rate curves.

Our starting point is to consider the following equations of HELMBOLD-type combat with "operational" losses (*cf*. Section 6.14 above)

$$\begin{cases} \dfrac{dx}{dt} = -a(t)\bullet\left(\dfrac{x}{y}\right)^{1-W_Y}\bullet y - \beta(t)x & \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b(t)\bullet\left(\dfrac{y}{x}\right)^{1-W_X}\bullet x - \alpha(t)y & \text{with} \quad y(0) = y_0 . \end{cases} \qquad (7.13.1)$$

In the above equations (7.13.1) we have added a feature not contained in the model of Section 6.14: each side has its own WEISS parameter, denoted as $W_X$ and $W_Y$ for the X and Y forces respectively. We also recall from Section 6.11 that, for example, such a parameter $W_Y$ allows one to account for inefficiencies of scale in producing casualties by the Y force when the two opposing forces are grossly unequal in size. In other words, the firepower-modification factors $E_X$ and $E_Y$ are no longer necessarily the same for both sides, i.e. $E_Y(u;W_Y) = u^{1-W_Y} \neq E_X(u;W_X) = u^{1-W_X}$ [cf. (6.11.1)]. Also, a term like $\beta(t)x$ may be considered to represent (here X's) "operational" losses (e.g. losses due to sickness, accidents, etc.; see Section 6.12 for further details).

For the case of constant attrition-rate coefficients, (7.13.1) becomes

$$\begin{cases} \dfrac{dx}{dt} = -a\bullet\left(\dfrac{x}{y}\right)^{1-d}\bullet y - \beta x & \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b\bullet\left(\dfrac{y}{x}\right)^{1-e}\bullet x - \alpha y & \text{with} \quad y(0) = y_0 , \end{cases} \qquad (7.13.2)$$

where for notational convenience we have denoted $W_Y$ simply as $d$ and $W_X$ as $e$. For our model (7.13.2), for example, X's fractional casualties per unit time are now given by

$$\left(-\frac{1}{x}\frac{dx}{dt}\right) = \left(\begin{matrix}\text{X's fractional casualties}\\ \text{per unit time}\end{matrix}\right)$$

$$= au^{-d} + \beta = av^{d} + \beta \ . \tag{7.13.3}$$

In Figure 7.12 we show the relation between X's fractional casualties per unit time and the force ratio $v = y/x$ for the case in which X defends (cf. our discussion in Section 5.2 (recall Figure 5.3) and see, in particular, Figure 6.15 of Section 6.11). Figure 7.13 shows the same type of relation when X attacks.

Essentially all of the principal large-scale ground-combat models currently in operational use in the world today[35] assess casualties using the firepower-score concept and (in one form or another) casualty-rate curves of the form shown in Figure 7.14, which is taken from documentation on ATLAS [64]. Such casualty-rate curves are typically plots of fractional casualties per unit time (or its equivalent) versus the force ratio (A/D) for different engagement types[36]. Thus, two such plots like those shown in Figure 7.14 are used to assess casualties, one curve for the attacker and one curve for the defender. It turns out now that the Helmbold-type model (7.13.3) gives a remarkably good fit to almost all these casualty rate curves, i.e compare Figures 7.12 and 7.13 with Figure 7.14 (i.e. Figure 6-5 on p. 6-5 of [64]), Figure 3 on p. 12 of [50], or pp. 28-31 of [51].

In other words, if (for a given engagement type) we assume that the fractional casualty rate depends on only the force ratio, then the so-called [17] asymptotic-power form (7.13.3) gives a very good fit to most such casualty-rate curves currently used, and thus the Helmbold-type equations (7.13.2) may be considered to model the attrition process, with

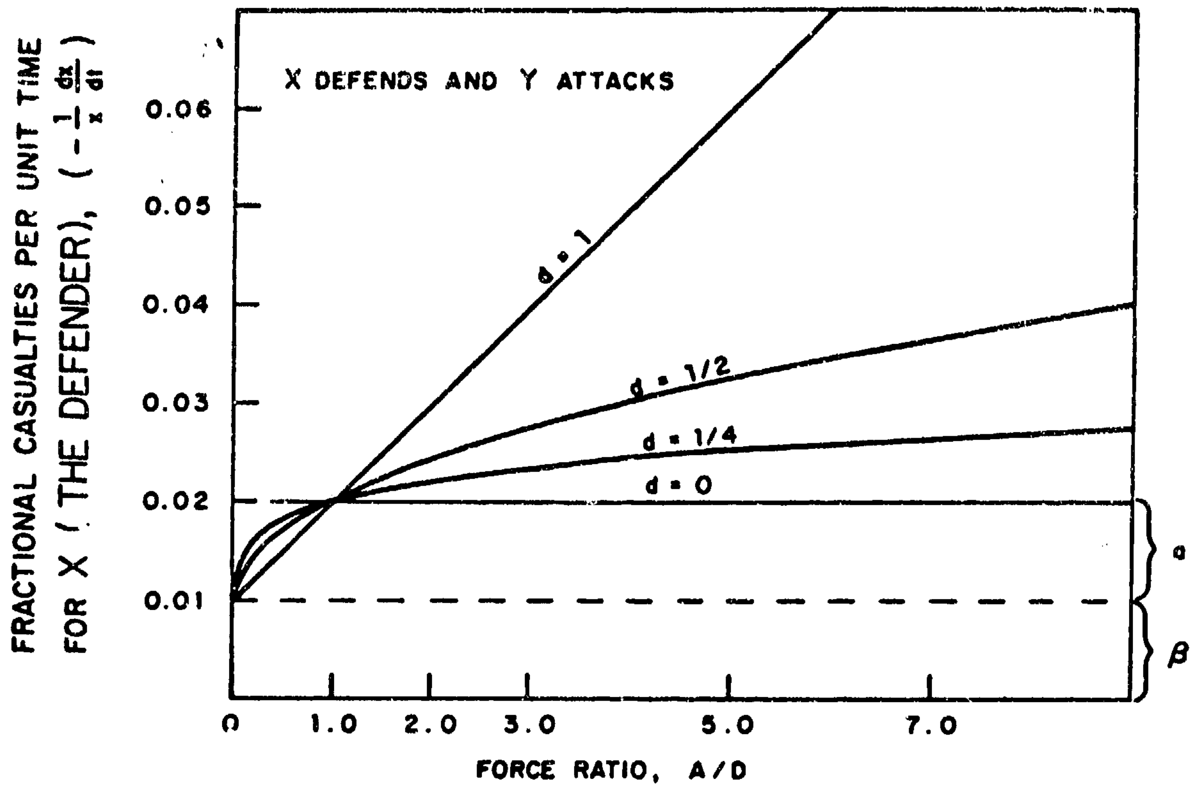


Figure 7.12. Relation between the defender's fractional casualty rate and the force ratio for the model $\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-d}\cdot y - \beta x$ with X defending.

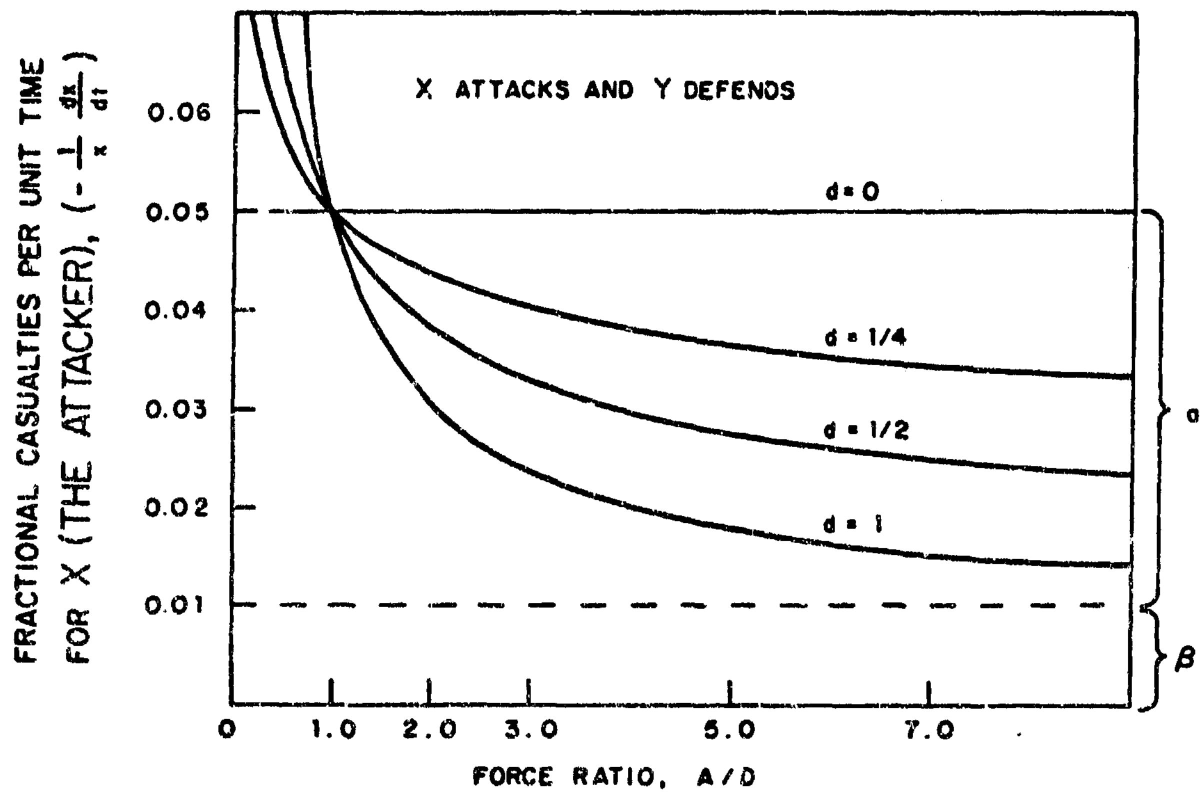


Figure 7.13. Relation between the attacker's fractional casualty rate and the force ratio for the model $\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-d}\cdot y - \beta x$ with X attacking.

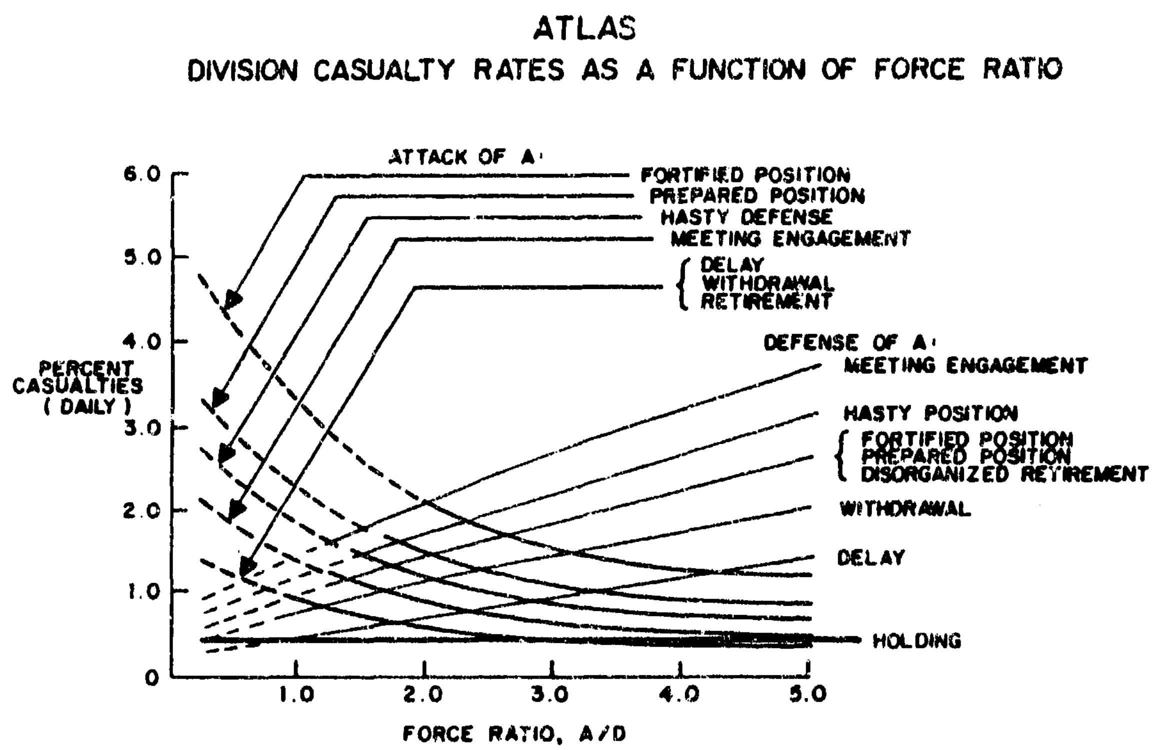


Figure 7.14. Typical casualty-rate curves used in ATLAS (from [64]).

the parameters a, b, $\alpha$, $\beta$, d, and e depending on the type of engagement. Moreover, there are even computerized routines available for the least-squares estimation of these parameters (e.g. see [17], especially Figure 1 on p. 6).

As we discussed in Section 6.11 above, the model (7.13.3), equivalently (7.13.2), can accommodate a wide variety of classic attrition-rate forms, and furthermore, a variety of attrition-rate forms have indeed been used in large-scale ground-combat models over the years. For example, ground-combat attrition in the original version of TAGS was assumed to follow the logarithmic law (see SISKA, GIAMBONI, and LIND [132, p. 29]), cf. $d = e = 0$ in (7.13.2). Today, attrition is usually modelled as being "intermediate" between the logarithmic and square laws. For example, comparing Figure above to Figure 7.14 (i.e. Figure 6-6 of [64]), we find that the casualty rate for a defending force is best fit by d near 1 (i.e. $dx/dt = -ay - \beta x$). However, comparing Figure 7.13 above to Figure 7.14, we find that a value for d around one-half seems more reasonable for the attrition-rate of an attacking force (i.e. $dx/dt = -ax^{1/2}y^{1/2} - \beta x$). All these attrition-rate functional forms may, of course, be handled by the HELMBOLD-type equations of warfare with operational losses (7.13.2) by taking the appropriate values for the fire-effectiveness-modification exponents d and e. Thus, this general model (7.13.2) has the flexibility of fitting a wide variety of attrition-rate forms that have been used to model large-scale ground combat.

Let us finally note here that the author knows of no acknowledgment of the possibility that the casualty-rate curves such as we have been discussing could be fit by a differential-equation model, or might even

have arisen from a formal or informal understanding of simple differential equations. Thus, we have developed an important simplified analytical model of large-unit attrition. A good analytical model, of course, should simplify, be transparent and easy to understand, be easy to manipulate, and increase our understanding of real-world processes (i.e yield important insights). In the next section we will develop from the model (7.13.1) and its constant-coefficient version (7.13.2) some important insights into the dynamics of combat that are not at all obvious from the above casualty-rate curves.

7.14. Changes over Time in the Force Ratio for the Above Model.

First, let us recall (see Section 6.14 above) that when $W_X = W_Y = W$ in (7.13.1), i.e. we have the equations

$$\begin{cases} \frac{dx}{dt} = -a(t) \cdot \left(\frac{x}{y}\right)^{1-W} \cdot y - \beta(t)x & \text{with} \quad x(0) = x_0 , \\ \\ \frac{dy}{dt} = -b(t) \cdot \left(\frac{y}{x}\right)^{1-W} \cdot x - \alpha(t)y & \text{with} \quad y(0) = y_0 , \end{cases} \tag{7.14.1}$$

the substitution $p = x^W$ and $q = y^W$ transforms this nonlinear combat model (7.14.1) into the following linear one

$$\begin{cases} \frac{dp}{dt} = -W\{a(t)q + \beta(t)p\} & \text{with} \quad p(0) = x_0^W , \\ \\ \frac{dq}{dt} = -W\{b(t)p + \alpha(t)q\} & \text{with} \quad q(0) = y_0^W . \end{cases} \tag{7.14.2}$$

Hence, we can invoke all the results of TAYLOR and PARRY [146] (see Section 6.13 above). In particular, if we let

$$R(t) = \frac{a(t)}{b(t)} \quad \text{and} \quad S(t) = \frac{\{\beta(t) - \alpha(t)\}}{\sqrt{a(t)\,b(t)}} , \tag{7.14.3}$$

and assume that (A1) $W \in (0,1]$, (A2) $R(t)$ and $S(t)$ are nondecreasing functions of time, (A3)[37] $\lim_{T\to+\infty}\int_0^T b(t)dt = +\infty$, and (A4) $R(t)$ is not identically equal to zero, then X will lose a fixed-force-ratio-breakpoint battle in finite time if

$$\left(\frac{x_0}{y_0}\right)^W < \sqrt{R_0} \quad S_0/2 + \sqrt{(S_0/2)^2 + 1} \quad , \tag{7.14.4}$$

where $R_0$ denotes $R(0)$ and $S_0$ denotes $S(0)$. Moreover, the force ratio $u = x/y$ is a strictly decreasing function of time in such a battle. When $W_X \neq W_Y$, the model (4.1) is, unfortunately, no longer transformable into a linear one, but we still can obtain similar results for constant attrition-rate coefficients by slightly different arguments.

We accordingly compute the rate of change of the force ratio $u = x/y$ for the model (7.13.2), namely

$$\frac{du}{dt} = bu^{1+e} + (\alpha - \beta)u - au^{1-d} = F(u) \ . \tag{7.14.5}$$

Computing $F''(u) = (1 + e)ebu^{e-1} + d(1-d)au^{-d-1}$, we find that $F(u) = F(u;d,e)$ is a strictly convex function of $u$ on $[0,+\infty)$ for $0 \leq d,\ e \leq 1$ but not both $d$ and $e$ simultaneously equal to zero. Let us therefore assume that this condition is satisfied, i.e. $0 \leq d,\ e \leq 1$ but not both $d$ and $e$ are simultaneously equal to zero. Observing that $F(0) < 0$ and $\lim_{u \to +\infty} F(u) = +\infty$, we see that there exists a unique positive value for $u$ such that $F(u) = 0$, since $F(u)$ is strictly convex on $[0,+\infty)$. Let us denote this unique positive root of $F(u) = 0$ as $u_+$. Then we have

$$F(u) \begin{cases} < 0 & \text{for} \quad 0 \leq u < u_+, \\ > 0 & \text{for} \quad u_+ < u \ . \end{cases} \tag{7.14.6}$$

It follows that if $u_0 < u_+$, then $du/dt(t) < 0$ as long as $u \geq 0$, since although $u(t)$ changes (decreases) over time, it still $\in [0,u_+)$. Also,

if $u > u_+$, then $du/dt(t) > 0$ as long as $u = x/y$ remains finite. Thus, we have proved (cf. Theorem 6.13.1).

THEOREM 7.14.1: For the nonlinear HELMBOLD-type combat model (7.13.2), $du/dt(t) < 0$ for all $t \geq 0$ as long as $u \geq 0$ if and only if $du/dt(0) < 0$, i.e. $u_0 < u_+$.

We observe that when $d = e$ and $d \in (0,1]$, then

$$u_+ = \left[\left\{\sqrt{R} \quad S/2 + \sqrt{(S/2)^2 + 1}\right\}\right]^{1/d} , \qquad (7.14.7)$$

where $R$ now denotes $a/b$ and $S$ denotes $(\beta-\alpha)/\sqrt{ab}$.

Theorem 7.14.1 not only is of intrinsic interest, but it also forms the basis of important results about the dynamics of FEBA movement given in Section 7.16 below. Theorem 7.14.1 also leads to

THEOREM 7.14.2: For the nonlinear HELMBOLD-type combat model (7.13.2), X will lose a fixed-force-ratio-breakpoint battle in finite time if and only if $u_0 < u_+$.

PROOF. By Theorem 7.14.1 we know that $du/dt(t) < 0$ for all $t \geq 0$ as long as $u \geq 0$ if and only if $u_0 < u_+$. It remains to show that $u \to 0+$ in finite time. Since $F(u)$ is convex, we know that its maximum value occurs at the end points of the interval $[0,u_0]$. Denote this maximum value as $-M$ with $M > 0$. Then $F(u) \leq -M < 0$ for all $u \in [0,u_0]$. Hence,

$$u(t) = u_0 + \int_0^t \left(\frac{du}{dt}\right) dt \leq u_0 - Mt .$$

It follows that $u(t) \to 0+$ in finite time, and we have proven the theorem. Q.E.D.

The following theorem is then an immediate corollary to Theorem 7.14.2 and (7.14.7).

THEOREM 7.14.3: Assume that $d = e$ and $d \in (0,1]$ for the nonlinear HELMBOLD-type combat model (7.13.2). Then X will lose a fixed-force-ratio-breakpoint battle in finite time if and only if

$$(u_0)^d < \sqrt{R} \left\{ S/2 + \sqrt{(S/2)^2 + 1} \right\}. \tag{7.14.8}$$

7.15. FEBA-Movement Modelling.

Although the fundamental role of ground-combat troops (in the U. S. Army's own words, e.g. see [164,p. iv]) is to "shoot, move, and communicate," one may think of the Army's mission as being ground control. All ground-combat models must consequently in one way or another reflect the control of territory by the opposing forces. Many large-scale ground-combat models (e.g. ATLAS, CEM, and TAGS) assume that a "contact zone" (or FEBA) separates the two opposing forces and runs in a more or less continuous line between them. These models divide the tactical battlefield into strips called "sectors," and the fighting forces are generally constrained to move within these sectors, which correspond to axes of advance or withdrawal (e.g. see [25, pp. 9-13 and p. 82]. Combat operations in such a sector are then more or less independent of those in adjacent sectors, with the exact details varying significantly from model to model (e.g. between ATLAS [98] and CEM [25]). For our purposes here, however, we assume that there are no interactions between sectors, and let us then focus on an individual sector.

In such a sector, the forces are separated by a FEBA (see Figure 7.15), and during an engagement changes in the rate of FEBA movement are primarily caused by changes in the force ratio[38]. FEBA position is then calculated as the integral of a rate-of-advance equation, i.e.

$$s = \int_0^t \left(\frac{ds}{dt}\right) dt \ , \qquad (7.15.1)$$

where

$$\frac{ds}{dt} = f(u;\tau) \ ,$$

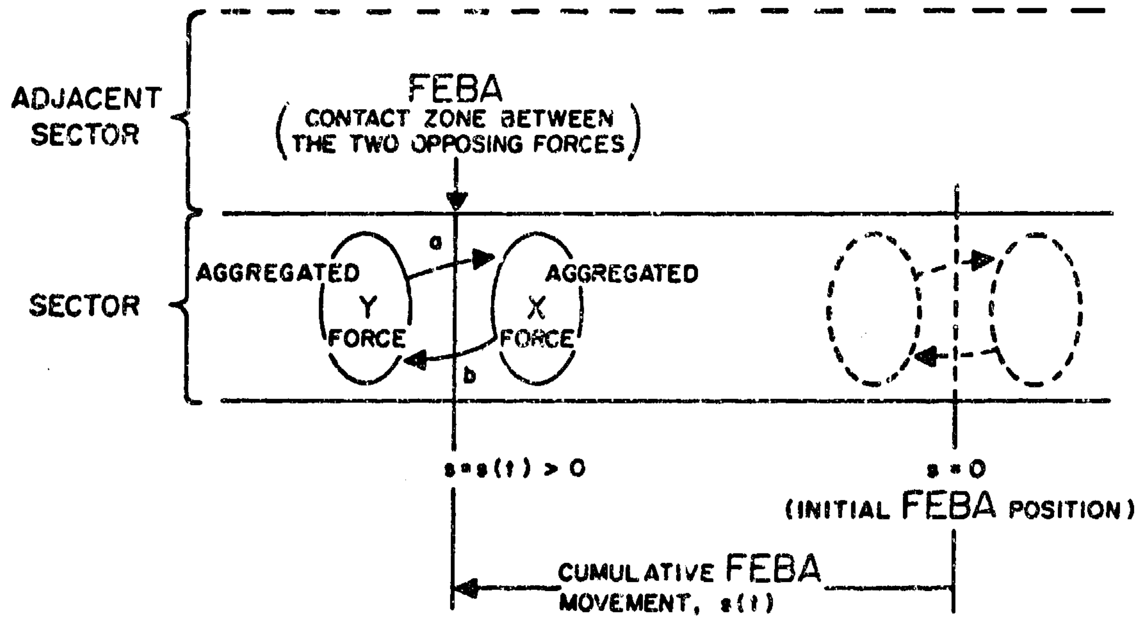


Figure 7.15. Conceptualization of aggregated-force combat in a sector.

$s = s(t)$ denotes cumulative FEBA movement from its initial position, $ds/dt$ denotes the rate of advance (taken to be positive for $X$), and $u = x/y$ denotes the force ratio (usually the ratio of the firepower indices $x$ and $y$ of the opposing forces). In other words, we have adopted the convention that $ds/dt > 0$ means that $X$ is advancing against the enemy ($Y$). In current aggregated-force models (e.g. ATLAS and CEM) it is assumed that the rate of advance also depends on additional tactical factors such as: (1) terrain trafficability, (2) unit types in the attacking force, and (3) the engagement type[39] (e.g. route, retirement, delay, meeting engagement, attack of a hasty defense, prepared position, or fortified zone). In equation (7.15.2), $\tau$ denotes all these other tactical factors. For a fixed value of $\tau$, the rate of advance consequently depends on only the force ratio, and this dependence (at least for most of the rate-of-advance curves seen by this author) may be characterized as follows:

(C1) a threshold force ratio is required for an advance to start,

(C2) above this threshold value, the rate of advance increases as the force ratio increases, but at a decreasing rate (i.e. above the threshold value, the rate of advance is a convex function of the force with essentially a horizontal asymptote).

A sector such as depicted in Figure 7.15 is one-dimensional in the sense that only a single number $s(t)$ is used to specify FEBA position at time $t$. We may think of this $s(t)$ as representing an average FEBA position within the sector (i.e. variations in FEBA position within the sector are

not considered). Although we have depicted the sectors shown in Figure 7.15 as being straight and of uniform width, this need not be the case (e.g. see [25, p. 10 or p. 82]).

Let us now consider an example of a rate-of-advance equation that has been suggested by historical data and used in various forms in many RAND studies. We use this example in the next section to show that we need to know only the above two general characteristics of a rate-of-advance curve (and not numerical particulars as long as the curve has these general characteristics) and, for example, the fact that the force ratio is a strictly increasing function of time (see Section 7.14 above) in order to develop some important insights into the dynamics of FEBA movement. We therefore consider (see Figure 7.16)

$$\frac{ds}{dt} = \begin{cases} \dfrac{V_{max}^R}{u_R}\left(\dfrac{u - u_R}{u + 1}\right) & \text{for } 0 \leq u < u_R , \\ 0 & \text{for } u_R \leq u \leq u_A , \\ V_{max}^A\left(\dfrac{u - u_A}{u + 1}\right) & \text{for } u_A < u , \end{cases} \qquad (7.15.3)$$

where $V_{max}^R$ denotes the maximum speed for retreat of the X force, $u_R$ denotes the force ratio at which retreat begins, $V_{max}^A$ denotes the maximum speed for advance of the X force, and $u_A$ denotes the force ratio at which advance begins. We should think of the parameters $V_{max}^R$, $u_R$, $V_{max}^A$, and $u_A$ as depending on the tactical variables (i.e. terrain type, attacking unit types, and engagement type), denoted as $\tau$ above. The functional form (7.15.3)

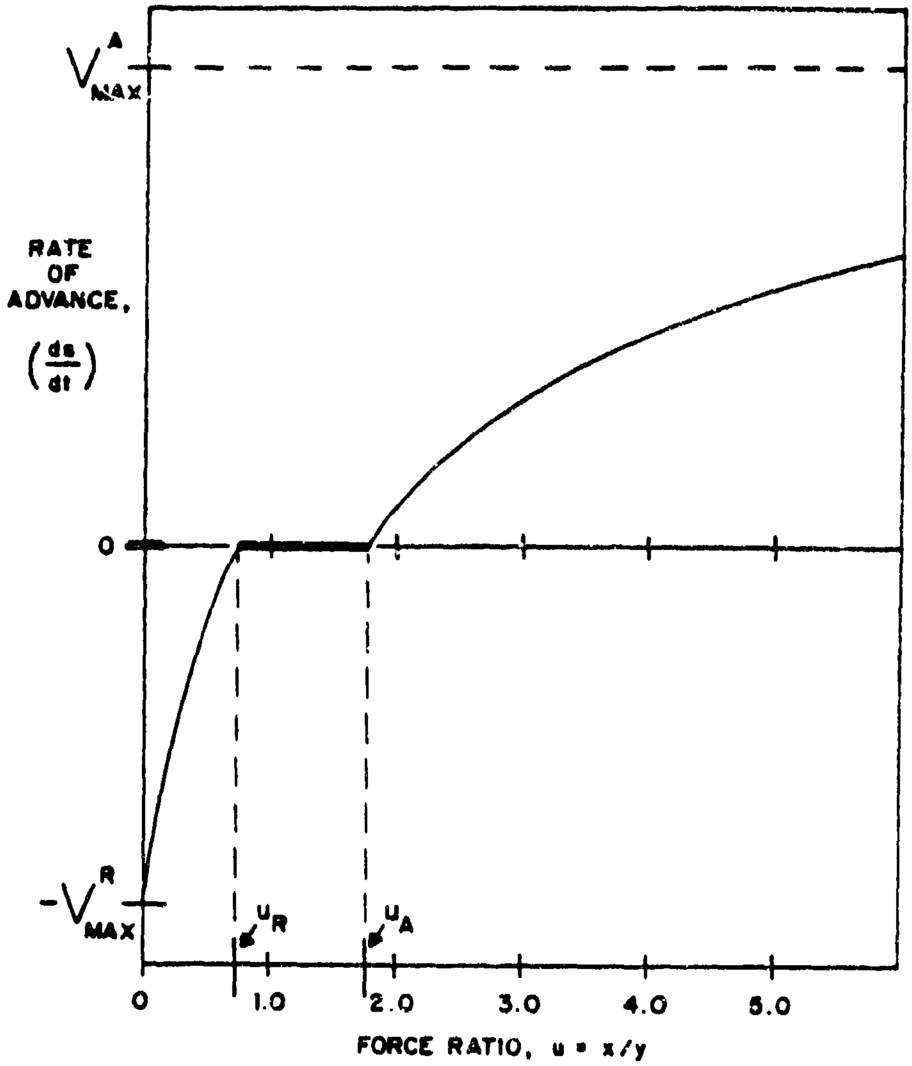


Figure 7.16. Rate of advance versus force ratio for the model (7.15.3) with all other tactical factors held constant.

is suggested by a model that fits data on operations in Western Europe during Wrold War II (see [116] and [66, pp. 17-18]). We have chosen to consider the functional form (7.15.3) because (1) it provides a good fit to many rate-of-advance curves[40] currently in use, and (2) it yields an analytically tractable model when combined with attrition equations such as (7.13.2).

From (7.15.3) we see that for a given set of tactical conditions (denoted as $\tau$ above), FEBA motion depends on the force ratio, and consequently we are interested in how the force ratio behaves over time (see Section 7.14 above). In the next section we will show how the equations (7.13.2) and (7.15.3) provide some valuable insights into the dynamics of ground combat.

## 7.16. Dynamics of FEBA Movement in Large-Scale Ground-Combat Models[41].

As discussed above, in an engagement FEBA movement is governed by the force ratio, which in turn varies with time due to losses on both sides. We will now show how the analytical formations for attrition and FEBA motion (given above in Sections 7.13 and 7.15) lead to some valuable insights into the dynamics of FEBA movement as portrayed in current large-scale ground-combat models.

It will be convenient to restate our combat model here, since its component parts are widely scattered above. As we have seen above, conventional-force combat in large-scale operations may be modelled by (7.13.2) and (7.15.3). Unfortunately, we have not been able to obtain explicit analytical results concerning FEBA position, combat capabilities (i.e. the two firepower indices of the opposing forces), and the force ratio for the general version of this model. However, by choosing the appropriate values for certain parameters, we are able to obtain such explicit analytical results: thus, we assume that $d = e$ in (7.13.2) and denote this common value as $W$. Also, for convenience and simplicity, we assume that $u_A = u_R$ and $V^A_{max} = V^R_{max} = V_{max}$ in with extension to the general case of $u_A > u_R$ and $V^A_{max} \neq V^R_{max}$ being straightforward but messy. Our model for conventional combat between large ground-force units in a sector may then be written as the three coupled equations

$$\left\{\begin{array}{lll} \dfrac{dx}{dt} = -a \left(\dfrac{x}{y}\right)^{1-W} y - \beta x & \text{with } x(0) = x_0 , & \\ \dfrac{dy}{dt} = -b \left(\dfrac{y}{x}\right)^{1-W} x - \alpha y & \text{with } y(0) = y_0 , & (7.16.1) \\ \dfrac{ds}{dt} = V_{max}\left(\dfrac{u - u_A}{u + 1}\right) & \text{with } s(0) = 0 , & \end{array}\right.$$

where $0 < W \leq 1$.

One important characteristic of the analytical model (7.16.1) is its transparency (cf. the last paragraph of Section 7.13): we explicitly see all hypothesized functional relations. For a special case of this particular model (the author currently knows of no other), explicit analytical results are readily available, and we will develop them below. The author conjectures (but cannot prove) that analytical results take their simplest form for the model (7.16.1). Even in this "simplest" case, however, the analytical expression for FEBA position [see (7.16.8) and (7.16.9) below] is so complicated that computational results are required to provide any insight into the dynamics of FEBA movement. However, the qualitative behavior of FEBA position over time is readily discernible for the more general case of $d \neq e$ in (7.13.2) by combining results on changes over time in the force ratio (see Theorem 7.14.1) with the general characteristics of rate-of-advance equations [see (C1) and (C2) in Section 7.15 above]. Thus knowledge about how the force ratio changes over time is a key piece of information for understanding the dynamics of large-scale combat as currently represented in many large-scale ground-combat models. Analysts should therefore become familiar with how various functional forms for attrition rates yield different types of temporal variations in the force ratio.

We will now develop the explicit analytical results for the model (7.16.1). If we let $u = x/y = v^Z$, where $Z = 1/W$, then the above model may be written in the equivalent form

$$\frac{dv}{dt} = \frac{1}{Z}\{bv^2 + (\alpha - \beta)v - a\} \qquad \text{with} \quad v(0) = \left(\frac{x_0}{y_0}\right) ,$$

$$\frac{ds}{dt} = V_{max}\left(\frac{v^Z - u_A}{v^Z + 1}\right) \qquad \text{with} \quad s(0) = 0 , \tag{7.16.2}$$

where $1 \leq Z < +\infty$. The first equation of (7.16.2) is readily integrated to yield the force ratio as a function of time, namely

$$u^W(t) = v_M \frac{\{(u_0^W - v_P) - (v_P/v_M)(u_0^W - v_M)\ e^{-2W\theta t}\}}{\{(u_0^W - v_P) - (u_0^W - v_M)\ e^{-2W\theta t}\}} , \tag{7.16.3}$$

where

$$I = \sqrt{ab} , \qquad R = a/b , \qquad S = \frac{\beta - \alpha}{\sqrt{ab}} , \tag{7.16.4}$$

$$v_P = \sqrt{R}\left\{S/2 + \sqrt{(S/2)^2 + 1}\right\} > 0 , \tag{7.16.5}$$

and

$$v_M = \sqrt{R}\left\{S/2 - \sqrt{(S/2)^2 + 1}\right\} < 0 , \tag{7.16.6}$$

Because of the coupling of equations (7.16.2) we have not been able to develop an explicit expression for FEBA position as a function of time, $s(t)$. It is possible, however, to express FEBA position as a function of the force ratio. Thus, we may eliminate time from (7.16.2) to obtain

$$ds = \frac{Z\ V_{max}}{b}\left\{\frac{dv}{(v - v_P)(v - v_M)} - \frac{(u_A + 1)\,dv}{(v^Z + 1)(v - v_P)(v - v_M)}\right\} . \tag{7.16.7}$$

For $Z = n = 1/W$, we may use a partial fraction expansion of (7.16.7) to obtain after some rather lengthy computations

$$s = n\,V_{\max}\left\{ C_n \,\ln\left(\frac{u^{1/n}+1}{u_0^{1/n}+1}\right) + D_n \,\ln\left(\frac{u^{1/n}-v_P}{u_0^{1/n}-v_P}\right) + E_n \,\ln\left(\frac{u^{1/n}-v_M}{u_0^{1/n}-v_M}\right) \right.$$

$$\left. + \sum_{k=1}^{[n/2]} \left( F_k^n \,\ln\left[\frac{P_k^n(u^{1/n})}{P_k^n(u_0^{1/n})}\right] + G_k^n [\tan^{-1}\{Q_k^n(u)\} - \tan^{-1}\{Q_k^n(u_0)\}] \right) \right\} , \qquad (7.16.8)$$

where $[n/2]$ denotes "the integer part of" $n/2$, $\theta = \sqrt{(S/2)^2 + 1}$ , and the other coefficients are given in Table 7.VI. When $n$ is odd and $v_M = 1$, the above expression (7.16.8) reduces to

$$s = n\,V_{\max}\left\{ \tilde{C}_n \,\ln\left(\frac{u^{1/n}+1}{u_0^{1/n}+1}\right) + \tilde{D}_n \,\ln\left(\frac{u^{1/n}-v_P}{u_0^{1/n}-v_P}\right) + \tilde{E}_n \left(\frac{1}{\{u_0^{1/n}+1\}} - \frac{1}{\{u^{1/n}+1\}}\right) \right.$$

$$\left. + \sum_{k=1}^{[n/2]} \left( \tilde{F}_k^n \,\ln\left[\frac{P_k^n(u^{1/n})}{P_k^n(u_0^{1/n})}\right] + \tilde{G}_k^n [\tan^{-1}\{Q_k^n(u)\} - \tan^{-1}\{Q_k^n(u_0)\}] \right) \right\} , \qquad (7.16.9)$$

where the modified coefficients $\underset{\sim}{C}_n$ through $\underset{\sim}{G}_k^n$ are given in Table 7.VII.

Thus, we see that explicit analytical results are readily obtainable for the model (7.16.1), although the FEBA-position results only hold for $W = 1/n$. Unfortunately, even these explicit results do not readily reveal the dynamics of FEBA movement. We will now show how the qualitative behavior of the force ratio over time as determined from a force-ratio equation like (7.14.5) may be coupled with a rate-of-advance equation to yield some important insights into the dynamics of FEBA movement. This approach also allows us to consider more general models of both

TABLE 7.VI. Coefficients for Relation (7.16.8) Between FEBA Position and the Force Ratio $u = x/y$.

$$C_n = \sqrt{R}\,(u_A + 1)\{(-1)^n - 1\}/\{2nI(v_P + 1)(v_M + 1)\}$$

$$D_n = (v_P^n - u_A)/\{2\theta(v_P^n + 1)\}$$

$$E_n = -(v_M^n - u_A)/\{2\theta(v_P^n + 1)\}$$

$$F_k^n = -\sqrt{R}\,(u_A + 1)\{S\sqrt{R} + (R - 1)\cos\theta_k^n\}/\{nIP_k^n(v_P)\,P_k^n(v_M)\}$$

$$G_k^n = 2\sqrt{R}\,(u_A + 1)(R + 1)(\sin\theta_k^n)/\{nIP_k^n(v_P)\,P_k^n(v_M)\}$$

$$P_k^n(q) = q^2 - 2q(\cos\theta_k^n) + 1$$

$$Q_k^n(u) = (u^{1/n} - \cos\theta_k^n)/(\sin\theta_k^n)$$

and

$$\theta_k^n = (2k - 1)\pi/n$$

TABLE 7.VII. Modified Coefficients for Relation (7.16.9) Between FEBA Position and the Force Ratio $u = x/y$ When $n$ is Odd and $v_M = 1$.

$$\tilde{C}_n = -\{nv_P - u_A + (n-1)[1 - (v_P + 1)(u_A + 1)/2]\}/\{n(v_P + 1)^2\}$$

$$\tilde{D}_n = (v_P^n - u_A)/\{(v_P^n + 1)(v_P + 1)\}$$

$$\tilde{E}_n = (u_A + 1)/\{n(v_P + 1)\}$$

$$\tilde{F}_k^n = -\sqrt{R}(u_A + 1)\ \{S\sqrt{R} + (R-1)\cos\theta_k^n\}/\{2nI(1 + \cos\theta_k^n)\ P_k^n(v_P)\}$$

and

$$\tilde{G}_k^n = 2\sqrt{R}\ (u_A + 1)\ (R + 1)\ (\sin\theta_k^n)/\{2nI(1 + \cos\theta_k^n)\ P_k^n(v_P)\}$$

attrition and also FEBA motion for conventional combat between large units in a sector.

We therefore consider the more general version of

$$\begin{cases} \dfrac{dx}{dt} = -a\left(\dfrac{x}{y}\right)^{1-d} y - \beta x & \text{with} \quad x(0) = x_0\,, \\ \dfrac{dy}{dt} = -b\left(\dfrac{y}{x}\right)^{1-e} x - \alpha y & \text{with} \quad y(0) = y_0\,, \\ \dfrac{ds}{dt} = \begin{cases} f_R(u;\tau) < 0 & \text{for} \quad 0 \le u < u_R \\ 0 & \text{for} \quad u_R \le u \le u_A \\ f_A(u;\tau) > 0 & \text{for} \quad u_A < u \end{cases} & \text{with} \quad s(0) = 0\,, \end{cases} \qquad (7.16.10)$$

where $0 \le d, e \le 1$, with $d$ and $e$ not simultaneously equal to zero. Here the attrition-rate coefficients $a$, $b$, $\alpha$, and $\beta$ also depend on the tactical parameters, denoted as $\tau$. For understanding how the trading of casualties interacts with the rate-of-advance equation to determine the dynamics of FEBA movement, we need consider only the force-ratio equation in conjunction with the rate-of-advance equation, however. Thus we consider

$$
\begin{cases}
\dfrac{du}{dt} = bu^{1+e} + (\alpha-\beta)u - au^{1-d} & \text{with } u(0) = u_0 = \dfrac{x_0}{y_0}, \\[2ex]
\dfrac{ds}{dt} = \begin{cases} f_R(u;\tau) < 0 & \text{for } 0 \le u < u_R \\ 0 & \text{for } u_R \le u \le u_A \\ f_A(u;\tau) > 0 & \text{for } u_A < u \end{cases} & \text{with } s(0) = 0,
\end{cases}
\tag{7.16.11}
$$

where $0 \le d, e \le 1$, with $d$ and $e$ not simultaneously equal to zero. Let us assume that $X$ is the attacker. Consequently it is not unreasonable to expect that $u_+ > u_A$. For example, $a/b = 9$, $\alpha = \beta$, $d = e$, and $u_A = 1.7$ leads to this situation. In this case (i.e. when $u_+ > u_A$), recalling Theorem 7.14.1, we can obtain some important insights into the dynamics of FEBA movement by considering the second differential equation in (7.16.11) (see Figure 7.17). In other words, the FEBA-movement information shown in Figure 7.17, has been obtained by combining the strictly-monotonic behavior of the force ratio over time (cf. Theorem 7.14.1) with the general characteristics (C1) and (C2) (see Section 7.15 above) of the rate of change of FEBA position (cf. the second differential equation in 7.16.11).

Figure 7.17 shows us that there are several critical initial-force-ratio threshold values that bound regions of quite different subsequent evolution for the course of combat. If the initial force ratio $u_0$ exceeds $u_+ > u_A$, then the X-force attack will continue to advance against increasingly more favorable force ratios, i.e. the attack "breaks out" in the sector. If $Y$ does not, for example, commit reserves or allocate air strikes to the sector, then (according to the model) his forces will continue to retreat in the face of an increasingly more unfavorable force ratio until

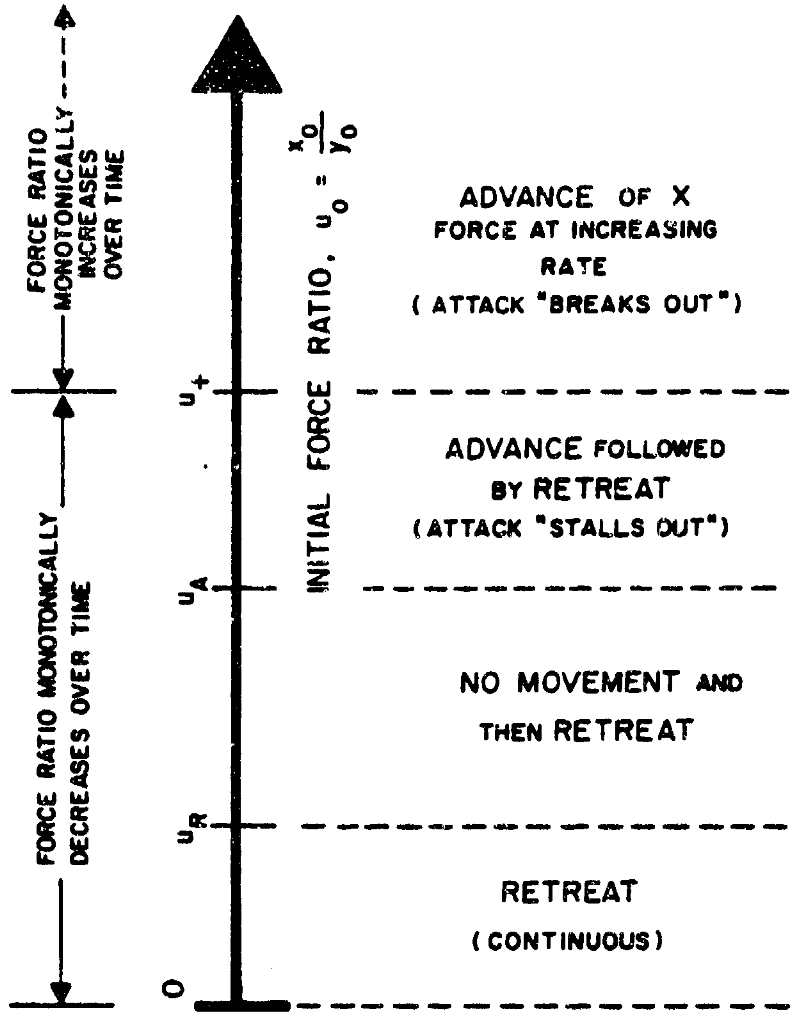


Figure 7.17. Qualitative behavior of FEBA position over time for combat modelled with (7.16.11).

they are eventually annihilated. If $u_S < u_0 < u_+$, we have the most interesting (and enlightening) case: the X-force attack will continue to push forward but at increasingly more unfavorable casualty-exchange ratios until the force ratio is no longer such that an advance can be sustained, i.e. the attack "stalls out." Our model then says that the contact zone will remain stationary for a while until the force ratio is further worn down enough for the Y force to counterattack and being to advance.

Although the model considered here is quite an idealization and simplification of operational models such as ATLAS, CEM, and TAGS, this basic trading of space for time (in the case in which $u_A < u_0 < u_+$) in order to wear down the force ratio and then to subsequently counterattack has been a basic premise of NATO defense planning for years. Thus our simple model has revealed this important structure of large-scale operations. It should, of course, be noted that this structure (i.e. the combat dynamics portrayed in Figure 7.17) is not directly discernable from any of the complex operational models from which we have distilled our simplified auxiliary model.

7.17. Current Complex Aggregated-Force Operational Models of Large-Scale Tactical Engagements.

The following are currently operational theater-level combat models that use the firepower-score approach to aggregate forces for assessing casualties in the manner discussed above[42]:

TAGS,

ATLAS,

CEM,

IDAGAM,

and TACWAR.

These are essentially the only operational models currently available in the United States for analyzing simulated theater-level combat. It was estimated [9] that as of August 1977 the approximate frequency of use of ATLAS was 600 times per year, that of CEM was 25, and that of IDAGAM II was between 150 and 200.

## *7.18. A Linear Model for Imputing Values to Weapon-System Types Based on Their LANCHESTER Attrition-Rate Coefficients.

One significant and basic criticism [90, pp. II-C-3] of the firepower-score[43] approach is that the effectiveness (or value) of a weapon-system type depends on the circumstances of its employment and that any methodology for quantifying the combat capability of a weapon-system type should result in each weapon being assigned a number representing that weapon-system type's value in a particular combat situation relative to all other weapon-system types being employed. Consequently, there have been several attempts to impute value to a weapon-system type based on the particular circumstances of that system's fighting capability relative to that of other systems on the battlefield. This value is then treated like a firepower score for aggregating forces in models of large-scale combat operations for purposes of modelling combat processes such as attrition, FEBA movement, and tactical decision making[44]. Thus, in this section we will examine an approach for imputing value (i.e. assigning a firepower score) to a weapon-system type based on the circumstances of its employment and its casualty-producing capability relative to that of all other weapon-system types in the particular combat environment under consideration. The basic idea[45] of this approach is to use a linear model for transforming all the LANCHESTER attrition-rate coefficients[46] of a combined-arms team fighting against a heterogeneous enemy force into a set of values for these weapon-system types.

This approach for imputing values to weapon-system types based on their single-system kill rates is important because it has been and continues to be used in so-called weapon-system equivalence studies by the U. S. Army [89,; 149], and it also forms the basis in IDAGAM[47] [5; 6; 130] for computing

force ratios that are used for scaling casualties, determining FEBA movement, modelling tactical decision making, etc. (see ANDERSON et al. [6] or SHUPACK [130] for further details). It has also served as the theoretical basis for aggregating forces in a hierarchical combat-modelling approach developed in the United Kingdom (see DARE and JAMES [43] and DARE [42] for further details; see also Section 7.20 below). Unfortunately, different authors have used different names for referring to this method (and certain of its variants based on how weapon-system-type value is "scaled"): HOLTER [89] has used the terms weapon effectiveness value (WEV) and unit effectiveness value (UEV), ANDERSON [5] has called it the antipotential potential method, while HOWES and THRALL [90; 92] have referred to it as the method of ideal linear weights.

The rest of this section is organized in the following fashion. First, we will present the basic linear model for imputing values to weapon-system types based on their LANCHESTER attrition-rate coefficients. Next, we will show how these weapon-system-type values allow one to consider the evolution of aggregated-force value without having to keep track of individual weapon-system types in detail when it is assumed that all attrition occurs according to the equations for a heterogeneous-force F|F LANCHESTER-type attrition process. This result leads to several important interpretations for parameters of the linear-valuation model, including that of the square root of the eigenvalue of maximum magnitude from an associated eigenvalue-problem as representing the intensity of combat between the aggregated forces. Additionally, the evolution over time of the force ratio for this associated aggregated-force model is examined. The imputed weapon-system-type

values for each force are only determined up to a constant multiple by the basic linear-valuation model. Various methods for scaling the two opposing force-value vectors determined by the basic model are reviewed, and an alternative scaling scheme that avoids certain difficulties is suggested.

We begin by considering a linear model for imputing values to weapon-system types based on their heterogeneous-force single-system kill rates against opposing enemy weapon-system types. Let us first consider a few heuristics to foster an understanding of the linear-valuation model's fundamental premise: namely, that weapon-system types are valued in direct proportion to the rate at which they destroy the value of opposing enemy weapon-system types. Assume that you are in combat against an enemy combined-arms team composed of various weapon-system types. Would you value an enemy machine gun more than, say, a rifle? Without doubt, one will value the machine gun more than a rifle because it is more "dangerous," i.e. it will hurt us more in combat by destroying more of our systems. Since different types of systems are involved here in the list of machine-gun kills, one will have to pick some common denominator, aggregate target-type kills accordingly, and consider the overall value of targets destroyed. Thus, one is very naturally led to the following general principle for assigning value to weapon-system types.

FUNDAMENTAL PRINCIPLE OF WEAPON-SYSTEM VALUATION: <u>The value of a weapon system is directly proportional to the value of enemy weapon systems that it destroys</u>.

This qualitative maxim will now be developed into a quantitative model for determining weapon-system-type values. In order to have a common basis for comparing different weapon-system types one should consider the number of kills by a particular weapon-system type in some standard unit of time, and thus we are led to consider the _rate_ at which the value of enemy weapon systems is destroyed. Thus, we see that a very natural and intuitively appealing basic premise upon which to build a model for determining weapon-system-type value is the following[48].

> BASIC MODELLING HYPOTHESIS FOR IMPUTING VALUES TO WEAPON-SYSTEM TYPES: _The value of a weapon-system type is directly proportional to the rate at which it destroys the value of opposing enemy weapon-system types_.

We will now translate the above intuitively appealing basic hypothesis into a quantitative model.

Consider two opposing heterogeneous forces: an $X$ force consisting of $m$ different types of weapon systems (denoted as $X_1, X_2, \ldots, X_m$) opposed by a $Y$ force consisting of $n$ different types of weapon systems (denoted as $Y_1, Y_2, \ldots, Y_n$) (_recall_ Figure 7.11). If we assume that the total value of a collection of different weapon-system types is a _linear_ function of the number of each of these different types of systems, then we can express the model's basic hypothesis given in the preceding paragraph as follows

$$\begin{pmatrix}\text{value of the}\\ i^{\underline{th}}\ X\ \text{weapon-}\\ \text{system type}\end{pmatrix} = \begin{pmatrix}\text{value of}\\ \text{one } X_i\\ \text{system}\end{pmatrix} = (\text{CONSTANT}) \sum_{j=1}^{n} \begin{pmatrix}\text{rate at which}\\ \text{one } X_i \text{ system}\\ \text{destroys } Y_j\\ \text{systems}\end{pmatrix} \begin{pmatrix}\text{value of}\\ \text{one } Y_j\\ \text{system}\end{pmatrix} . \tag{7.18.1}$$

As we have done in Section 7.7, we will always let (if it is at all possible) the subscript $i$ refer to the $X$ force and the subscript $j$ refer to the $Y$ force. Thus, if nothing else is said, the index $i$ will always take on the integer values 1 through $m$, and the index $j$ will always take on the integer values 1 through $n$. If we let $a_{ij}$ denote the rate[49] at which one $Y_j$ system kills $X_i$ systems in a particular combat situation and similarly let $b_{ji}$ denote the rate at which one $X_i$ systems kills $Y_j$ systems, then we may express (7.18.1) in mathematical terms as

$$s_i^X = K_X \sum_{j=1}^{n} b_{ji} s_j^Y \quad , \tag{7.18.2}$$

where $s_i^X$ denotes the value of one $X_i$ weapon system, $K_X$ denotes a constant of proportionality which will be given an operational interpretation below, and similarly $s_j^Y$ denotes the value of the $j^{\underline{th}}$ Y weapon-system type. Unfortunately, our model is so far incomplete, since not only are there m unknown values $s_i^X$ for the $X$ weapon-system types but also $n$ unknown values $s_j^Y$ for the $Y$ weapon-system types. This indeterminant situation is readily alleviated by observing that an analogous system of equations holds for the

Y weapon-system types. Thus, it is convenient to write the basic linear model (founded upon the above basic hypothesis) for imputing values to weapon-system types based on their single-system kill rates as follows

$$\begin{cases} s_i^X = K_X \sum_{j=1}^{n} b_{ji} s_j^Y , \\ s_j^Y = K_Y \sum_{i=1}^{m} a_{ij} s_i^X , \end{cases} \tag{7.18.3}$$

where (on physical/operational grounds we must have) $a_{ij}$ and $b_{ji} \geq 0$.

Equations (7.18.3) are (m+n) equations in the (m+n+2) unknowns $s_i^X$, $s_j^Y$, $K_X$, and $K_Y$. Thus, two more equations must be given before a determinant system can be obtained. On the other hand, if we consider that $K_X$ and $K_Y$ have been determined, then we have (m+n) linear equations in (m+n) unknowns $s_i^X$ and $s_j^Y$. On physical/operational grounds it only makes sense to have $s_i^X$ and $s_j^Y \geq 0$, with a zero value meaning that the model has imputed absolutely no value to the weapon-system type in question. Thus, we should inquire whether the linear equations (7.18.3) possess such a nonnegative solution. It is indeed remarkable that as long as $a_{ij}$ and $b_{ji} \geq 0$ we are guaranteed of always being able to find such desired nonnegative solutions to (7.18.3) without any further assumptions about the single-system kill rates $a_{ij}$ and $b_{ji}$. To prove this latter assertion, one substitutes the second equation of (7.18.3) into the first to obtain

$$s_i^X = K_X K_Y \sum_{k=1}^{m} \{ \sum_{j=1}^{n} a_{kj} b_{ji} \} s_k^X , \tag{7.18.4}$$

and similarly

$$s_j^Y = K_X K_Y \sum_{k=1}^{n} \{\sum_{i=1}^{m} b_{ki} a_{ij}\} s_k^Y , \tag{7.18.5}$$

which are more easily to be recognized as a pair of so-called eigenvalue problems (e.g. see HILDEBRAND [82], MIRSKY [111], or SAMELSON [123] by writing

$$(AB)^T \underset{\sim}{s}_X = \lambda \underset{\sim}{s}_X \tag{7.18.6}$$

and

$$(BA)^T \underset{\sim}{s}_Y = \lambda \underset{\sim}{s}_Y \tag{7.18.7}$$

where

$$\lambda = 1/(K_X K_Y), \tag{7.18.8}$$

$\underset{\sim}{s}_X$ denotes a column vector of the $m$ X-weapon-system-type values [i.e. $\underset{\sim}{s}_X^T = (s_1^X, s_2^X, \ldots , s_m^X)$], $A$ denotes an $m \times n$ matrix of attrition-rate coefficients (i.e. $A = [a_{ij}]$), $A^T$ denotes the transpose of $A$ obtained by interchanging its rows and columns, and similarly for $\underset{\sim}{s}_Y$ and $B$ (with $B$ being an $n \times m$ matrix). We will see that by invoking the so-called PERRON-FROBENIUS theorem[50] for nonnegative matrices that one can guarantee that (without any further assumptions about $A$ and $B$) there always exists a vector of nonnegative values such that, for example, (7.18.6) holds.

Before we state the PERRON-FROBENIUS theorem for nonnegative matrices, it will be convenient to state a few basic definitions from matrix theory. Our discussion here follows VARGA [152, Chapters 1 and 2]. For $n \geq 2$, an $n \times n$ matrix $C$ is called reducible if there exists an $n \times n$ permutation matrix. $P$ such that

$$PCP^T = \begin{bmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{2,2} \end{bmatrix}.$$

where $C_{1,1}$ is an $r \times r$ submatrix and $C_{2,2}$ is an $(n-r) \times (n-r)$ submatrix, with $1 \leq r < n$. If no such permutation matrix exists, then $C$ is called <u>irreducible</u>. Any reducible $n \times n$ matrix $C$ may consequently be written in the following <u>normal form</u>

$$PCP^T = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,m} \\ 0 & R_{2,2} & \cdots & R_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{m,m} \end{bmatrix}, \tag{7.18.9}$$

where $P$ is an $n \times n$ permutation matrix and each square submatrix $R_{j,j}$ for $1 \leq j \leq m$ is either irreducible or a $1 \times 1$ null matrix. Also, an $n \times n$ matrix $M = [m_{ij}]$ is called strictly upper triangular only if $m_{ij} = 0$ for all $i \geq j$. Finally, the spectral radius of a square matrix is defined to be the maximum of the absolute values of the matrix's eigenvalues. We now state here without proof the PERRON-FROBENIUS theorem for nonnegative matrices (<u>see</u> VARGA [152] for a proof of this important theorem).

THEOREM 5.18.1 (PERRON [121] and FROBENIUS [60]): Let $C \geq 0$ be an $n \times n$ matrix. Then,

1. $C$ has a nonnegative real eigenvalue equal to its spectral radius. This eigenvalue is positive unless $C$ is reducible and the normal form (7.18.9) of $C$ is strictly upper triangular.

2. To the spectral radius, there corresponds a nonnegative eigenvector. If $C$ is irreducible, then this eigenvector is positive and the corresponding eigenvalue is simple.

3. The spectral radius of $C$ increases when any entry of $C$ is increased unless $C$ is reducible, and then it does not decrease.

The above Theorem 5.18.1 tells us that since $AB \geq 0$, we can always find a nonnegative vector of weapon-system-type values $\underset{\sim}{s}_X \geq 0$, which is unique only up to a constant multiple, for the X force such that

$$(AB)^T \underset{\sim}{s}_X = \lambda^* \underset{\sim}{s}_X \tag{7.18.10}$$

holds, where $\lambda^*$ denotes the nonnegative real eigenvalue of AB with largest absolute value. If AB is an irreducible $n \times n$ matrix, then $\underset{\sim}{s}_X$ and $\lambda^* > 0$. Similarly, $BA \geq 0$ guarantees that we can also find a nonnegative vector of weapon-system-type values $\underset{\sim}{s}_Y \geq 0$, which is unique only up to a constant multiple, for the Y force such that

$$(BA)^T \underset{\sim}{s}_Y = \lambda^* \underset{\sim}{s}_Y , \tag{7.18.11}$$

and if BA is an irreducible $m \times m$ matrix, both $\underset{\sim}{s}_Y$ and $\lambda^*$ are positive.

It should be noted that under the present scheme of things $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ are each only unique up to a constant multiple, i.e. unique up to a scale factor. In other words, if (for example) $\underset{\sim}{s}_X$ satisfies (7.18.10), then so will $k\underset{\sim}{s}_X$ where k is an arbitrary constant. By scaling these value vectors in some appropriate fashion, one can make them be uniquely determined, but we will see that this scaling is not really necessary, although it may be convenient.

To summarize, we have shown that we can always solve (7.18.10) and (7.18.11) to determine $s_i^X$ and $s_j^Y \geq 0$ (with, for example, $s_i^X > 0$ if AB is an irreducible matrix), but that these weapon-system-type scores are only unique up to a constant multiple. Thus, the weapon-system-type-valuation scheme given by (7.18.3) is a "reasonable" model for imputed

valuation of weapon-system types, since it does yield values that do not obviously violate any paradigms of rationality (such as a negative value occurring).

Let us now consider what happens to the total value of each of the two opposing forces in the special case in which all attrition occurs according to a heterogeneous-force $F|F$ attrition process (see Section 7.7), and all such attrition is accounted for by the A and B matrices of attrition-rate coefficients. We will see that in such cases the total value of each force undergoes a homogeneous-force $F|F$ attrition process and that the quantities $K_X$, $K_Y$, and $\lambda^*$ may be given simple operational interpretations. Thus, instead of having to analyze heterogeneous-force combat, one can examine a derived homogeneous-force model for total force capability (i.e. value). Additionally, we will find that certain model quantities are invariant under admissible[51] changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, and we will be led to a very convenient scaling scheme for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ which in many senses is the "best" scaling scheme. It should be pointed out here that within the context of aggregated-force value, the existence of quantities that are invariant under (admissible) changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ is of the greatest significance because it allows us to deduce system behavior that is fundamental in the sense of not depending on the particular scaling assumptions (i.e. scaling method) adopted. All other quantities (i.e those not invariant under the group of transformations effecting admissible changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$) depend on the choice of scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, and consequently different results will be obtained for them with different scaling schemes. Thus, one has motivation for looking for quantities that are invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$.

Thus, we will assume that the X and Y forces undergo heterogeneous-force F|F attrition (see Section 7.7 for a discussion of the operational assumptions associated with this attrition process), i.e. for $x_i$ and $y_j > 0$

$$\begin{cases} \dfrac{dx_i}{dt} = -\sum_{j=1}^{n} a_{ij}y_j & \text{with } x_i(0) = x_i^0 , \\ \dfrac{dy_j}{dt} = -\sum_{i=1}^{m} b_{ji}x_i & \text{with } y_j(0) = y_j^0 . \end{cases} \tag{7.18.12}$$

Let us consider now the total value of the X force, denoted as $V_X$, which (if we assume that the aggregated-force value is a linear function of the number of each component-weapon-system type in the combined-arms team) is given by

$$V_X = \sum_{i=1}^{m} s_i^X x_i . \tag{7.18.13}$$

Similarly, we take the total value of the Y force $V_Y$ to be given by

$$V_Y = \sum_{j=1}^{n} s_j^Y y_j . \tag{7.18.14}$$

The reader should recognize (7.18.13) and (7.18.14) as the usual linear scoring scheme for determining a single index number to represent the total combat capability or worth of a heterogeneous force (see Sections 7.11 and 7.12 for further details). It follows from (7.18.3), (7.18.13), and (7.18.14) that as long as $\underset{\sim}{x}$ and $\underset{\sim}{y} > 0$

$$
\begin{cases}
\dfrac{dV_X}{dt} = -\left(\dfrac{1}{K_Y}\right) V_Y & \text{with } V_X(0) = V_X^0 , \\[2ex]
\dfrac{dV_Y}{dt} = -\left(\dfrac{1}{K_X}\right) V_X & \text{with } V_Y(0) = V_Y^0 ,
\end{cases}
\tag{7.18.15}
$$

where

$$
V_X^0 = \sum_{i=1}^{m} s_i^X x_i^0 \qquad \text{and} \qquad V_Y^0 = \sum_{j=1}^{n} s_j^Y y_j^0 \quad . \tag{7.18.16}
$$

From (7.18.15) we see that it is convenient to let

$$
C_X = 1/K_X \qquad \text{and} \qquad C_Y = 1/K_Y \tag{7.18.17}
$$

and write (7.18.15) as

$$
\begin{cases}
\dfrac{dV_X}{dt} = -C_Y V_Y & \text{with } V_X(0) = V_X^0 , \\[2ex]
\dfrac{dV_Y}{dt} = -C_X V_X & \text{with } V_Y(0) = V_Y^0 .
\end{cases}
\tag{7.18.18}
$$

It follows from (7.18.8) and (7.18.17) that the maximal eigenvalue $\lambda^*$ determining the weapon-system-type scores $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ in (7.18.10) and (7.18.11) is related to $C_X$ and $C_Y$ by

$$
\lambda^* = C_X C_Y . \tag{7.18.19}
$$

Thus, the square root of the PERRON-FROBENIUS eigenvalue $\sqrt{\lambda^*} = \sqrt{C_X X_Y}$ may be interpreted as the intensity of combat attriting the values of the aggregated X and Y forces (cf. our discussion in Section 2.2 of the intensity of combat for the F|F attrition process). Furthermore, $C_X$ and $C_Y$ may be interpreted as LANCHESTER attrition-rate coefficients in the process by which aggregated-force value is diminished over time. Thus, for example, $C_X$ may be thought of as the rate at which one unit of aggregated-X-force value (or combat capability) is destroying aggregated-Y-force value.

At this juncture it is convenient to use (7.18.17) to rewrite the fundamental equations for weapon-system-type worth imputed by attrition as

$$\begin{cases} C_X s_i^X = \sum_{j=1}^{n} b_{ji} s_j^X , \\ C_Y s_j^Y = \sum_{i=1}^{m} a_{ij} s_i^X . \end{cases} \tag{7.18.20}$$

Although we will have no immediate use for them, it is convenient for future purposes to record here the "summed results" that follow from (7.18.20)

$$C_X = \frac{\sum_{j=1}^{n} \{ \sum_{i=1}^{m} b_{ji} \} s_j^Y}{\sum_{j=1}^{n} s_j^X} ,$$

and (7.18.21)

$$C_Y = \frac{\sum_{i=1}^{m} \{ \sum_{j=1}^{n} a_{ij} \} s_i^X}{\sum_{j=1}^{n} s_j^Y} ,$$

from which it follows that the quantity $C_X C_Y$ <u>is invariant under changes in scale for</u> $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, i.e. $C_X C_Y$ remains the same when $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ are replaced by $k_1\underset{\sim}{s}_X$ and $k_2\underset{\sim}{s}_Y$ where $k_1$ and $k_2$ are arbitrary positive constants.

<u>Example 7.18.1</u>. For the 2 × 2 case, i.e. two weapon-system types on each side (m = n = 2), one can obtain explicit (but rather complicated and generally unenlightening by themselves) results:

$$\lambda^* = \begin{cases} \frac{1}{2}\{c_{11} + c_{22} + \sqrt{(c_{11}-c_{22})^2 + 4c_{12}c_{21}}\} & \text{for } c_{12}c_{21} > 0, \\ \max(c_{11}, c_{22}) & \text{for } c_{12}c_{21} = 0\ , \end{cases} \tag{7.18.22}$$

or, equivalently,

$$\lambda^* = \begin{cases} \frac{1}{2}\{d_{11} + d_{22} + \sqrt{(d_{11}-d_{22})^2 + 4d_{12}d_{21}}\} & \text{for } d_{12}d_{21} > 0, \\ \max(d_{11}, d_{22}) & \text{for } d_{12}d_{21} = 0, \end{cases} \tag{7.18.23}$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21}, & d_{11} &= a_{11}b_{11} + a_{21}b_{12}, \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22}, & d_{12} &= a_{12}b_{11} + a_{22}b_{12}, \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21}, & d_{21} &= a_{11}b_{21} + a_{21}b_{22}, \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22}, & d_{22} &= a_{12}b_{21} + a_{22}b_{22}. \end{aligned} \tag{7.18.24}$$

We find that

$$s_2^X = \begin{cases} \left(\dfrac{\lambda^* - c_{11}}{c_{21}}\right) s_1^X & \text{for } c_{21} > 0 , \\ \\ \left(\dfrac{c_{12}}{c_{11} - c_{22}}\right) s_1^X & \text{for } c_{21} = 0 \text{ and } c_{11} > c_{22}, \end{cases} \tag{7.18.25}$$

with $s_1^X = 0$ for $c_{21} = 0$ and $c_{11} \leq c_{22}$, and

$$s_2^Y = \begin{cases} \left(\dfrac{\lambda^* - d_{11}}{d_{21}}\right) s_1^Y & \text{for } d_{21} > 0 , \\ \\ \left(\dfrac{d_{12}}{d_{11} - d_{22}}\right) s_1^Y & \text{for } d_{21} = 0 \text{ and } d_{11} > d_{22} , \end{cases} \tag{7.18.26}$$

with $s_1^Y = 0$ for $d_{21} = 0$ and $d_{11} \leq d_{22}$.

Let us now turn to consideration of the evolution of the total-aggregated-force values $V_X$ and $V_Y$ over time. Since these values satisfy the LANCHESTER-type equations (7.18.18) for a F|F attrition process, we can invoke all the results that we developed in Chapter 2. In particular, the total-aggregated-X-force value as a function of time $V_X(t)$ is given by

$$V_X(t) = V_X^0 \cosh \sqrt{\lambda^*} \ t - V_Y^0 \sqrt{\frac{C_Y}{C_X}} \sinh \sqrt{\lambda^*} \ t. \tag{7.18.27}$$

From (7.18.27) the interpretation of $\sqrt{\lambda^*}$ as the intensity of aggregated-force combat should be obvious. However, if we consider the fraction of the initial total-aggregated-X-force value, denoted as $f_X(t) = V_X(t)/V_X^0$, we will learn much more about this aggregated-force model. Hence, we consider

$$f_X(t) = \frac{V_X(t)}{V_X^0} = \cosh \sqrt{\lambda^*}\ t - \frac{V_Y^0}{V_X^0} \sqrt{\frac{C_Y}{C_X}} \sinh \sqrt{\lambda^*}\ t, \tag{7.18.28}$$

from which we will deduce that <u>the normalized force ratio</u> $\rho(t)$, defined by

$$\rho(t) = \sqrt{\frac{C_X}{C_Y}} \left\{ \frac{V_X(t)}{V_Y(t)} \right\}, \tag{7.18.29}$$

<u>is invariant under changes in scale for the value vectors</u> $\underset{\sim}{s}_X$ <u>and</u> $\underset{\sim}{s}_Y$ by the following argument. Consider the fraction of the initial total-aggregated-X-force value

$$f_X(t) = \frac{V_X(t)}{V_X^0} = \frac{\underset{\sim}{s}_X^T \underset{\sim}{x}(t)}{\underset{\sim}{s}_X^T \underset{\sim}{x}_0}, \tag{7.18.30}$$

and observe that it is invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, i.e. $f_X(t)$ remains the same when $\underset{\sim}{s}_X$ is replaced by $k\underset{\sim}{s}_X$ where $k$ is an arbitrary positive constant. Consequently, from the right-hand side of (7.18.28) we may conclude that the same is true for $\sqrt{C_X/C_Y}\ (V_X^0/V_Y^0) = \rho(0) = \rho_0$. Thus, the same invariance must hold for the normalized force ratio defined by (7.18.29), and our above assertion has been proven.

It is instructive for future purposes to consider a second proof of the stated invariance of the normalized force ratio $\rho(t)$. As we have seen previously in this chapter, the force ratio is used for many key purposes in aggregated force-on-force combat modelling (e.g. casualty assessment, FEBA-movement determination, simulation of tactical decision making, etc.). Therefore, let us consider the force ratio $F_R(t)$ defined by

$$F_R(t) = \frac{V_X(t)}{V_Y(t)} . \tag{7.18.31}$$

We observe that the ordinary force ratio $F_R(t)$ is not invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, since the substitution $\underset{\sim}{s}'_X = k_1\underset{\sim}{s}_X$ and $\underset{\sim}{s}'_Y = k_2\underset{\sim}{s}_Y$ transforms it in the following way

$$F'_R = \left(\frac{k_1}{k_2}\right) F_R . \tag{7.18.32}$$

From (7.18.18) and (7.18.31) it follows that the force ratio $F_R(t)$, as usual, satisfies a RICCATI equation which in this case takes the form

$$\frac{dF_R}{dt} = C_X(F_R)^2 - C_Y \quad \text{with} \quad F_R(0) = V_X^0/V_Y^0 . \tag{7.18.33}$$

Let us observe that by (7.18.21) neither of $C_X$ and $C_Y$ is invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$. Furthermore, any quantity possessing such invariance cannot satisfy any (differential) equation with coefficients that do not themselves possess such invariance. From this last observation and inspection of (7.18.33) we are led to discover that $p(t)$ defined by

$$p(t) = C_X F_R(t) \tag{7.18.34}$$

possesses the desired invariance by seeking to transform (7.18.33) into a differential equation whose coefficients are invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$. Considering (7.18.33), we see that an obvious thing to do is to multiply both sides of it by $C_X$ and to use (7.18.19) to find that

$$\frac{dp}{dt} = p^2 - \lambda^* \qquad \text{with} \quad p(0) = p_0 \, . \tag{7.18.35}$$

The conjecture that $p(t)$ possesses the desired invariance is readily confirmed by using (7.18.21) to write (7.18.34) as

$$p(t) = \left\{ \frac{\sum_{i=1}^{m} s_i^X x_i}{\sum_{i=1}^{m} s_i^X} \right\} \left\{ \frac{\sum_{j=1}^{n} \left[ \sum_{i=1}^{m} b_{ji} \right] s_j^Y}{\sum_{j=1}^{n} s_j^Y y_j} \right\} \, . \tag{7.18.36}$$

It is clear from (7.18.35) that $p(t)$ remains the same when we replace $s_X$ and $s_Y$ by $k_1 s_X$ and $k_2 s_Y$.

Thus, we have proven that both $C_X C_Y$ and $C_X F_R(t)$ are invariant under such changes of scale. Since the same must also be true for any function of these two invariants, we have consequently shown that the normalized force ratio $\rho(t) = C_X F_R(t)/(\sqrt{C_X C_Y})$ possesses the desired invariance (which we have previously shown by other means). This invariance may, of course, also be proven directly by using (7.18.13), (7.18.14), (7.18.21), and (7.18.29). From (7.18.29) and (7.18.33) it follows that the normalized force ratio $\rho(t) = \rho(s(t))$ satisfies the following very simple RICCATI equation

$$\frac{d\rho}{ds} = \rho^2 - 1 \qquad \text{with} \quad \rho(0) = \rho_0 \, , \tag{7.18.37}$$

where

$$\rho_0 = \sqrt{\frac{C_X}{C_Y}} \left\{ \frac{\sum_{i=1}^{m} s_i^X x_i^0}{\sum_{j=1}^{n} s_j^Y y_j^0} \right\} ,$$

and

$$s = \sqrt{\lambda^*}\, t .$$

Invoking results about the force ratio from Section 2.2, we may conclude the possession of the following important properties by the normalized force ratio $\rho(t)$, which we have shown to be invariant under changes in scale for $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$:

(P1) $\rho(t)$ is a strictly decreasing function of time if and only if $\rho_0 < 1$;

(P2) $\rho(t)$ is constant over time if and only if $\rho_0 = 1$;

(P3) $Y$ will win any aggregated-force fixed-force-ratio-breakpoint battle if and only if $\rho_0 < 1$;

(P4) $\rho(t)$ is given by

$$\rho(t) = \left\{ \frac{(\rho_0 + 1)\exp(-2\sqrt{\lambda^*}t) + (\rho_0 - 1)}{(\rho_0 + 1)\exp(-2\sqrt{\lambda^*}t) - (\rho_0 - 1)} \right\} ; \qquad (7.18.38)$$

and (P5) the time $t_f$ that it will take for the normalized force ratio to reach any specified final value $\rho_f \neq 1$ is given by

$$t_f = \frac{1}{2\sqrt{\lambda^*}} \ln \left\{ \left( \frac{\rho_f - 1}{\rho_f + 1} \right) \left( \frac{\rho_0 + 1}{\rho_0 - 1} \right) \right\} , \tag{7.18.39}$$

where only one of the following two situations is possible:

either (S1) $\rho_f < \rho_0 < 1$,

or (S2) $\rho_f > \rho_0 > 1$.

It remains for us to discuss the normalization (or scaling) of the weapon-system-type-value vectors $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ determined by the linear model (7.18.3). Accordingly, we will first review how various authors have scaled these value vectors, and then (based on being able to circumvent certain observed apparent antimonies of imputed weapon-system-type valuation) we will suggest an alternative scaling scheme that avoids some difficulties observed for the other scaling schemes.

Two additional conditions (one for each vector) are needed to uniquely specify the weapon-system-type-value vectors $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ that have been each determined up to a scale factor by the linear imputed-value model (7.18.3). Different normalization (scaling) schemes that have been proposed and tried by various authors are shown in Table 7.VIII, with the $C_X$ and $C_Y$ proportionality constants of (7.18.20) [equivalently, the aggregated-force LANCHESTER attrition-rate coefficients of (7.18.18)] that arise from these various scaling schemes being shown in Table 7.IX. It should be noted that when the HOLTER-ANDERSON approach to scaling is used, the usual force ratio $F_R = V_X/V_Y$ is equal to the normalized force ratio $\rho = (\sqrt{C_X/C_Y})V_X/V_Y$, since one has chosen to scale the value vectors in such a way that $C_X = C_Y$. Consequently,

TABLE 7.VIII. Normalization (Scaling) of Imputed Values for Weapon-System Types.

SPUDICH[†] (1968)

$$\sum_{i=1}^{m} \{ \sum_{j=1}^{n} a_{ij} y_j^0 \} {}^S s_i^X = \lambda^* \qquad \sum_{j=1}^{n} \{ \sum_{i=1}^{m} b_{ji} x_i^0 \} {}^S s_j^Y = \lambda^*$$

DARE and JAMES (1971)

$$\sum_{i=1}^{m} {}^{DJ} s_i^X = 1 \qquad \sum_{j=1}^{n} {}^{DJ} s_j^Y = 1$$

HOWES and THRALL (1972)

$$\sum_{i=1}^{m} \{ \sum_{j=1}^{n} a_{ij} \} {}^{HT} s_i^X = \lambda^* \qquad \sum_{j=1}^{n} \{ \sum_{i=1}^{m} b_{ji} \} {}^{HT} s_j^Y = \lambda^*$$

HOLTER (1973) and ANDERSON (1979)

$$ {}^{HA} s_1^X = 1 \qquad \sum_{j=1}^{n} b_{j1} \, {}^{HA} s_j^Y = \sqrt{\lambda^*}$$

[†] Here SPUDICH (1968) = the document published by SPUDICH in 1968 (see list of references at the end of this chapter).

TABLE 7.IX. Proportionality Constants (Aggregated-Force LANCHESTER Attrition-Rate Coefficients) that Arise from the Various Normalization (Scaling) Schemes for Imputed Values of Weapon-System Types.

SPUDICH[†] (1968)

$$C_X^S = \sum_{j=1}^{n} y_j^0 \, {}^S s_j^Y \qquad C_Y^S = \sum_{i=1}^{m} x_i^0 \, {}^S x_i^X$$

DARE and JAMES (1971)

$$C_X^{DJ} = \sum_{j=1}^{n} \{\sum_{i=1}^{m} b_{ji}\} \, {}^{DJ}s_j^Y \qquad C_Y^{DJ} = \sum_{i=1}^{m} \{\sum_{j=1}^{n} a_{ij}\} \, {}^{DJ}s_i^X$$

HOWES and THRALL (1972)

$$C_X^{HT} = \sum_{j=1}^{n} {}^{HT}s_j^Y \qquad C_Y^{HT} = \sum_{i=1}^{m} {}^{HT}s_i^X$$

HOLTER (1973) and ANDERSON (1979)

$$C_X^{HA} = C_Y^{HA} = \sqrt{\lambda^*}$$

---

[†]As in the preceding table, SPUDICH (1968) = the document published by SPUDICH in 1968 (see list of references at the end of this chapter).

$$F_R^{HA} = \rho^{HA} = \rho^{S} = \rho^{DJ} = \rho^{HT} , \tag{7.18.40}$$

where the superscript denotes which scaling method is being used to uniquely determine the value vectors $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ in conjunction with the basic model (7.18.20), and

S = the scaling method of SPUDICH [134],

DJ = the scaling method of DARE and JAMES [43],

HT = the scaling method of HOWES and THRALL [91] (see also [92]),

and HA = the scaling method of HOLTER [89] and ANDERSON [5].

We will also use this superscript notation for referring to various other quantities of interest computed according to these different scaling methods, e.g. ${}^{HA}s_i^X$ will denote the value of the $i^{\underline{th}}$ X-weapon-system type computed by (7.18.20) with the HOLTER-ANDERSON scaling method.

It is also instructive to investigate how results for these various scaling schemes are related to one another. Using (7.18.20), one can easily show that if

$$\underset{\sim}{s}'_X = k_1 \underset{\sim}{s}_X \qquad \text{and} \qquad \underset{\sim}{s}'_Y = k_2 \underset{\sim}{s}_Y , \tag{7.18.41}$$

then

$$\left( \frac{C_X}{C_Y} \right)' = \left( \frac{k_2}{k_1} \right)^2 \frac{C_X}{C_Y} , \tag{7.18.42}$$

and (as we have already shown above)

$$F_R' = \left( \frac{k_1}{k_2} \right) F_R . \tag{7.18.43}$$

Recalling that $C_X C_Y = \lambda^*$ is invariant under such changes in scale, we may also deduce from (7.18.42) that

$$C_X' = \left( \frac{k_2}{k_1} \right) C_X , \tag{7.18.44}$$

and

$$C_Y' = \left( \frac{k_1}{k_2} \right) C_Y . \tag{7.18.45}$$

Using the above equations (7.18.43) through (7.18.45), one can easily develop relations between these various quantities of interest for the different scaling methods shown in Table 7.VIII. Such relations (except those pertaining to SPUDICH's scaling method) are given in Table 7.X.

The HOLTER-ANDERSON scaling method is to be preferred over the others mentioned above (<u>see</u> also Table 7.VIII) because it allows the X and Y weapon-system types to be compared with each other, not just among themselves (<u>see</u> ANDERSON [5] for further details). It is the approach taken to scaling weapon-system-type-value vectors determined by the linear model (7.18.20) that is used by IDAGAM [5; 6; 130]. Consequently, we have

TABLE 7.X. Relations Between Various Quantities of Interest for Different Normalization (Scaling) Schemes for Imputed Values of Weapon-System Types.

$$\underset{\sim}{s}_X^{DJ} = \frac{1}{C_Y^{HT}}\, \underset{\sim}{s}_X^{HT} = \left\{ \frac{1}{\sum_{i=1}^{m} {}^{HA}s_i^X} \right\} \underset{\sim}{s}_X^{HA}$$

$$\underset{\sim}{s}_Y^{DJ} = \frac{1}{C_X^{HT}}\, \underset{\sim}{s}_Y^{HT} = \left\{ \frac{1}{\sum_{j=1}^{n} {}^{HA}s_j^Y} \right\} \underset{\sim}{s}_Y^{HA}$$

$$F_R^{DJ} = \frac{C_X^{HT}}{C_Y^{HT}}\, F_R^{HT} = \left\{ \frac{\sum_{j=1}^{n} {}^{HA}s_j^Y}{\sum_{i=1}^{m} {}^{HA}s_i^X} \right\} F_R^{HA}$$

$$\rho^{DJ} = \rho^{HT} = \rho^{HA} = F_R^{HA}$$

$$C_X^{DJ} = C_Y^{HT} = \left\{ \frac{\sum_{i=1}^{m} {}^{HA}s_i^X}{\sum_{j=1}^{n} {}^{HA}s_j^Y} \right\} C_X^{HA}$$

$$C_Y^{DJ} = C_X^{HT} = \left\{ \frac{\sum_{j=1}^{n} {}^{HA}s_j^Y}{\sum_{j=1}^{m} {}^{HA}s_i^X} \right\} C_Y^{HA}$$

NOTE: The superscripts denote whose scaling method is being used for uniquely determining the value vectors $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$, with: S = the scaling method of SPUDICH [134]; DJ = the scaling method of DARE and JAMES [43]; DJ = the scaling method of HOWES and THRALL [91] (<u>see</u> also [92]); and HA = the scaling method of HOLTER [89] and ANDERSON [5].

worked out explicit results for the 2 × 2 case in the following example. In more complex cases with more weapon-system types on each side, the eigenvalue problems (7.18.10) and (7.18.11) may be solved by iterative methods[52] (e.g. see HILDEBRAND [82, pp. 68-74] or ANDERSON [5]) or some type of iterative procedure may be used to solve the original linear system (7.18.20) (see HOLTER [89] for further details).

Example 7.18.2. For the 2 × 2 case with the HOLTER-ANDERSON scaling applied, the general results of Example 7.18.1 take the particular form

$$s_1^X = 1,$$

$$s_2^X = \begin{cases} \left(\dfrac{\lambda^* - c_{11}}{c_{21}}\right) & d_{11} > d_{22}, \\[2ex] \left(\dfrac{c_{12}}{c_{11} - c_{22}}\right) & \text{for} \quad c_{21} = 0 \quad \text{and} \quad c_{11} > c_{22}, \end{cases}$$

$$s_1^Y = \begin{cases} \dfrac{d_{21}\sqrt{\lambda^*}}{\{b_{11}d_{21} + b_{21}(\lambda^* - d_{11})\}} & \text{for} \quad d_{21} > 0 \\[2ex] \dfrac{(d_{11} - d_{22})\ \sqrt{d_{11}}}{\{b_{11}(d_{11} - d_{22}) + b_{21}d_{12}\}} & \text{for} \quad d_{21} = 0 \quad \text{and} \quad d_{11} > d_{22}, \\[2ex] 0 & \text{for} \quad d_{21} = 0 \quad \text{and} \quad d_{11} \leq d_{22}, \end{cases}$$

$$s_2^Y = \begin{cases} \dfrac{(\lambda^*-d_{11})\sqrt{\lambda^*}}{\{b_{11}d_{21} + b_{21}(\lambda^*-d_{11})\}} & \text{for } d_{21} > 0, \\[2ex] \dfrac{d_{12}\sqrt{d_{11}}}{\{b_{11}(d_{11}-d_{22}) + b_{21}d_{12}\}} & \text{for } d_{21} = 0 \text{ and } d_{11} > d_{22}, \\[2ex] 0 & \text{for } d_{21} = 0 \text{ and } d_{11} \le d_{22}. \end{cases}$$

Unfortunately, this method of imputing value with the HOLTER-ANDERSON scaling scheme[53] sometimes produces results that at first sight seem counterintuitive (see the next section, however). For example, increasing the kill rate of a weapon-system type for one side may actually increase the force ratio in favor of the other side. This apparently paradoxical behavior is shown by the following example.

Example 7.18.3. For the special $2 \times 2$ case in which $a_{21} = a_{22} = b_{12} = b_{22} = 0$, i.e. two $Y$ weapon-system types against a single $X$ weapon-system type (see Figure 7.18), the imputed weapon-system-type values determined with the HOLTER-ANDERSON scaling reduce from the general expressions given in Example 7.18.2 to

$$s_1^X = 1\,, \qquad s_2^X = 0\,, \tag{4.18.46}$$

$$s_1^Y = \frac{a_{11}}{\sqrt{a_{11}b_{11} + a_{12}b_{21}}}\,, \qquad s_2^Y = \frac{a_{12}}{\sqrt{a_{11}b_{11} + a_{12}b_{21}}}$$

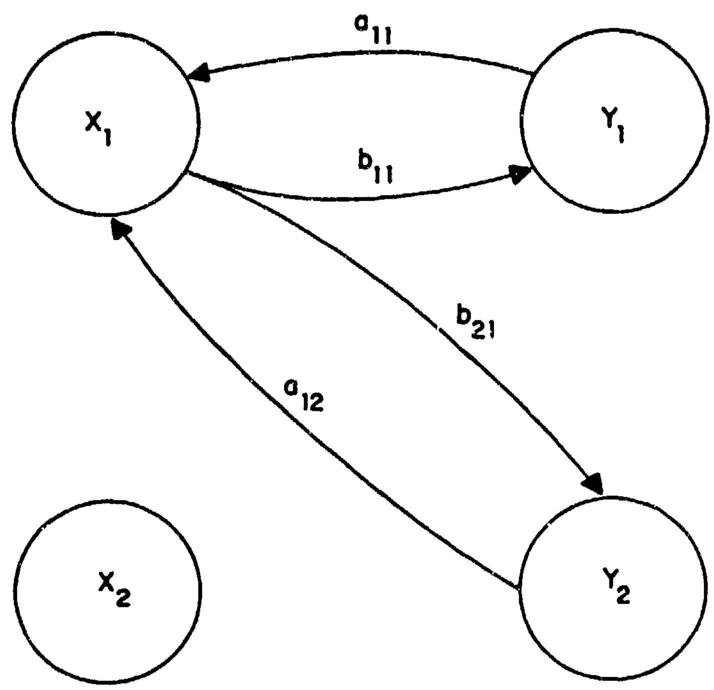


Figure 7.18. Diagram of heterogeneous-force interactions considered in Example 7.18.3.

Computing the force ratio $F_R = s_1^X x_1/(s_1^Y y_1 + s_2^Y y_2)$, we find that

$$\frac{\partial F_R}{\partial a_{11}} = \frac{a_{12}b_{11}x_1 \left\{ y_2 - \frac{a_{11}}{a_{12}} \left(1 + \frac{2a_{12}b_{21}}{a_{11}b_{11}}\right) y_1 \right\}}{2\sqrt{a_{11}b_{11} + a_{12}b_{21}}\ (a_{11}y_1 + a_{12}y_2)^2} .$$

Thus, we see that there are circumstances, i.e. $y_2/y_1 > (a_{11}/a_{12})\ \{1 + 2a_{12}b_{21}/(a_{11}b_{11})\}$, under which increasing the kill rate of a Y weapon-system type actually increases the force ratio in X's favor, i.e. $\partial F_R/\partial a_{11} > 0$.

Although such apparently paradoxical behavior cannot entirely be eliminated from the imputed valuation of weapon-system types by the linear model, it is eliminated in a few special cases (such as that of Example 7.18.3) by the following proposed scaling system. First let us recall, though, that the HOLTER-ANDERSON scaling method picks one of the X weapon-system types (taken to be the first X weapon-system type here) as a reference point, and that the other X-weapon-system-type values, which are determined by (7.18.20) only up to a constant multiple, are then scored (i.e. scaled) relative to this standard. The reference weapon-system type must be a "major system" in order for this scaling method to work[54]. The Y-weapon-system-type values, which are also (of course) only determined by (7.18.20) up to a constant multiple, are then scaled by using

the first of equations (7.18.20) with $i = 1$ and the assumption that $C_X^{HA} = C_Y^{HA}$ (see Tables 7.VIII and 7.IX). Considering the above, we propose here the following scaling scheme: choose both an X and also a Y reference-weapon-system type; assign a value of 1.00 to the X weapon-system type and score the Y weapon-system type according to its relative effectiveness against this reference X weapon-system type in a $1 \times 1$ duel (i.e. the ratio of single-opposing-reference-system kill rates). Thus, we would have

$$ {}^{T}s_1^X = 1 \qquad \text{and} \qquad {}^{T}s_1^Y = \frac{a_{11}}{b_{11}} \quad . \tag{7.18.47} $$

The basic idea here is that for each force a weapon-system type is selected for scaling purposes as a reference point, the X-reference-weapon-system type is assigned a value of 1.00, and the Y-reference-weapon-system type is scored relative to this arbitrary X-reference-weapon-system-type value.

Example 7.18.4. For the $2 \times 2$ case with the above scaling method (7.18.47), the general results of Example 7.18.1 take the particular form

$$ s_1^X = 1, $$

$$ s_2^X = \begin{cases} \left(\dfrac{\lambda^* - c_{11}}{c_{21}}\right) & \text{for } c_{21} > 0 \, , \\ \left(\dfrac{c_{12}}{c_{11} - c_{22}}\right) & \text{for } c_{21} = 0 \text{ and } c_{11} > c_{22} \, , \end{cases} $$

$$ s_1^Y = \frac{a_{11}}{b_{11}} \, , $$

and

$$s_2^Y = \begin{cases} \dfrac{\lambda^* - d_{11}}{d_{21}} & \text{for } d_{21} > 0, \\[2ex] \dfrac{d_{12}}{d_{11} - d_{22}} \quad \dfrac{a_{11}}{b_{11}} & \text{for } d_{21} = 0 \text{ and } d_{11} > d_{22}. \end{cases}$$

For the special case in which $a_{21} = a_{22} = b_{12} = b_{22} = 0$ (again, <u>see</u> Figure 7.18), the above imputed weapon-system-type values reduce to

$$s_1^X = 1 , \qquad s_2^X = 0 ,$$
$$s_1^Y = a_{11}/b_{11} , \qquad s_2^Y = a_{12}/b_{12} , \tag{7.18.48}$$

and we then find that $\partial F_R/\partial a_{1j} = -b_1 x_1 y_j/(a_{11}y_1 + a_{12}y_2)^2 < 0$ .

Which scaling method is "best"? This important question should undoubtedly be answered by investigating which scoring scheme [i.e. combination of basic model (7.18.20) and scaling method] provides the "best" model for imputing weapon-system-type values, i.e. produces the best results according to some criteria. However, this type of investigation has apparently never been completely carried out, and it does appear that alternate scaling methods are quite naturally suggested. For example, besides the above one (7.18.47), another very reasonable scaling method would be to assign a value of 1.00 to the X-reference-weapon-system type

and then score the Y-reference-weapon-system type on the basis of its relative effectiveness against this weapon-system type but weighted by the intensity of combat in this $1 \times 1$ duel relative to the intensity of combat in the overall battle, i.e.

$$TT_{s_1^Y} = \frac{a_{11}}{b_{11}} \left\{ \frac{a_{11} b_{11}}{\lambda^*} \right\} = \frac{a_{11}^2}{\lambda^*} \quad . \tag{7.18.49}$$

However, it should be noted that the HOLTER-ANDERSON scaling method is more natural in the sense of using the first of equations (7.18.20) with $i = 1$ and $s_1^X = 1$ to scale $\underset{\sim}{s}_Y$. In the last analysis, though, the choice of scaling method should be based on consideration of the properties of the induced results.

*7.19. Critique of Such Methodology for Imputing Values to Weapon-System Types.

It is only fair to alert the reader to the fact that there is far from universal agreement about the usefulness and validity of the methodology described in the previous section for imputing values to weapon-system types. Although it is beyond the scope of our current investigation to examine in detail criticism of and issues associated with this methodology for valuating forces in aggregated-force analyses, we will try to outline the salient features of such discourse and identify sources of further information for the reader who desires additional details. It should be born in mind, though, that (irrespective of such criticism) comparing, equating, or quantifying in some way the relative performance of diverse weapon systems is one of the key tasks in the evaluation of weapon systems for defense planning (e.g. see [149, Chapter 30] for further details), and frequently such analysis must be done within such stringent resource and time constraints that the use of any type of detailed combat model is precluded (see below for further discussion).

The above method for imputing values to weapon-system types based on their LANCHESTER attrition-rate coefficients has evolved out of previous attempts to use the index-number approach to quantify military capabilities: it was apparently developed in response to the criticism of the old firepower-score approach that it did not value (or score) weapon-system types based on the circumstances (i.e. combat environment, friendly force structure, and enemy force structure) of employment for a weapon system [90, p. II-C-3] (see also [39, p. 15], LESTER and ROBINSON [105], and [150, p. 56]). Thus, in order to properly assess the usefulness of this new weapon-system-valuation

methodology one should review critical appraisals of the old firepower-score methodology: the interested reader can find critical reviews of the firepower-score approach in HONIG et al. [90, Appendix C to Chapter II], BODE [10], and STOCKFISCH [135] (see also [150, pp. 54-56]). It appears to this author that the model considered in the last section for imputing values to weapon-system types does respond favorably to the criticism that the old firepower-score approach, which essentially judgmentally determined the values of weapon-system types, was not a transparent model of weapon-system valuation [150, p. 56], and also did not reflect changes in the circumstances of combat (e.g. enemy force mix or distribution of fire over enemy target types) in the valuation of weapon-system types. See, however, FARRELL [56] and ANDERSON [5] for critiques of the imputed-value method.

No discussion about the pros and cons of index-number approaches used in general-purpose-force analyses and/or models can be considered to be complete without placing it in the perspective of noting that it may be viewed as part of a broader debate over whether corps-level and theater-level combat operations should be represented by aggregated or detailed models for purposes of defense analyses (e.g. see STOCKFISCH [135, pp. 9-10] or [150, pp. 54-56]). It has been argued that detailed models are to be preferred[55] because they make judgment (and the use of judgment in an immature field such as combat modelling apparently cannot be avoided[56]) explicit and hopefully transparent. Due to the almost complete lack of relevant combat data to empirically test whether detailed or aggregated combat models yield better predictions (at least when tested within the context of past historical combat), the debate has become essentially metaphysical, with many people seemingly arguing

that more detail is necessarily better. A more germane question is: How much detail is relevant? And an even more practical question is: How much detail can one afford? A recent U. S. General Accounting Office (GAO) report [150, pp. 28-29] points out that there is a strong inconsistency between people wanting more detail in combat models and yet resenting having to pay for it by spending more man-years of effort to have analysts understand such a detailed combat model and learn how to use it. In other words, more support is required in terms of people (i.e. analysts) to maintain and use a more detailed model, particularly if another agency or company developed the model. The transfer of a complex model from one installation to another is frequently an insuperable problem (e.g. see SZYMCZAK [139] for further details).

Many people today feel that combat models have become too complicated[57], and there has consequently been talk of a "complexity crisis" (see Section 7.23 below). One suggested way out of this dilemma of requiring both model detail and also ease of running and understanding has been to use a hierarchical modelling approach in which the output from detailed combat models of small-unit operations is used to generate various combat-results tables for a large-scale aggregated combat model. Thus, the output from one model is the input to another model. Well-developed hierarchies of combat models exist in the United Kingdom and West Germany, and also to a lesser extent in the United States (see Section 7.20 for further details). Within this context the above weapon-system-valuation model provides an essential interface between a small-unit detailed model and a large-scale aggregated one by converting heterogeneous-force single-system kill rates (determined by the detailed model) into firepower scores[58] (i.e. weapon-system-type values)

that are sensitive to the physical and operational circumstances of battle (e.g. see DARE [42, pp.294-295]). Thus, these imputed weapon-system-type values in some sense combine the best of the detailed- and aggregated-combat-modelling worlds by explicitly considering the physical and operational factors of a combined-arms-team engagement but yet aggregating all the forces on each side in some geographical region. Within this context, these new imputed weapon-system-type values apparently are a distinct improvement over the old firepower scores which were essentially judgmentally determined.

With the above as general background, let us now briefly turn to the problem of evaluating the merits of the above methodology for imputing values to weapon-system types. Four criteria that one can use for this evaluation are as follows:

(C1) internal consistency,

(C2) external validity,

(C2a) prima-facie validity,

(C2b) empirical validity,

(C3) transparency,

and (C4) computational efficiency.

The first criterion (C1) asks that such a methodology is logically consistent and produces no contradictions or paradoxes, while the second (C2) requires that if such weapon-system-type scores (i.e. values) are used in a model of

some combat process (e.g. attrition, FEBA movement, tactical decision making), the results produced are consistent with evidence from the real world. In the latter instance (as well as the next two), the use to which the weapon-system-type scores are being put must be considered. The last two criteria (C3) and (C4) are particularly important for any quantitative methodology that is to be used for defense planning/defense decision making (e.g. see [150, pp. 25-31]). They are apparently particularly well satisfied by the above methodology for imputing values to weapon-system types in relation to other modelling approaches (especially the computational efficiency of index-number-based models of such combat processes as aggregated-force attrition, FEBA movement, and tactical decision making), and consequently they will not be further discussed here. Thus, it remains to discuss the internal consistency and external validity of the imputed-value method.

R. L. FARRELL [56] has investigated the internal consistency of the above weapon-system-type-valuation scheme and concluded[59] that this valuation method does not satisfy the elementary properties that one would desire for a weapon- and force-evaluation methodology. He used the following four criteria for evaluating the methodology:

(FC1) consistency,

(FC2) regularity,

(FC3) tactical meaningfulness,

(FC4) dependence on effectiveness parameters and independence of nuisance parameters.

FARRELL argued that the methodology failed to be tactically meaningful by exhibiting the following "paradoxes":

(P1) increasing the kill rate against an enemy system sometimes actually increases the value of that system,

and (P2) a shift in fire distribution to cause more attrition to a higher-value enemy target can sometimes reduce the value of the firing force.

We will now show by considering a simple example that a little further analysis reveals that neither instance is really a paradox.

Example 7.19.1. Consider the special $2 \times 2$ case in which $a_{12} = a_{22} = b_{21} = b_{22} = 0$, i.e. two X weapon-system types against a single Y weapon-system type (see Figure 7.19). The imputed weapon-system-type values determined with the HOLTER-ANDERSON scaling are given by

$$s_1^X = 1 , \qquad s_2^X = \frac{b_{12}}{b_{11}} ,$$
$$s_1^Y = \frac{1}{b_{11}} \sqrt{a_{11}b_{11} + a_{21}b_{12}} , \qquad s_2^Y = 0 . \tag{7.19.1}$$

It is readily shown that

$$\frac{\partial s_1^Y}{\partial b_{11}} < 0 , \tag{7.19.2}$$

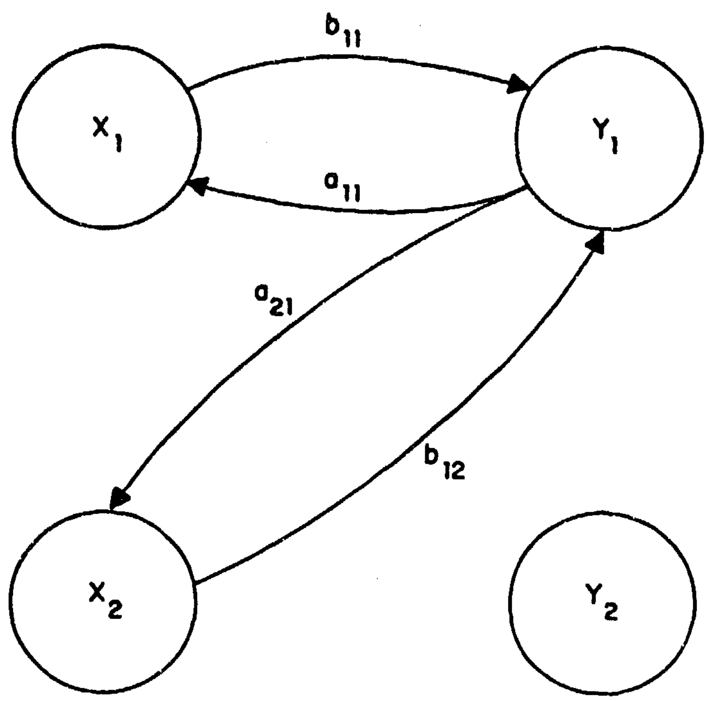


Figure 7.19. Diagram of heterogeneous-force interactions considered in Example 7.19.1.

but that

$$\frac{\partial s_1^Y}{\partial b_{12}} > 0 \ . \tag{7.19.3}$$

What does not seem to have been previously noted, though, is that

$$\frac{\partial}{\partial b_{12}} \left( \frac{s_1^Y}{s_2^X} \right) < 0 \quad . \tag{7.19.4}$$

Thus, the value of a $Y$ target type is increased when it is inflicted with a higher loss rate by any other $X$ weapon-system type except the reference one $X_1$, since the value of the firing $X$ system goes up and consequently the $Y$ system kills a higher value target type and hence increases in value. However, the target type always increases in value less rapidly than the firer type [see (7.19.4) above], and this result is quite plausible and intuitively appealing. Computing the force ratio $F_R = V_X/V_Y = (b_{11}x_1 + b_{12}x_2)/(y_1 \sqrt{a_{11}b_{11} + a_{21}b_{12}})$, we find that

$$\frac{\partial F_R}{\partial b_{11}} = \frac{a_{11}b_{12}}{2(a_{11}b_{11} + a_{21}b_{12})^{3/2}\ y_1} \left\{ x_1 \frac{b_{11}}{b_{12}} \left( 1 + \frac{2a_{21}b_{12}}{a_{11}b_{11}} \right) - x_2 \right\}. \tag{7.19.5}$$

Thus, we see that there are circumstances, i.e. $x_2/x_1 > (b_{11}/b_{12})\{1 + 2a_{21}b_{12}/(a_{11}b_{11})\}$ under which increasing the kill rate of an $X$ weapon-system type actually reduces the force ratio against $X$, i.e. $\partial F_R/\partial b_{11} < 0$. To understand why this has happened, let us observe that

$$\frac{\partial s_2^X}{\partial b_{11}} < 0 \,, \tag{7.19.6}$$

i.e. increasing the kill rate of $X_1$ against $Y_1$ decreases the value of $X_2$ relative to that of $X_1$ (see Figure 7.19). Hence, increasing the kill rate of $X_1$ can actually decrease the force ratio against $X$ when there are not enough $X_1$ systems present to overcome the decrease in value of the $X_2$ systems.

The above example provides much insight into the imputed-valuation scheme (7.18.20) with HOLTER-ANDERSON scaling and raises the question (at least in this author's mind) of whether the "paradoxes" (P1) and (P2) above are really paradoxes at all. Some further discussion of Example 7.19.1 within this context therefore seems to be in order. Further investigation has revealed that more generally[60] (at least for the 2 × 2 case)

$$\frac{\partial s_j^Y}{\partial b_{ji}} > 0 \quad \text{for} \quad i \neq 1 \quad \text{but} \quad \frac{\partial}{\partial b_{ji}}\left(\frac{s_i^X}{s_1^X}\right) < 0 \,, \tag{7.19.7}$$

i.e. increasing the single-system kill rate $b_{ji}$ of $X_i$ (with the exception of $i = 1$, the reference-weapon-system type for the HOLTER-ANDERSON scaling scheme) against $Y_j$ increases not only the value of the firer-type weapon system $s_i^X$ but also the value of the $Y_j$ target-weapon-system type $s_j^Y$. This is not unreasonable, since the $Y_j$ system now kills a more valuable $X_i$

target type. However, the firer type increases in value more than the target type, i.e. $\partial(s_j^Y/s_i^X)/\partial b_{ji} < 0$, as is eminently reasonable. Furthermore, $\partial s_k^X/\partial b_{ji} < 0$ for $k \neq i$ or 1, i.e. increasing the single-system kill rate of $X_i$ against any target type decreases the value $s_k^X$ of any other X firer type $X_k$ (except for, of course, $k = i$ or 1) because it has become less effective relative to $X_i$ [cf. (7.19.6) above in Example 7.19.1]. Furthermore, this last result explains the second apparent paradox (P2), since

$$\frac{\partial F_R}{\partial b_{ji}} = \frac{1}{\{\sum_{\ell=1}^{n} y_\ell s_\ell^Y\}} \left\{ \sum_{k=2}^{m} x_k \frac{\partial s_k^X}{\partial b_{ji}} - F_R \sum_{\ell=1}^{n} y_\ell \frac{\partial s_\ell^Y}{\partial b_{ji}} \right\} \tag{7.19.8}$$

In particular, recalling (7.19.5) and the subsequent discussion in Example 7.19.1, we see that increasing the fire effectiveness of one weapon-system type decreases the relative effectiveness of other weapon-system types (except for, of course, the X-reference-weapon-system type) against the enemy weapon-system type, with the attendant consequence that total force value may actually decline[61] if the relative numbers of these diminished-value weapon-system types are sufficient to outweigh the total value of the weapon-system type whose fire effectiveness has been increased. It should be noted that this situation occurs when a weapon-system type with relatively small numbers on the battlefield is increased in effectiveness (i.e. single-system kill rate), while relatively more numerous weapon-system types remain at their previous effectiveness and therefore decrease in relative value.

It consequently does seem to be perfectly reasonable to this author that increasing the single-system kill rate of a particular weapon-system type could actually decrease the total value of a force due to weapon-system types that are more numerous becoming less valuable. In this context, it should be born in mind that increasing the capability of a single particular weapon-system type in a combined-arms team historically has not always increased total-force effectiveness. (Here we have taken some literary license in the phrasing of this argument, but in any case the model here indicates that more detailed analysis of interactions is required for assessing total-force effectiveness.) Thus, the model (7.18.20) for imputing values to weapon-system types based on their single-system kill rates not only does not apparently produce any serious paradoxes but also yields some interesting and important insights into weapon-system valuation. In retrospect, it does not seem intuitively obvious that one could increase the value of a single particular weapon-system type (as the old judgmentally-based firepower-score methodology allowed) in isolation from its interactions with other weapon-system types.

Thus, the above paradoxes (P1) and (P2) produced by this model for imputing weapon-system-type values appear to this author to be more illusionary than real, just as have so many other paradoxes of rationality that have, for example, been noted for game-theoretic models of political behavior[62] (e.g. see BRAMS [21]). The brief remarks made in this section about the internal consistency of this methodology are not meant to be definitive but to stimulate further detailed analysis and discourse. Thus, it does appear to be premature to dismiss the weapon-system-valuation model presented in

the preceding section as not being a satisfactory quantitative tool for defense planning because it fails to satisfy elementary properties that one would desire for such weapon-system-valuation methodology (although indeed one cannot guarantee that it may not eventually turn out to be so). Further investigation, thought, communication, and discussion of such results are definitely required.

It remains for us to very briefly discuss the external validity of the above weapon-system-valuation methodology. It seems appropriate to consider both the valuation methodology itself and also the use in models of combat processes (e.g. attrition, FEBA movement, tactical decision making) of index numbers developed from these weapon-system-type values. Concerning the valuation methodology itself, it easily passes the test of prima-facie validity, but to date no experiments about whether tactical commanders, defense planners, battlefield soldiers, etc. actually value weapon-system types this way have been conducted to establish its empirical validity (cf. SHUBIK's [129] remarks on experimental gaming). Concerning the use of index numbers derived from these weapon-system-type values in combat-process models, such models again easily pass the test of prima-facie validity[63]. As with any type of combat model, however, empirical validity is an open question because of the scarcity of combat data (recall our discussion in Section 1.2 above and see Section 7.22 below). One point that is rather ironic in view of the current fashionability of detailed models today and bears special note is the fact the available real combat data does not support investigating the empirical validity of detailed combat models but only that of relatively simple, aggregated large-unit models (see Section 7.22 and HUBER, LOW, and TAYLOR [95] for further details).

Finally, a very important point that has not been mentioned and apparently has been overlooked is that aggregated-force casualty-rate and FEBA-movement curves (see Sections 7.13 and 7.15) that were developed for one set of firepower scores must be recalibrated for these new imputed values based on single-system kill rates. For example, if one uses the ATLAS casualty-rate curves as IDAGAM [6, p. 53] does but with weapon-system-type scores developed by the antipotential-potential method, then the casualty-rate curves must be revalidated for the new weapon-system-type scores, since different firepower scores originally produced the derived data points upon which the curves are based (see [84]). In other words, firepower scores (called theoretical lethality indices in [84]) were used to convert raw historical data (numbers of men and material) into derived historical data (force ratios and combat-environment descriptors) from which the casualty-rate curves were developed (see Figure 7.20 and also [84]). It certainly is not obvious a priori that a different set of firepower scores (such as produced by the antipotential-potential method) would lead to the same curves, and this point regarding the validity of empirically-based functional relations developed for one set of firepower scores when different scores are later used to compute force ratios should be further investigated.

Thus, we have exposed the reader to a number of objections that have been raised against this new weapon-system-valuation methodology. The interested reader can find further discussions of this matter in the references cited in this section. However, the author does not believe that these objections are any more serious than can be raised against essentially any other combat-modelling methodology. Furthermore, there are times when aggregated-force models based on index numbers must be used, and this new methodology appears to overcome many of the shortcomings of the old purely-judgmentally-based firepower-score method.

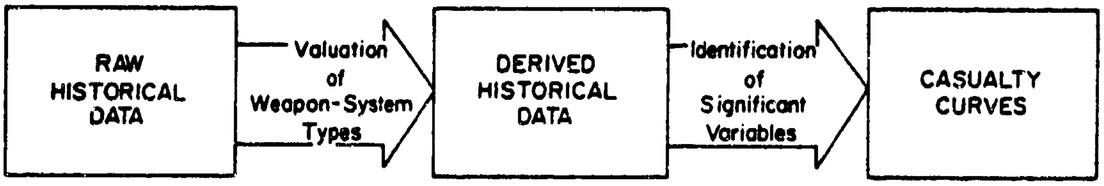


Figure 7.20. Process of developing casualty curves from raw historical data via valuation of weapon-system types.

## 7.20. Hierarchical-Modelling Approaches.

As we have seen above, one can either model the force-on-force combat attrition process in detail or use some type of aggregation approach to model it in not so much detail. Each approach has its strengths and weaknesses. Modelling in detail produces very complex models that are more credible[64] to many people, apparently mainly because they do contain more detail. However, for many (of these very same) people such detailed models of large-scale combat operations are far too complicated to be understood, require too much input data, and (in general) are just not responsive enough. On the other hand, aggregated combat models are fast running, do not require as large data bases, and are much more responsive. However, they do lack a certain amount of credibility, and many of their inputs are not derivable from physically measurable quantities [14]. But yet for many defense-planning purposes there is a need for large-scale (e.g. theater-level) fast-running models (e.g. see DARE [42, pp. 286-287]).

How can one represent large-scale combat in an aggregated fashion and still maintain credibility? The hierarchical-modelling approach attempts to solve this formidable problem by combining the strengths of high-resolution detailed combat models of small-unit operations with those of low-resolution aggregated models of large-scale combat operations. The basic idea is to run the detailed model (or models) to generate data for estimating parameters (i.e. input data) for an aggregated model. In this way, the output data of a high-resolution combat model is used as the input data for a low-resolution combat model. This is also the basic idea behind the fitted-parameter analytical model which was discussed in Sections 5.1 and especially 5.15 above (see Figures 5.1 and 5.12 again).

Although (to the best understanding of this author) the idea of such a hierarchy of models has been around for some time, recent interest in the United States and an attendant analytical framework apparently dates from the Ph.D. thesis of G. CLARK [34] in 1969 (<u>see</u> also [35]). Subsequently, CLARK's ideas have been used by a couple of organizations in the United States. For example, Research Analysis Corporation (RAC) (later GRC) has employed this approach (<u>see</u> STOCKTON [137]) to use output from CARMONETTE to develop combat-results tables for assessing engagement outcomes in the Division Battle Model (DBM) [47] (<u>see</u> also [64]).

Apparently, however, such a hierarchical approach has been much more widely used in NATO countries for a variety of reasons. There are well-developed hierarchies of models in both the United Kingdom (UK) and also the Federal Republic of Germany (FRG) [41] (<u>see</u> also DARE [42], FISCHER and HUBER [57], and NIEMEYER [119]). In fact, the best conceptual discussion of the hierarchical-combat-modelling approach known to this author is the recent one by D. P. DARE [42] of the UK (<u>see</u> [64, Appendix A], however).

7.21. Significant Modelling Issues.

We have briefly touched upon the conceptual bases (i.e. methodologies) for assessing casualties in tactical engagements in war games and other combat simulations in the above sections. However, there remain a number of significant problems involved with the implementation of such methodologies and building operational models of combat (cf. our discussion of the art of modelling in Section 7.1 above). Here we will briefly indicate what some of the issues are. The following is therefore a list of what appear to the author to be some of the significant modelling issues:

(1) scale of operations to be represented,

(2) significant factors (i.e variables) to be represented,

(3) degree of resolution versus amount of detail,

(4) representation of time and space,

(5) assessment of battle outcomes.

Time prohibits any detailed discussion of all these important issues so let us focus on one area that holds particular promise but (unfortunately) has apparently not been appreciated by military OR workers as much as it should have been: namely, the identification and classification of the significant variables in combat. The American military historian and combat analyst COL TREVOR N. DUPUY [86] (U. S. Army, ret.) has developed the following classification of combat variables:

(1) environmental variables-those which affect the effectiveness of weapons,

(2) operational variables-those which influence the employment of weapons and forces,

A. tangible

B. intangible.

DUPUY [48; 49] has developed methodology for systematically applying the effects of such variables (<u>see</u> Table 7.XI) to his own fire-power-score method of combat analysis, which he calls the Quantified Judgment Method of Analysis (QJMA). He has the advantage of apparently being essentially the only person in the United States to have generated new primary combat data from historical records, and combat modellers and analysts should get many new ideas from his work.

TABLE 7.XI. The Significant Combat Variables of T. N. DUPUY [86].

| | |
|---|---|
| | A. Weapons effects |
| Environmental Variables | B. Terrain factors |
| | C. Weather factors |
| Operational Variables | D. Posture factors |
| | E. Mobility effects |
| | F. Tactical decision-making effects |
| | G. Vulnerability factors |
| | H. Tactical air effects |
| | I. Intengible factors |

7.22. <u>Historical Validation of Attrition Models.</u>

What confidence do we have that our models can actually predict what might happen in future possible combat? What is the basis of our knowledge about military combat that is represented by these models? Following STUART CHASE [29], it is possible for us to identify at least seven methods for obtaining such knowledge:

(M1) appeal to the supernatural,

(M2) appeal to worldly military authority -- the higher ranking the better,

(M3) listen to the claims of the most compelling contractor or advisor (i.e. the best "snake-oil salesman"),

(M4) intuition,

(M5) common sense,

(M6) pure logic,

and (M7) the Scientific Method.

These approaches are, of course, not mutually exclusive and often overlap. Unfortunately, the Scientific Method has not always been the source of knowledge in defense-planning work[65], and the simple fact is that if we are honest, there are some severe limitations on the current state-of-the-art as far as how literally we should believe

model outputs. The main problem is that the nature and quality of the available combat data is so extremely poor[66] that we have no reliable "bench mark" against which to "calibrate" our combat models. Compared with the physical sciences, there is an almost complete lack of historical combat data (see Section 1.2 above). Although future combat may be quite unlike that of the past due to the introduction of new technologies and weaponry, it does seem desirable to (in some sense) calibrate our models with past military operations.

Does such a model (necessarily an abstraction) agree (or, at least, not disagree) with the realities of the physical world (either now or in a possible future)? Thus, the combat scientist is faced with the very practical problem of verifying a combat model, perhaps with respect to future possible circumstances and not even the realities of today. In general, the problem of verifying models of man/machine systems is quite difficult (e.g. see NAYLOR and FINGER [118] or VAN HORN [151]), and combat models in particular present a number of special subtleties (see also HUBER, LOW, and TAYLOR [95, Appendix C]), although the process of model verification[67] frequently appears to the uninitiated to be straight-forward. We will now discuss a few of these subtle points, but more careful reflective discussion is needed on this difficult subject.

Special subtleties present in the scientific verification of combat models are as follows:

(1) principle of uniformitarianism does not hold,

(2) systems are only partially observable,

(3) conceptual basis of knowledge is more like that in the social sciences than that in the physical sciences.

The physical sciences are essentially based on the principle of uniformitarianism, which holds that physical and biological processes, conditions, and operations do not change over time (i.e. uniformity over time). For example, in geology the doctrine of uniformitarianism holds that the present is the key to the past [112]. This principle, of course, does not hold for planning models of new future environments (e.g. see HOWLAND [93]). Thus, the combat modeller faces a special problem (which has gone largely unnoticed) in verifying his models: the empirical data base for the testing of such a model is from the real world (past), whereas the prediction from the model is for the real world (future). What is meant by the verification of such a planning model is in need of critical examination. Additionally, in contrast to the modelling of purely physical systems, combat models involve (1) hardware (e.g. weapons) and physical processes, (2) people, and (3) organizational structures. Although human behavior in combat may not change appreciably over time, weapons (i.e. hardware) and organizational structures have and will continue to change appreciably. Thus, the principle of uniformitarianism does not hold for combat analysis, and we cannot use the past by itself to predict the future for combat operations.

Furthermore, since wars are fought for reasons other than just for collecting combat data, even our knowledge as to what has occurred in past combat is imperfect and incomplete. One might even say in technical jargon that military systems in combat are only "partially observable." Finally, since combat models resemble social-science models more than physical-science ones, the standards of knowledge about combat should be more like those of the social sciences than those of the physical sciences. Unfortunately, this has caused difficulties, since the backgrounds of most military OR workers are most closely related to the latter field (i.e. the physical sciences). It appears that epistemological concepts from the social sciences should be quite useful and possibilities in this direction should be further explored in the future.

Before we consider the specifics of the verification of combat models, it seems appropriate for us to briefly consider the sources, nature, and availability of combat data. Firstly, one should distinguish between two types of combat data:

(T1) real combat data,

(T2) simulated combat data (i.e. data generated in a simulated combat environment by field experiments, field exercises, war games, machine simulations, etc.)

The two basic primary sources of real combat data are (see McQUIE et al. [110] or McQUIE [109] for further details):

(S1) archives,

(S2) official military histories.

Unfortunately, quantitative data that is needed from these primary sources for verification of mathematical models of combat is not readily available: the extraction of such quantitative data from archives requires great investment in manpower of a highly specialized nature (one essentially needs a military historian), while the official histories (at least those for the U. S. Army) are purely narrative and do not contain tables, graphs, or appendices with data [109]. (Moreover, a glance at Russian works like SIDORENKO [131] indicates that such quantitative historical studies have been undertaken with vigor in the Soviet Union.) COL T. N. DUPUY (U. S. Army, ret.) and his associates at the Historical Evaluation and Research Organization (HERO) are some of the few people to have conducted research on the archival data (e.g. <u>see</u> [84] or [85]; <u>see</u> also DUPUY [49]) and must be considered the only bona fide experts on it. Moreover, HERO has provided (from winter 1975 until spring 1978) a "Combat Data Subscription Service," whose volumes contain quantitative data (laboriously) extracted from archives[68]. Finally, secondary sources of real combat data are discussed in many of the papers mentioned later in this section.

After a thorough study of the sources, nature, and availability of real combat data, McQUIE et al. [110] concluded that for the purposes of statistical analysis, the data available on World War II and Korea are "inadequate, incomplete, and probably biased." Incompleteness is a particular problem with data measured for one engagement

frequently not available for others [110]. Moreover, the available real combat data is essentially of an aggregated (as opposed to detailed) nature, i.e. "bean counts" for the larger combat units (*see* McQUIE et al. [110] or McQUIE [109] for further details). In other words, the available historical records do not provide detailed combat data such as the positions of individual weapons, targets engaged, engagement conditions for individual target-firer combinations (including the number of rounds expended at each target), etc. Thus, *the available real combat data does not support verification of detailed combat models, but it only supports such investigations of relatively simple aggregated large-unit models* (*see* Section 7.3 (also TAYLOR [145]) for a discussion of detailed versus aggregated combat-attrition models).

However, using simulated combat data, one can in principle verify either detailed small-unit (or even many-on-many) models or the submodels used in such models. There have apparently been some efforts along these lines (e.g. by the U.S. Army's Combat Developments Experimentation Command (CDEC)) but information dissemination about them is poor to nonexistent. The author can supply no specific references outside of mentioning the relatively recent TETAM (Tactical Effectiveness of Antitank Missiles) study by the U.S. Army [31-33] (*see* also BRYSON [26] and THORP [147]).

There have been some (but surprisingly few) attempts to verify combat models. To place this work in proper perspective, it is convenient to conceptually factor the overall combat process into the following four components (*see* HUBER, LOW, and TAYLOR [95] for further details):

(1) attrition,

(2) movement,

(3) $C^3I$ (command, control, communications, and intelligence),

(4) support.

Verification efforts have concentrated on the first of these four processes, and for present purposes so will we. We may also consider that there are different organizational levels at which combat can be represented. One example of such a set of levels is as follows:

(1) force-on-force $\begin{cases} \text{(a) large scale,} \\ \text{(b) small scale,} \end{cases}$

(2) many-on-many,

(3) few-on-few,

(4) one-on-one,

(5) engineering design.

The available (real) combat data[69] is only on Level 1 of the above classification scheme, i.e. force-on-force operations, and then apparently predominantly for large-scale operations. Generally speaking, one can develop both detailed and also aggregated models

of combat processes at each of these five levels[70] (cf. Section 7.3 above). Model verification efforts, moreover, have primarily considered the attrition process[71] for such large-scale force-on-force combat. Furthermore, there are essentially only two general approaches for verifying[72] (or testing) such large large-scale attrition models:

(A1) "replay" some particular historical battle(s) to see whether or not the model satisfactorily "reproduces" the historical outcome(s),

and (A2) find regularities or "patterns" in historical battle data, and then determine whether or not the model exhibits a similar "pattern."

The first approach has generally involved large-scale detailed models and large-scale aggregated data (e.g. see FAIN et al. [54], and one can raise serious objections about its scientific validity (see below). The second approach has generally involved large-scale aggregated models and large-scale aggregated data and has by and large only considered the classic constant-coefficient LANCHESTER-type equations for modern warfare, with rather mixed results being reported (see below for further details). To this author, the general consensus seems to be that such a simple functional form is not violently contradicted by the available combat data but that the consequent model predictions are statistically too inaccurate for practical use [77] (cf. McQuie et al. [110, p. 93]). A careful review and integration of such past work is lacking and seems to be in order before plowing any new ground.

Now that we have established the contextual setting for the historical validation of combat models, let us consider a few particulars. A number of studies (see Table 7.XII) have considered verification of very simple LANCHESTER-type models, i.e. LANCHESTER's classic formulations (2.2.1) and (2.4.1) and simple variations thereof. In Table 7.XII we give the authors' names and publication date of every empirical-verification examination appearing in the open literature and known to the author. The exact reference to each piece of work may be obtained by consulting the list of references at the end of this chapter. All this work has considered secondary sources and combat data, i.e. data available from other sources such as history books. Usually considering only initial and final strengths in numbers, it has generated results that at best may be called inconclusive. This result is not too surprising, since "aggregated" forces were considered without any type of "scoring" (i.e. weighting) of the various different weapon-system types comprising the opposing heterogeneous forces.

Positive results (i.e. reports of theoretical consequences not at variance with the available combat data) have been reported by ENGEL [52], WEISS [158; 160], HELMBOLD [74-76; 79-80], SCHMIEMAN [126], BUSSE [27], and SAMZ [124]. For example, WEISS [158] reports that there is some justification for using LANCHESTER-type equations of modern warfare (2.2.1) "as a point of departure" in modelling combat. On the other hand, after a rather lengthy and comprehensive analysis, WILLARD [162, p. 4] concluded that his analysis did not justify the use of LANCHESTER's classic equations (2.2.1) and (2.4.1) for modelling large-scale combat. This conclusion is not at all surprising, since heterogeneous forces were aggregated on the

TABLE 7.XII. Authors Who Have Investigated the Empirical Verification of LANCHESTER-Type Models of Warfare.

J. H. ENGEL (1954)

H. K. WEISS (1957, 1966)

R. L. HELMBOLD[†] (1961a, 1961b, 1964a, 1964b, 1969, 1971a, 1971b)

D. WILLARD (1962)

W. A. SCHMIEMAN (1967)

W. W. FAIN, J. B. FAIN, L. FELDMAN and S. SIMON (1970)

J. J. BUSSE (1971)

R. W. SAMZ (1972)

J. B. FAIN (1977)

[†] Here HELMBOLD (1961b) = the second paper published by HELMBOLD in 1961 (see list of references at the end of this chapter).

basis of numbers alone without any "scoring" of the various different weapon-system types. Moreover, when such "scoring" is used, much more positive results have been reported (see FAIN [53, pp. 38-39]).

As we have previously discussed above, HELMBOLD [80, pp. 1-3] has emphasized that there are only the two general approaches (A1) and (A2) for verifying combat models: (A1) the approach of "replaying" some particular battle(s), and (A2) the approach of looking for regularities, or "patterns," in the historical battle data. The usual difficulty with the first appraoch (A1) is that insufficient data is available on any one historical battle to carry out the proposed comparison (see HELMBOLD [80]; also McQUIE [109]). Even when sufficient data is available, rather restrictive assumptions must be made about the conduct of battle, and critical appraisal of these assumptions leads one to raise serious objections about generalizations based on such an examination (see HELMBOLD [80, pp. 1-2] for further details). The work by ENGEL [52], FAIN et al. [54], BUSSE [27], and SAMZ [124](see also BOULTON et al. [19]) falls into this first category (A1), while that by WEISS [158; 160], HELMBOLD [74-77; 79-81], and SCHMIEMAN [126]falls into the second category (A2). This second approach (A2) is nothing more than the Scientific Method of verifying a model indirectly through checking testable consequesces against observations, the so-called hypothetico-deductive method (see MORRIS [113,pp. 101-103]).

ENGEL's work [52] gets more attention from the uninitiated than it probably should. Its weakness is that he estimated parameters and also tested the model with the same set of data and forced a fit through the initial and final force levels for the battle of

Iwo Jima. In fact, all such attempts at model verification by method (M1), i.e. historical "replay," suffer from such deficiencies (see HELMBOLD [80, pp. 1-2] for a further discussion). On the other hand, HELMBOLD's work [74-77; 79-81] has been much more comprehensive. He has sought to indirectly test LANCHESTER-type combat models against the available historical data by empirically examining the testable consequences of such models (see Footnote 40 of Chapter 2 for further details). He has applied this approach not only to ground battles [74-76] but also to air battles [80] and has reported positive results concerning the validity of LANCHESTER-type combat models. More recently, he [81] has examined the validity of "breakpoint-type" hypotheses (see Chapter 3) and found that "the breakpoint hypothesis yields theoretical implications that are at variance with the available battle termination data in several essential respects."

On the other hand, T. N. DUPUY [83-86] has examined combat data from primary sources and has in some sense shown the validity of the firepower-score approach (see also [69]). His work apparently is the original empirical basis for both the ATLAS and also TBM (see [164]) casualty-rate curves. Subsequently, J. FAIN [53] has analyzed HERO (Hisotrical Evaluation and Research Organization) World War II data on 60 engagements in four major Italian campaigns and has reported positive results concerning the scientific validity of LANCHESTER-type models of warfare (particularly when a "scoring" system is used to aggregate the heterogeneous forces). She [53, p. 34] has emphasized that the HERO data (of which she examined only a small part) is the most nearly complete and accurate collection of combat data. Most recently, DUPUY [49] has published a book Numbers, Predictions

*and War*, which may be considered to be the culmination of about fifteen years of historical research by DUPUY and his associates at HERO and makes their work available to the general public. Much more work should be done in this area. It is encouraging that today HERO offers a "Combat Data Subscription Service"[73] and a journal entitled *History, Numbers, and War*.

7.23. The Complexity Crisis.

It appears that the trend for the future is for the development and use of more detailed and complex combat models. This trend has, however, caused an unanticipated result: it has created a complexity crisis. In fact, this complexity crisis was even the theme of the U. S. Army's Fifteenth Annual Operations Research Symposium held in 1976 (see HARDISON [71]). The complexity crisis has manifested itself in several significant and far-reaching ways such as the inability of various DoD agencies to use their complicated computer-based models to their maximum potential, or by the inability of military OR analysts to communicate model methodology (and hence the quality of study-generated information) to decision makers[74]. This communication problem is especially acute because of the high degree of labor differentiation and specialization in DoD analysis activities (e.g. KAPPER [97] identifies the following different participants: users, designers, developers, producers, and managers of models and data bases, and decision/policy makers; see also BREWER and SHUBIK [24].

The operational combat models that we have mentioned in Sections 7.9 and 7.17 above are very complex, particularly detailed models. Such complicated combat models must be implemented on a digitial computer, and without the modern high-speed large-scale digitial computer they would be impossible. Consequently, detailed combat models (not only the Lanchester-type ones we have discussed above but also high-resolution Monte Carlo simulations) are quite costly to build, costly to run, and generate quite demanding data-base requirements (see [9] for further details). In other words, such complicated operational combat models are rather demanding in resources (especially highly technically qualified peopole to maintain, exercise, and modify them).

In fact, just evaluation[75] of such complex models is a significant and by no means completely solved problem (e.g. <u>see</u> GASS [62]). Additionally, the complexity of a model limits one's ability to conduct useful sensitivity and other parametric analyses. Thus, there is a definite price to pay for complexity, and those who demand more detail are frequently not willing to pay the price for it (e.g. <u>see</u> the discussion by BONDER [14]).

How should one go about resolving this complexity crisis? This is a very difficult and subtle question that is far beyond the scope of our modest efforts here. If the reader has become aware that more detail is not always better, that too much detail can cause a problem, and that serious thought should be devoted to this problem, then this section has achieved its goal. Now that the modelling community has proven that it can build very detailed and complicated combat models, how should it manage their use? This is not purely a technical question, but one with organizational, professional, managerial, and sociological aspects (<u>cf</u>. STOCKFISCH [135; 136], BREWER [22], and BREWER and SHUBIK [24].

The hierarchical-modelling approach (<u>see</u> Section 7.20) may be thought of as one possible way to overcome the complexity crisis: a detailed model is used to support a more aggregated model. Along the same lines, a colleague of the author[76] has suggested that the complex model should be used to educate the analyst, while a simple model should be used to communicate with the decision maker. In other words, complex combat models should be used as research tools to determine basic relations that can be presented to decision makers with simple, transparent, easily-understood models. The detailed combat model could be used as a device for developing confidence in the ability of the simple model

to reflect the same trends as the complex one and consequently for giving credibility to the simple model. In this context the complex model serves as the "back-up" for the simple model[77]. The reader will, of course, recognize this approach as being essentially the coordinated use of the large-scale complex operational model with a simple auxiliary model (<u>see</u> Section 7.1 above; also IGNALL, KOLESAR, and WALKER [96] for a lucid discussion not in a defense context). It should be clear to the reader that more work on such modelling strategies for large-scale systems is desperately needed.

FOOTNOTES FOR CHAPTER 7

1. As the author's colleague Professor C. J. ANCKER of the University of Southern California has pointed out, it is not generally true that a so-called mean-value model (obtained by replacing a random variable in a stochastic model with its mean value) yields a good approximation to the mean value of the corresponding stochastic process. However, the results of Section 4.16 indicate that if the initial force levels are "not small" and the forces are "not near parity," a deterministic LANCHESTER-type combat model may be considered to approximately yield the mean course of combat in the sense that it yields very nearly the same expected values for the force levels as does the corresponding continuous-parameter MARKOV chain (see Section 4.2) for the same values of model inputs. Thus, in this very special case of exponentially-distributed times between casualties, such a deterministic LANCHESTER-type model may indeed be considered to yield the mean course of combat (see Section 4.16 for further details). In other cases (e.g. some other distribution for the times between casualties), however, this is not always true. Thus, without the appropriate qualifications being observed, it is simply not true that such a deterministic model invariably yields the same results for the mean course of combat as do corresponding stochastic attrition models (e.g. a Monte Carlo simulation). Hopefully, we will see further clarification of this important point in the literature in the future.

2. The reverse process of starting with a simple model and then elaborating upon it and enriching it in details is, of course, the approach usually used by model developers to build their models. See W. T. MORRIS [114] for a lucid discussion of this enrichment process. It is discussed later in this section.

3. Our discussion here follows that in TAYLOR [143], where these ideas were apparently first articulated.

4. GEOFFRION [65] has suggested a similar conceptual approach of using a simple auxiliary model to generate tentative hypotheses to be tested in a full-scale operational model and thus to provide guidance for further (computerized) higher-resolution investigations. We also have felt (see TAYLOR [140]) that the use of relatively simple auxiliary models in conjunction with complex operational models has much to offer for the analysis of military operations (see also NOLAN and SOVEREIGN [120] and WEISS [159]).

5. Documentation about these models has been discussed in Chapter 1 (see Footnote 23 of Chapter 1). For the reader's easy reference, however, let us point out that information about ATLAS may be found in KERLIN and COLE [98] or [64]. Also, information about BONDER/IUA and its various derivative models may be found in [9; 15-16; 72; 153], while that about VECTOR-2 may be found in [39] (for VECTOR-1, see [154] or [117]).

6. See Footnote 1 above. Further information about the comparison of deterministic and stochastic LANCHESTER-type models (in particular, about the comparison of a deterministic force-level trajectory with the mean course of combat for a corresponding MARKOV-chain model) is to be found in Section 4.16.

7. VECTOR-2 promises [155] detailed representation of the $C^3$ process, combat intelligence, and further refinements in target acquisition (see [39] for the final product). These processes were apparently not modelled in detail in VECTOR-1 (see [117; 154]) but require user-supplied tactical decision rules for their representation. Also, see TIEDE and LEAKE [148] for some related ideas concerning the modelling of tactical information systems.

8. The command and control system tries to avoid wasting fire by engaging killed targets or false ones. The uniform distribution of fire over surviving enemy targets reflects this mission.

9. Thus, the target-acquisition, allocation, and attrition processes are represented by analytical submodels, while movement (which causes changes in the positions of weapons) is represented in a simulatory manner. Bonder [13]has consequently referred to a model like BONDER/IUA or one of its many derivatives as a hybrid analytical-simulation model.

10. This is the approach apparently taken in AMSWAG (a derivative of BONDER/IUA) [72]. A more sophisticated approach would be to also modify the appropriate LANCHESTER attrition-rate coefficients to reflect decreased vulnerability of suppressed combatants.

11. The firepower-score approach has been briefly discussed in Chapter 1, and we will discuss it further in this chapter. Indices of the relative combat capabilities of military units (based on a "scoring system" for the weapons employed in the units) have been used by military gamers and force planners

in the United States for at least thirty five years. We are here generically referring to both such indices and the associated scores as firepower scores. (see Section 1.3, STOCKFISCH [135, pp. 7-9], and Section 7.11 below for a discussion of the difference in meaning between the words score and index as generally used in defense analyses). Members of this family of scores and indices are firepower score/index of combat effectiveness (FS/ICE), firepower potential/unit firepower potential (FP/UFP), firepower potential score/index of firepower potential (FPS/IFP), weapon effectiveness index/weighted unit value (WEI/WUV), weapon effectiveness value/unit effectiveness value (WEV/UEV), antipotential potential, etc. (see STOCKFISCH [135] for further references and a guide to the literature about firepower scores; also see HONIG et al. [90, Appendix C to Chapter II] and HOLTER [89]). When two names (separated by a "slash") are given above, the first name (e.g. FS) denotes the scoring system for weapon-system types, while the second (e.g. ICE) identifies the index number for a unit's capability. The firepower-score approach has also been used in NATO countries (e.g. see WOLF [163], HUBER et al. [94], or DARE [42]).

12. We are calling both differential-equation and also difference-equation models LANCHESTER-type models. In practice, all operational models of combat systems of any degree of complexity use finite-difference methods for computation and thus are really difference-equation models. However, for purposes of model building, it is much more convenient to think in terms of differential equations.

13. Again (also see Footnote 5 above), most of these models have been discussed in Chapter 1 (see Footnotes 17 and 23 of Chapter 1). However, information about T3M-68 (as well as a discussion of the concept of a theater-level "quick game") may be found in [164].

14. DEITCHMAN's [44] analysis neglected many important factors of guerrilla-counterguerrilla operations (particularly the effect of the attitude and support of the local population, for which the two sides must contend by political, economic, and psychological as well as military means). However, such factors may be represented in the model's parameters (e.g. fighting effectiveness or size of the group). Also, they might be expressed in probabilistic terms, but DEITCHMAN did not consider this aspect (see KISI and HIROSE [103] for an examination of the probability of winning for the MARKOV-chain analogue of DEITCHMAN's ambust model).

15. Thus, DEITCHMAN's [44] model is purely deterministic. Stochastic aspects have been investigated by KISE and HIROSE [103], who considered the MARKOV-chain version of DEITCHMAN's ambush model and determined expressions (both exact and a POISSON approximation) for the probability of winning a fixed-force-level-breakpoint battle.

16. The concept of phases of insurgency is apparently due to MAO TSE-TUNG (see SCHAFFER [125,p. 458]). There are three such phases, with Phase III being traditional national warfare. The first two phases of insurgency are characterized by small-force ground-yielding operations by the insurgents but overall military superiority of the counterinsurgents. During Phase II the insurgents' operations escalate in military character but remain basically small-force guerrilla activities designed to cause the defense to fragment (i.e. the engagements are localized and relatively isolated). During Phase III the insurgents take the strategic offensive and operate with larger, more conventional forces in more traditional military ways (see also [70] or [108]).

17. Thus, one obtains valuable guidance for selecting numerical values for the coefficients in (7.6.2): pick larger values for the coefficients $p$ and $q$ corresponding to troops that are poorer in motivation and discipline.

18. To determine whether or not the solution to a particular differential equation is expressible in terms of "elementary" functions is a very difficult advanced-mathematical task (<u>see</u> Footnote 5 of Chapter 6 for a further discussion). Here all we mean is that (based on our mathematical experience and intuition) we feel that the statement is very likely to be true.

19. Here we mean "primary" (as opposed to "supporting") weapons system. The reader may think of a force composed entirely of primary weapon systems as being infantry (<u>see</u> WEISS [159, p. 180] for further details).

20. SCHAFFER [125, p. 470] stated that (7.6.8) holds approximately if $v_U a_{L_U} \lesssim 0.2A_Y$ and that a "more exact formula accounting for overlapping effects would be"

$$S_c = \{1 - (1 - a_{L_U}/A_Y)^{v_U}\} y/T_v ,$$

where $T_v$ denotes the "time it takes to fire $v_U$ rounds." SCHAFFER also gave a more precise definition of $a_{L_U}$.

21. Essentially all complex operational LANCHESTER-type combat models that represent engagements in detail (i.e. do not aggregate forces with fire-power scores) and are in current operational use in the United States have been developed by the principals of Vector Research, Inc. The discussion here follows that of BONDER and FARRELL [15, pp. 11-17].

22. The value of such an allocation factor may, of course, change during an engagement, and thus we should denote it as being a function of time, e.g. $\psi_{ij} = \psi_{ij}(t)$.

23. We are justified in doing so because each of the variables upon which such an attrition-rate coefficient directly depends (see Section 5.11) may be considered to be a function of time. Hence, it is possible to explicitly determine the value of such an attrition-rate coefficient as a function of time (cf. Section 6.2).

24. Actually, the results were apparently obtained by others and summarized by SNOW [133, p. iii].

25. Documentation about these models have been discussed above in Footnote 5 (see also Footnote 23 of Chapter 1, BOSTWICK et al. [18], CORDESMAN [40], and FARRELL [55]).

26. Here we mean a model that represents some of the complexities of actual combat operations. Such a model may be used to address operational problems.

27. Our discussion here follows that of BONDER and FARRELL [15, pp. 11-12].

28. Military planners have apparently used the firepower-score approach (see below in the main text) for at least thirty years (see MULHOLLAND and SPECHT [116] to plan operations and to plan and control tactical exercises. Although the origins of using firepower scores for these purposes are somewhat obscure, they are still in use today (see the U. S. Army's field manual FM 105-5 [73]). Furthermore, it appears as though such use of firepower scores in planning was the origin of their use by OR workers for modelling large-scale ground combat.

29. Examples of such scores/indices are given in Footnote 11 above. BODE [10] has given an excellent discussion of the use of such index numbers in general-purpose force analysis, while ALDRICH and BODE [1] have given a lucid discussion of the conceptual problems of aggregation in theater-level combat models.

30. The one exception is the antipotential potential or WEV/UEV (see Footnote 11 above, HOWES and THRALL [92], and ANDERSON [3-4]; see also Section 7.18), which may be exercised in the running of IDAGAM (see ANDERSON et al. [6]). ATLAS and other models that employ the firepower-score approach have, however, been in the recent past much more widely used in the United States than IDAGAM (see [9]).

31. Our discussion here follows that already given in Section 1.3, but we have repeated part of it here in order to give the reader a complete and unified overview of the topic of aggregation of forces.

32. Many times the first assessment (i.e. determination of engagement outcome) is omitted. For example, ATLAS and IDAGAM only do the last two assessments. However, some models (e.g. Theater Battle Model (TBM-68) [164]) determine the outcome of an engagement (e.g. whether or not an attack is successful) before assessing casualties. In this case, the casualty-assessment curves depend on the engagement's outcome (see Figures 4 through 7 of [164]).

33. For a slightly different discussion of the developments of this section, see TAYLOR [142].

34. Examples of such casualty-rate curves may be found in the documentation for the following large-scale ground-combat models (see also Footnote 5 above): ATLAS [18; 98]; CEM [25; 106], TBM-68 [164] and TAGS [50-51]. See HONIG et al. [90] for a general discussion about such large-scale models (but for the period before 1971). Although IDAGAM does not use firepower scores (see Footnote 11 above), it uses the same casualty-rate curves as ATLAS (see [6, p. 53]). In fact, it is stated on p. 53 of [6] that until better historical data is available, the standard functional relationships (used in ATLAS) between force ratios and percent casualties must still be used. Finally, models used for NATO planning also employ the firepower-score approach and similar casualty-rate curves (e.g. see [94, pp. 287-298]).

32. See Footnote 32 and also Footnote 5.

33. For example, as shown in Figure 7.14, ATLAS [64] distinguishes between seven different types of engagements.

37. Subsequent research by the author (see TAYLOR [144]) has shown this assumption to be necessary. It was not orignally given by TAYLOR and PARRY [146] (see also Sections 6.6 and 6.13 above).

38. Here, again, force ratio means the ratio of firepower indices (A/D).

39. In CEM [25, p. 21; 106, p. 35], for example, the type of engagement is determined by the missions of opposing forces and, where appropriate, the type of defensive position. In this fashion the tactical decisions (i.e. mission assignments) of commanders influence FEBA movement through the determination of engagement type (see the last paragraph of Section 7.12 for further details).

40. Rates of advance for simulated large-scale ground-combat operations are usually given as tables or curves (e.g. see [25; 46; 64; 90; 106; 164] WAINSTEIN [156-157] and not as mathematical relations. See EMERSON [51], however, for some other functional relations. An excellent survey of rate-of-advance modelling (with some European perspectives) is to be found in GOAD [68].

41. See also TAYLOR [142].

42. With the exception of that for TACWAR [100] (formerly called TACNUC [102]) (also see KERLIN et al. [101]), references to documentation about these models has already been given in Footnotes 5, 13, and 34 above (see also Footnote 23 of Chapter 1).

43. It should be emphasized to the reader here that we are generically using the term firepower-score approach to refer to any one of a family of index-number approaches for determining the value (or score) of an individual weapon-system type and then the combat capability (or value) of the military unit employing them (see Footnote 11 above). A simple linear model is used to aggregate the firepower capabilities of all the different weapons in the unit (recall the example given in Table 1.II).

44. In IDAGAM [6] (see also SHUPACK [130]) tactical decisions such as allocation and movement of reserve divisions, to attack (or not) and where, and withdrawal of divisions from a sector are handled by the theater-control model. Force ratios (based on some type of scoring for weapon-system types) are one of several factors considered in algorithms modelling these tactical decisions. Moreover, there are a number of different options (in all 13) available to the user of IDAGAM (see SHUPACK [130, pp. 86-97]), all but one of which use force ratios to scale the magnitude of combat losses. It is therefore possible to use LANCHESTER-type equations by themselves without any such scaling (i.e. use the 13th attrition option) to model combat losses and use force ratios only for modelling tactical decisions. The CEM model [25; 106] does something similar in not using force ratios for the assessment of casualties but using them only in the modelling of tactical decisions. Thus, the possibility exists of using a detailed (e.g. LANCHESTER-type) model of attrition in conjunction with a tactical-decision model that uses force ratios. It is interesting to note that the need for some aggregation method for quantifying the military

capability of fighting units for use in a tactical-decision algorithm in a closed (i.e. no human intervention) model of large-scale combat operations is never mentioned by critics of the firepower-score approach.

45. This idea was apparently independently proposed by SPUDICH [134], DARE and JAMES [43], and HOWES and THRALL [91] (see also [92]). Early work was done by ANDERSON [2] (see also [5]) and HOLTER [89]. Some further references to work done by U. S. Army analysts is to be found in [149]. See also ANDERSON [3; 4] for ome further background material and references.

46. Here we are using the term LANCHESTER attrition-rate coefficient in its broadest sense to denote the kill rate of a single weapon-system type against a particular enemy weapon-system type. Consequently, no assumption at all is being made here that any LANCHESTER-type model be used or even represents the attrition for such an engagement. For example, in several U. S. Army studies [89; 149, Chapter 30] the Division Battle Model (DBM) was used to generate the "casualty data" from which single-system kill rates were computed. In other cases, detailed Monte Carlo combat simulations have been used to generate "killer-victim scoreboard" (i.e. a matrix whose elements show how many of each weapon-system type were destroyed in a battle by each weapon-system type on the opposing side) in so-called weapon-equivalence studies (see [149, Chapter 30] for further details). Further information on approaches for determining single-system kill rates is to be found in Chapter 5.

47. IDAGAM is a theater-level combat model that is widely used in the United States and elsewhere (<u>see</u> Section 7.17). It is one of the major models of theater-level combat and is principally used at the joint-service level of studies and analyses.

48. Some alternative hypotheses for imputing values to weapon-system types are discussed in HOWES and THRALL [91; 92]. These authors, however, recommend the one we have given here.

49. For notational convenience, we have denoted here as $a_{ij}$ an attrition-rate coefficient that includes the effects of the fire-allocation process and that we have denoted above as $A_{ij}$. Thus, the reader should bear in mind that such an attrition-rate coefficient as $a_{ij}$ changes when the distribution of fire by a $Y_j$ firer type changes.

50. For further information and background about the PERRON-FROBENIUS theorem, which goes back to results of PERRON [121] and FROBENIUS [60], <u>see</u> GANTMACHER [61, Chapter 13], VARGA [152, Chapter 2], and SENETA [128].

51. Here we have used the term "<u>admissible</u>", since we must limit ourselves to those transformations of scale that preserve the fundamental requirement that $\underset{\sim}{s}_X$ and $\underset{\sim}{s}_Y$ must always be nonnegative. Henceforth we will omit the word "admissible" when referring to such transformations of scale, but the reader should keep the above restriction in mind.

52. All modern computer centers have "canned" algorithmic routines available for numerically solving such eigenvalue problems and determining the eigenvector associated with a particular eigenvalue.

53. Such apparent antimonies as discussed here are, unfortunately, inherent to this linear model for imputing values to weapon-system types. However, the choice of scaling method evidently does influence which particular cases will be plagued by such apparently anomalous behavior. Furthermore (and more importantly), we show in Section 7.19 that such antimonies are more apparent than real.

54. In the 2 × 2 case (<u>see</u> Example 7.18.2), one must have $(a_{11}b_{11} + a_{12}b_{21}) > (a_{21}b_{12} + a_{22}b_{21})$ in order that $s_2^X$ be defined when $a_{21}b_{11} + a_{22}b_{21} = 0$.

55. In real-world studies, the time and resources available invariably dictate whether or not a detailed model can be used. A detailed model like VECTOR-2 requires approximately five to ten times the number of data inputs as does an aggregated model like IDAGAM (see [150, p. 53]). Even a relatively simple theater-level model as ATLAS requires a fair amount of resources just to be prepared for a new set of production runs: it requires 2-4 months to acquire a fresh data base and 1 man-month to structure this data in the model's input format [9, p. 38]. A detailed model like VECTOR-2 requires infinitely more time for the preparation of inputs. Thus, there is a need for theater-level models that are fast running (including data-base preparation) and easily modified, i.e.

so-called "quick games" (see [164]). It has always struck this author as being rather unfair to criticize ATLAS because it is a relatively simple model that does not demand a lot of time and resources to be run. Such critics appear to have forgotten that ATLAS was developed as a "quick-game" model (see [164]) (it evolved out of a model called computerized QUICK GAME [99] (see also LOW [107, Appendix D])) and that it was not developed for detailed investigations of theater-level combat [98, p. 5].

56. STOCKFISCH [135, p. 6] has used the term immaturity to denote the state of affairs in which the phenomenological bases of the field are not well established. In such a field (as combat or conflict analysis), epistemological questions abound (often in the guise of questions about methodology) because the correspondence between the real world and the model world has not been irrefutably established. This situation should be contrasted to that for classical physics in which (within their realm of applicability) physical laws are so well established that one does not suggest the use of alternative paradigms (i.e. questions about methodology do not arise). STRAUCH [138, pp. 13-15] has pointed out that in an immature field like defense analysis the application of quantitative methodology to a problem (denoted by him as a squishy problem) differs fundamentally from that for a rigorously quantifiable problem in a mature field because the analyst must exercise judgment (see, in particular, [138, p. xiii]) to abstract a formal problem and attendant mathematical model from an ill-defined problem regarding phenomena not well understood (see also [138, pp. 3-20]).

57. Besides being difficult and costly just to maintain and run, complex models are particularly difficult to evaluate (<u>see</u> GASS [62] (also [63]) for a further discussion), especially when documentation is lacking (<u>see</u> [150, pp. 25-31] for a particularly lucid discussion of documentation and other related management problems). As we have already noted many times, documentation is a particular problem for combat models (<u>see</u> SZYMCZAK [139] for not only a lucid discussion of problems within the defense-analysis community but also some interesting suggestions for improving current documentation practices).

58. <u>See</u> Footnote 43 above.

59. However, there is far from universal agreement concerning many of the details of FARRELL's investigation (e.g. <u>see</u> ANDERSON [5, pp. vii-viii]).

60. The statements made here are based on further investigations that time and space do not permit us to document in complete detail.

61. ANDERSON [5, p. viii] has pointed out that (if desired) there are straightforward ways of preventing such behavior, for example, with the antipotential-potential method in IDAGAM (<u>see</u> SHUPACK [130] for further details).

62. The author would like to thank his colleague G. OWEN for exposing him to the literature of paradoxes of rationality. Professor OWEN has emphasized that the occurrence of such paradoxes did not result in

researchers abandoning trying to apply game theory to problems of rational behavior but instead provided rationale for further (more sophisticated) analysis. He has added that what appears to be a beginner as a paradox invariably appears to the seasoned game theorist as perfectly intuitively obvious behavior.

63. Frequently, such models are challenged because they are too simple, but any experienced modeller can take the basic paradigm and build a more complicated model through the process of model enrichment (see Section 7.1 above and MORRIS [114] for further details).

64. It is interesting to note that determination of whether such a model is "convincing" or "credible" is apparently based on logical grounds and not based on testing against any empirical data. In Section 7.22 we will discuss the problem of historical validation of combat models. To the best of this author's knowledge, no detailed combat model has ever been validated against historical data, essentially because of the quality of available historical combat data (see Section 7.22, McQUIE et al. [110], McQUIE [109], and/or HUBER, LOW, TAYLOR [95] for further details).

65. A recent U. S. General Accounting Office (GAO) [150] study has emphasized that empirical study is necessary to strengthen the scientific foundation and objectivity of defense decision making (see also BREWER and HALL [23], STRAUCH [138], STOCKFISCH [135; 136], and BREWER and SHUBIK [24]).

66. See HELMBOLD [77], McQUIE et al. [110], and McQUIE [109] for discussions of the limited availability of historical combat data. HELMBOLD discusses the nature of data available from secondary sources (e.g. history books), while McQUIE [109] (see also [110]) discusses the nature of data available from primary sources (e.g. unit reports and official miltiary histories). Additionally, McQUIE discusses the shortcomings of the historical combat data that does exist. He provides an outstanding discussion of the nature, availability, and quality of historical data.

67. We are here using the words "verification" and "validation" interchangeably. Many authors distinguish between the verification and the validation of a model, but there is apparently no consistent use of these terms in the literature (see, for example, MORRIS [113], FISHMAN and KIVIAT [58], BONDER [11, pp. 68-70], VAN HORN [151], and NAYLOR and FINGER [118]). For our present purposes, however, such a distinction does not seem warranted, especially since there is not consistent use of these terms in the literature.

68. Unfortunately, this unique service had to be terminated after only two volumes of (quarterly) publication, apparently due to lack of support.

69. Simulated combat data of one form or another exists on essentially all levels, particularly the lower levels (i.e. few-on-few and below).

70. BONDER [13] has considered models of different combat processes at three different levels: (1) individual firer against a passive target, (2) small-unit combat (battalion and below), and (3) large-scale combat.

He has discussed the verification of models at these three system levels. Based on our knowledge of the available combat data, such verification can only pertain to simulated (and <u>not</u> real) combat data (here some type of field experimentation), but this fact is not explicitly pointed out to the reader. No references are given by BONDER [13].

71. Some notable exceptions have been the HERO ORALFORE study [85] and work by COCKRELL [36], GOAD [67] (<u>see</u> also [68]), and GRAVES [69], which have investigated historical FEBA movement (<u>see</u> also [46; 49]). Again, large-unit operations were considered.

72. Our discussion here follows HELMBOLD [80, pp. 1-3]. There are, of course, other positions that one can take concerning the verification of models (<u>see</u>, especially, NAYLOR and FINGER [118]). In the main text we have presented the two that are most germane to combat models.

73. <u>See</u> Footnote 68 above.

74. The author would like to thank LTC Richard S. Miller, U. S. Army, of the Naval Postgraduate School for many of the ideas discussed here, as well as elsewhere in this section. The author is, of course, solely responsible for the views expressed here.

75. GASS [62] (see also [63]) has considered the evaluation of computerized complex models to consist of the interrelated tasks of model verification and validation. Here, verification is taken to mean the attempt to ensure that a model behaves as the analysts (i.e. model formulators and computer programmers) intended, while validation is the testing of the agreement between the behavior of the model and the real-world system being modelled (see FISHMAN and KIVIAT [58]; see also, however, Footnote 67 above). As we have indicated above in Section 7.22, the validation of even simple combat models against historical data is a particularly difficult task.

76. LTC Richard S. Miller, U. S. Army, of the Naval Postgraduate School (see also Footnote 74 above).

77. For examples of the actual use of this approach, see NIEMEYER [119], WIEGAND [161], and ASBED [7]. Each of the first two West German authors [119; 161] has briefly discussed one so-called TREND model, which is a structurally rather simple aggregated deterministic simulation model that reproduces results of the more detailed interactive computerized theater-level war game RELACS (see also DARE [41]). ASBED [7] has similarly reported about the development of a relatively simple aggregated model and the comparison of its results with those obtained from IDAGAM (a much more detailed theater-level model).

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## APPENDIX E: FINITE-DIFFERENCE APPROXIMATIONS TO LANCHESTER-TYPE EQUATIONS

### 1. Introduction

As we have seen above in Chapters 6 and 7 (e.g. recall Figure 6.11), it is impossible for all practical purposes to solve analytically the differential equations for any but the most simple LANCHESTER-type combat models. In order for such models to have any practical value, there must be some convenient way to extract from them information that is needed for defense-planning purposes. Moreover, the solving of the differential equations for the dynamics of the force-on-force combat provides force-level information which many times forms the basis for extracting any further desired information from the model. Since analytical methods are usually of no avail in solving these differential equations (at least for models with any degree of operational realism), numerical methods for obtaining approximate results must be resorted to.

Thus, in this appendix we will consider so-called finite-difference methods for developing approximate solutions to LANCHESTER-type differential combat equations. We will see how LANCHESTER-type differential equations may be approximated by so-called difference equations, which can then be conveniently numerically solved by an automated computational procedure implemented on, for example, a modern digital computer. Moreover, the modern high-speed, large-scale computer

has made such recursive solution procedures computationally feasible, and without it current operational models like the BONDER/IUA and its many derivatives or the VECTOR series of models would be impossible. In this appendix, we will focus on the development of simple finite-difference approximations, with the mathematical proof of answers to attendant numerical-analysis questions such as convergence and stability of these approximations being beyond the scope of our examination here. Thus, the reader is referred to the numerical-analysis literature for a complete mathematical justification of the methods presented here (<u>see</u> the last section of this appendix).

## 2. A Simple Finite-Difference Approximation.

Let us consider the following general LANCHESTER-type homogeneous force equation for $t \geq 0$

$$\begin{cases} \dfrac{dx}{dt} = -G(t,x,y) & \text{with} \quad x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -H(t,x,y) & \text{with} \quad y(0) = y_0 , \end{cases} \tag{E.1}$$

where $x(t)$ and $y(t)$ denote the X and Y force levels, $t$ denotes time, and G and H denote force-change rates (which are net loss rates when G and $H \geq 0$). In this section we will show how to generate an approximate solution to (E.1) by first developing a simple finite-difference approximation to these LANCHESTER-type equations. Thus, we

will approximate the system of ordinary differential equations (O.D.E.s) (E.1) by a system of difference equations (i.e. equations that connect the force levels between only discrete points in time), which may then be recursively solved with the help of, for example, a modern automatic digital computer (or even a contemporary programmable hand-held calculator).

For any system of O.D.E.s such as (E.1), time is in essence allowed to vary continuously, i.e. in principle an analytical solution provides us with the X and Y force levels $x(t)$ and $y(t)$ at any desired time $t \geq 0$. For example, the successive-approximation solution (6.5.6), (6.5.16), and (6.5.18) to the F|F LANCHESTER-type equations (6.1.1), i.e. equations (E.1) with $G(t,x,y) = a(t)x$ and $H(t,x,y) = b(t)y$, in principle provides us with $x(t)$ and $y(t)$ at any time $t$ during the course of such a homogeneous-force battle. We will now consider an approach for numerically generating an approximation to the force levels at only discrete points in time.

Thus, we will consider a numerical-solution method that will enable us to generate approximate values for the force levels, but only at discrete points in time (as opposed to a point in time that can vary continuously over the course of the battle). Accordingly, we discretize time by introducing a finite number of so-called mesh points $t_n$ for the fixed interval $[0,T]$

$$t_0 = 0,$$

and (E.2)

$$t_n = t_{n-1} + \Delta t \qquad \text{for } n = 1,2,\ldots, N,$$

where

$$\Delta t = \frac{T}{N} \quad . \tag{E.3}$$

It then follows that (see Figure E.1)

$$t_n = n\Delta t \qquad \text{for} \quad n = 0,1,2,\ldots, N \; . \tag{E.4}$$

The time $t_n$ is commonly referred to as the $n^{\underline{th}}$ time step (i.e. the position of the $n^{\underline{th}}$ step in time), and the increment $\Delta t$ is referred to as the time-step size (here uniform). It is now convenient to introduce the notation

$$\begin{aligned} x(t_n) &= x_n \,, & y(t_n) &= y_n \,, \\ G(t_n,x,y) &= G_n(x,y), & H(t_n,x,y) &= H_n(x,y). \end{aligned} \tag{E.5}$$

The simplest way to generate a discrete-time approximation to the continuous-time equations (E.1) is to recall the definition of a derivative such as $\frac{dx}{dt}(t)$, i.e.

$$\frac{dx}{dt}(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \,, \tag{E.6}$$

and approximate the rate of change of the $X$ force level as

$$\widehat{\frac{dx}{dt}}(t) = \frac{x(t + \Delta t) - x(t)}{\Delta t} \; . \tag{E.7}$$

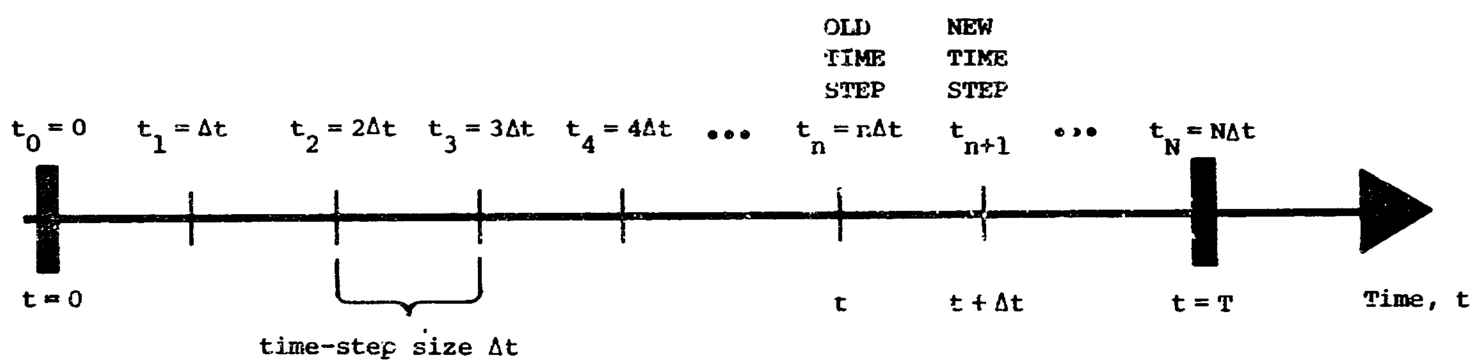


Figure E.1. Uniform computing grid composed of mesh points separated by time increments, each of the same size $\Delta t$. Such a subdivision of an interval like $[0, T]$ is also frequently called a net, lattice, or mesh; and the quantity $\Delta t$ is also referred to as the uniform time-step size. One also commonly refers to the time $t_n$ itself as the $n^{\underline{th}}$ time step (i.e. the position of the $n^{\underline{th}}$ step in time).

Here $\hat{\frac{dx}{dt}}(t)$ denotes an approximation to the value of the derivative $\frac{dx}{dt}$ at time $t$, i.e. the value of the rate of change of the $X$ force level at time $t$. If we use such an approximation for the rate of change of, for example, the $X$ force level at the battle point $(t_n, x_n, y_n)$, we obtain the following equation for an approximate value for the $X$ force level at the $(n+1)^{\underline{st}}$ time step in terms of previously determined approximate force-level values at the $n^{\underline{th}}$ time step

$$\frac{\hat{x}_{n+1} - \hat{x}_n}{\Delta t} = -G_n(\hat{x}_n, \hat{y}_n) \ , \tag{E.8}$$

where $\hat{x}_n$ and $\hat{y}_n$ denote approximate values for the $X$ and $Y$ force levels at time $t_n$, e.g. $\hat{x}_n$ represents an approximate value for $x_n = x(t_n)$. However, it is more convenient to write this latter finite-difference equation as

$$\hat{x}_{n+1} = \hat{x}_n - G_n(\hat{x}_n, \hat{y}_n)\Delta t \ . \tag{E.9}$$

Thus, by applying such approximations to our <u>continuous-time combat model</u> (E.1), we obtain a <u>discrete-time combat model</u> (E.10) for which values of the approximate force levels $\hat{x}_n$ and $\hat{y}_n$ may be generated recursively at a finite number of mesh points $t_n$ for the fixed interval $[0,T]$ by the following formulas for $n = 0,1,2,\ldots,N-1$

$$\begin{cases} \hat{x}_{n+1} = \hat{x}_n - G_n(\hat{x}_n, \hat{y}_n)\Delta t & \text{with } \hat{x}_0 = x_0, \\ \hat{y}_{n+1} = \hat{y}_n - H_n(\hat{x}_n, \hat{y}_n)\Delta t & \text{with } \hat{y}_0 = y_0 \ . \end{cases}$$

In the parlance of numerical analysis, the equations (E.10) are the difference equations of the EULER-CAUCHY method. As we will see below, there are not only several methods for developing such finite-difference approximations, but there are also many other such approximations possible. For convenience, we have assumed a uniform time-step size $\Delta t$, and it is an easy matter to extend these developments to cases of a variable time-step size $\Delta t_n = t_n - t_{n-1}$.

The most significant aspect about the finite-difference equations (E.10) is not that they are any easier to analytically solve than the original differential equations (E.1) (in matter of fact, they are not!) but that they may be recursively solved for any particular numerical values, a procedure that can be easily automated for use on, for example, a high-speed digital computer. As we will see below in Example E.1, automation is (in fact) quite necessary because although such recursive computation may be very straightforward to do, it is very tedious to carry out when it must be repeated a very large number of times. Thus, the approximation (E.10) may be considered to be the basis for a step-by-step solution procedure, which marches the battle results ahead in time: with the approximate force levels $\hat{x}_n$ and $\hat{y}_n$ known at the old time step $n$, equations (E.10) allow one to readily compute approximate values for the force levels $\hat{x}_{n+1}$ and $\hat{y}_{n+1}$ at the new time step $n+1$ and thus to "march ahead in time" (see Figure E.1).

We should now observe that since our original LANCHESTER-type equations (E.1) only hold for $x$ and $y \geq 0$, we must do some precautionary bookkeeping to prevent negative approximate force levels. This is readily done by interpreting (as we should) equations (E.10) as meaning for $n = 0,1,2,\ldots, N-1$

$$\begin{cases} \hat{x}_{n+1} = \max(0,\ \hat{x}_n - G_n(\hat{x}_n,\hat{y}_n)\Delta t) & \text{with } \hat{x}_0 = x_0 , \\ \hat{y}_{n+1} = \max(0,\ \hat{y}_n - H_n(\hat{x}_n,\hat{y}_n)\Delta t) & \text{with } \hat{y}_0 = y_0 . \end{cases} \tag{E.11}$$

For simplicity's sake, we will henceforth write an approximation in the form (E.10) with the understanding that an approximating system in the form (E.11) is meant.

From (E.6) it should be clear that the "goodness" of the approximate force levels $\hat{x}_n$ and $\hat{y}_n$ depends on the time-step size $\Delta t$ in the approximating finite-difference equations (E.10), which converge to the original LANCHESTER-type equations (E.1) as $\Delta t \to 0$. Indeed, it is not surprising that it may be shown (similar to how it is done in texts on the numerical solution to O.D.E.s) that, for example,

$$\lim_{\Delta t \to 0} \max_{0 \le n \le N} |x_n - \hat{x}_n| = 0 . \tag{E.12}$$

Unfortunately, it is a matter of artwork (as opposed to science) to pick a time-step size $\Delta t$ that yields "satisfactory" numerical results for the approximate force levels. Two heuristic methods for determining a satisfactory value for $\Delta t$ are accordingly suggested here:

(M1) compare exact analytical force-level results, i.e. numerical values for $x_n = x(t_n)$ and $y_n = y(t_n)$, for a simplified version of (E.1) (for which such analytical results are conveniently obtained) with the corresponding finite-difference-generated values for the approximate force levels $\hat{x}_n$ and $\hat{y}_n$ in order to find such a satisfactory value for $\Delta t$,

(M2) compare numerical values for $\hat{x}_n$ and $\hat{y}_n$ corresponding to the same value of $t$ but generated by several different mesh widths (or time-step sizes) $\Delta t$ (e.g. $\Delta t = h, h/2, h/4$, etc.) until such values, no longer "change appreciably for variations in $\Delta t$" in order to find such a satisfactory value for $\Delta t$.

Finally. let us note that so-called higher-order (i.e. more complicated and more accurate) approximations are possible (<u>see</u> below for a few brief comments), but we feel that they are not really jusitifed for a combat model such as (E.1) because the model itself is only a very rough approximation to reality. This situation should be contrasted to that in the physical sciences where the differential laws governing physical-system behavior are much more accurately known and in many uses (e.g. a mid-course correction for a space ship on a trip to the moon) must be very closely approximated.

3. <u>Extension to Heterogenous Forces</u>.

The above simple method of finite-difference approximation is both in principle and also in practice readily extended to heterogeneous-force combat. In fact, except for a relatively minor amount of notational complexity, it is essentially no more difficult to generate such numerical results for heterogeneous-force combat than for homogeneous-force combat.

Let us accordingly consider combat between an X force composed of $r$ different weapon-system types (denoted as $X_1, X_2, \ldots, X_r$) and a Y force composed of $s$ weapon-system types (denoted as $Y_1, Y_2, \ldots, Y_s$) (<u>cf</u>. Section 7.7 above). General LANCHESTER-type equations may be formulated for such combat by extension of the homogeneous-force model (E.1) and may be taken for $t \geq 0$ as

$$\begin{cases} \dfrac{dx_i}{dt} = -G_i(t,x_1,\ldots,x_r,y_1,\ldots,y_s) & \text{with} \quad x_i(0) = x_0^i, \\ \dfrac{dy_j}{dt} = -H_j(t,x_1,\ldots,x_r,y_1,\ldots,y_s) & \text{with} \quad y_j(0) = y_0^j, \end{cases} \tag{E.13}$$

where $x_i(t)$ denotes the number of $X_i$ at time t and analogously for $y_j(t)$. Here we have adopted the convention (cf. Section 7.7) that the index i will always take on the integer values 1 through r, and the index j will always take on the integer values 1 through s.

Discretizing time as above (recall Figure E.1) and introducing the notation

$$\begin{gathered} x_i(t_n) = x_n^i , \qquad\qquad y_j(t_n) = y_n^j , \\ G_i(t_n,x_1,\ldots,x_r,y_1,\ldots,y_s) = G_n^i(x_1,\ldots,x_r,y_1,\ldots,y_s) , \\ H_j(t_n,x_1,\ldots,x_r,y_1,\ldots,y_s) = H_n^j(x_1,\ldots,x_r,y_1,\ldots,y_s) , \end{gathered} \tag{E.14}$$

we may again introduce the above simple first-order finite-difference approximations to the force-level derivatives and analogously obtain the following simple approximation to the LANCHESTER-type heterogeneous-force combat equations

$$\begin{cases} \hat{x}_{n+1}^i = \hat{x}_n^i - G_n^i(\hat{x}_n^1,\ldots,\hat{x}_n^r,\hat{y}_n^1,\ldots,\hat{y}_n^s)\Delta t , \\ \hat{y}_{n+1}^j = \hat{y}_n^j - H_n^j(\hat{x}_n^1,\ldots,\hat{x}_n^r,\hat{y}_n^1,\ldots,\hat{y}_n^s)\Delta t , \end{cases} \tag{E.15}$$

with initial conditions

$$\hat{x}_0^i = x_0^i \qquad \text{and} \qquad \hat{y}_0^j = y_0^j ,$$

where $\hat{x}_n^i$ denotes the approximation to the $X_i$ force level $x_n^i = x_i(t_n)$ at $t = t_n$ and similarly for $\hat{y}_n^j$.

It should be clear to the reader that the approximating finite-difference equations (E.15) may again be numerically solved with a simple recursive algorithm. In fact, on a modern large-scale high-speed digital computer, they are essentially no more computationally difficult to solve than the homogeneous-force equations considered in the previous section.

## 4. General Approaches for Developing Finite-Difference Approximations

As we indicated in our examination of the simplest finite-difference approximation (E.10) to the homogeneous-force LANCHESTER-type equations (E.1), there are several approaches for developing such approximations to generate numerical solutions to such differential-equation combat models. In this section we will very briefly consider three basic approaches for developing such finite-difference approximations and will also mention a few specific methods that the reader may encounter elsewhere. In particular, all digital-computer computation centers today provide users with numerical differential-equation-solver routines (i.e. computer routines for the numerical solution of O.D.E.s) as part of their general scientific-computation package. Since a reader who attempts to numerically implement a LANCHESTER-type combat model on the computer is certain to encounter such methods and routines if he consults his computation center for assistance, a few general words about them seem in order. However, as we have discussed above, we suggest that the reader use the EULER-CAUCHY method in such computational work, since it is extremely convenient to implement and possesses

accuracy (crude as it may be) that is consistent with the scientific validity of the original LANCHESTER-type combat model. Of course, if one (for one reason or another) chooses to use one of the many numerical differential-equation-solver routines that are available from one's computation center (which usually supplies such differential-equation solvers to users in the physical sciences), one will undoubtedly wind up using some standard higher-order method, e.g. the so-called classical RUNGE-KUTTA method (see below).

The three general approaches that can be used to develop finite-difference approximations to O.D.E.s may be referred to as methods based on:

(M1) numerical differentiation,

(M2) numerical integration (combined with interpolation of the integrand),

(M3) TAYLOR-series expansion (either directly or indirectly).

We developed the above EULER-CAUCHY approximation (E.10) from the standpoint of numerical differentiation, although one could have equally well used either of the other two approaches (M2) and (M3) (e.g. see HENRICI [4, pp. 9-10]). Principal methods based on the numerical-integration approach are the ADAMS-BASHFORTH method, the ADAMS-MOULTON method, and the generalized MILNE-SIMPSON method (e.g. see HENRICI [4, especially Chapter 5] for further details; see also MILNE [11] and HILDEBRAND [6]); while the principal methods based on TAYLOR-series expansion are those of RUNGE-KUTTA type (e.g. see HENRICI [4, pp. 66-70]), especially the

classical RUNGA-KUTTA method which (next to the EULER-CAUCHY method) is probably the best known of all the so-called one-step methods.

How good is any particular finite-difference approximation? What is the basis for our recommendation that the reader should use the EULER-CAUCHY method? Would another approximation be better? The answers to such questions at least partially rest on certain concepts and results from the mathematical field of numerical analysis. It is beyond the scope of our present examination to provide a complete theoretical answer to these important questions, which are easy to answer but quite difficult to justify these answers (i.e. supply mathematical proofs). Thus, for present purposes we will go into numerical-analysis aspects just far enough to articulate issues and answers. Our goals then are (1) to expose the reader to numerical-approximation methods for O.D.E.s, (2) to suggest a general course of action (i.e. use the EULER-CAUCHY method) for satisfying computational requirements, and (3) to indicate to the reader that there are sound reasons for our suggestion.

How good is any particular finite-difference approximation? Three ways to answer such questions about the validity (or goodness) of a numerical-approximation technique are as follows:

(W1) compare exact and approximate results,

(W2) perform theoretical numerical-analysis investigations,

(W3) do experimental computing.

Similar to the intimate relation between game theory and war gaming (i.e. behavioral model building) (see Section 8.2 below), the theory of numerical analysis provides a fundamentally important methodological approach (i.e. concepts and results) to the study of computational algorithms. Thus the approach (W2) provides a basic framework (i.e. concepts and vocabulary) for pursuing the other two approaches (W1) and (W3), both of which have certain inherent shortcomings. For example, the comparison of exact and approximate results can only serve as a benchmark (i.e. a test in a specific known case), since the exact results are lacking when the approximate results are really needed. We thus turn to the theoretical investigation of the "goodness" of a given finite-difference approximation. A very reasonable criterion to consider in such an investigation is the magnitude of the error involved in using the approximation.

There are, in fact, several types of error involved in the numerical solution of O.D.E.s:

(T1) truncation (or discretization) error,

(T2) roundoff error,

(T3) approximate-solution error,

(T4) total error.

Moreover, the reader should be warned that not all authors define these terms in the same fashion or as we will here. The definitions given here by us for the above various types of errors are more or

less patterned after those of ISAACSON and KELLER [9, Chapter 8]. Let us now for illustrative purposes consider a finite-difference approximation to the homogeneous-force equations (E.1) and examine these various errors more closely. One important reason for doing this is that not all finite-difference methods yield satisfactory results: there do exist approximations with unsatisfactory error properties, and a potential user should be aware of this fact. In our examination here of these errors we will consider definitions and results for only the X force level, with similar results holding for the Y force level.

The local truncation error measures the error by which the exact force levels from (E.1) fail to satisfy the approximating difference equations, and consequently it depends on the finite-difference method used. Thus, when the EULER-CAUCHY method is used, the <u>local truncation error</u> for the X-force-level difference equation $\tau_n^X$ is defined (following ISAACSON and KELLER [9, Chapter 8]) as

$$\tau_n^X = \frac{x_{n+1} - x_n}{\Delta t} + G_n(x_n, y_n) \quad , \tag{E.16}$$

where we recall that $x_n = x(t_n)$ and similarly for $y_n$. If the $\tau_n^X$ vanish as $\Delta t \to 0$, we say that the difference equations are <u>consistent</u> with the differential equation (here for the X force level). Also of interest is how quickly such a limiting value is reached, and this speed of convergence may be expressed in mathematical terms as

$$\tau_n^X = O(h^P) \quad , \tag{E.17}$$

where $h = \Delta t$. Here the notation $f(h) = O(h^p)$ means that $\lim_{h \to 0} f(h)/h^p = c$, where $c$ is a constant independent of $h$, and is read "$f(h)$ is of the order $h^p$." For example, the EULER-CAUCHY method has truncation error $\tau_n^X = O(\Delta t)$ and is consequently called a first-order method. The classical RUNGE-KUTTA method has truncation error of order $(\Delta t)^4$ and is consequently called a higher-order method (here fourth-order).

The roundoff error for the X-force-level difference equation $r_n^X$ is defined as

$$r_n^X = \hat{X}_n - \hat{x}_n , \tag{E.18}$$

where $\hat{x}_n$ denotes the exact solution of the approximating equations and $\hat{X}_n$ denotes the numerical value that is actually calculated by the computing equipment in place of the $\hat{x}_n$. Roundoff errors exist because the number $\hat{x}_n$ cannot be calculated with infinite precision due to the limited accuracy of any computing equipment. The approximate-solution error for the X force level $e_n^X$ is defined as

$$e_n^X = \hat{x}_n - x(t_n) , \tag{E.19}$$

which measures the error made by taking the exact X-force-level solution of the approximating difference equations $\hat{x}_n$ in place of the exact solution to the X-force-level equation $x(t_n)$. For any finite-difference approximation to be any good, we require it to be convergent in the sense that we can make the approximate-solution error arbitrarily small by taking $\Delta t$ small enough, i.e. in analytical terms

$$\lim_{\Delta t \to 0} \max_{0 \leq n \leq N} |e_n^X| = 0 , \tag{E.20}$$

which is equivalent to (E.12) above, and similarly for $e_n^Y = \hat{y}_n - y(t_n)$. For example, the EULER-CAUCHY method is convergent, with an approximate-solution error of order $\Delta t$. The classical RUNGE-KUTTA method is also convergent [with error of order $(\Delta t)^4$], while the so-called MILNE method is not (there are differential equations for which spurious numerical approximations may be obtained). Finally, the <u>total error</u> in the approximate value for the X force level $E_n^X$ is defined as (<u>see</u> ISAACSON and KELLER [9, pp. 374-377] for a detailed analysis of the total error of the EULER-CAUCHY method)

$$E_n^X = \hat{X}_n - x(t_n) = e_n^X + r_n^X . \tag{E.21}$$

A word of caution, however, is in order on the indiscriminate use of higher-order finite-difference-approximation methods. Consider, for example, the single differential equation

$$\frac{dx}{dt} = -x \qquad \text{with} \quad x(0) = x_0 , \tag{E.22}$$

and approximate the derivative by the central-difference formula

$$\widehat{\frac{dx}{dt}}(t_n) = \frac{x_{n+1} - x_{n-1}}{2\Delta t}$$

to obtain the finite-difference approximation

$$\frac{\hat{x}_{n+1} - \hat{x}_{n-1}}{2\Delta t} = -\hat{x}_n ,$$

or

$$\hat{x}_{n+1} = -2\hat{x}_n \Delta t + \hat{x}_{n-1} \qquad \text{with} \quad \hat{x}_0 = x_0 , \tag{E.23}$$

which is well-known (e.g. <u>see</u> HILDEBRAND [7, pp. 132-133]) to have truncation error of order $(\Delta t)^2$. Nevertheless, although the truncation error for (E.23) is of higher order than for the EULER-CAUCHY approximation $\hat{x}_{n+1} = (1 - 2\Delta t)\hat{x}_n$, this finite-difference approximation is not convergent and hence is not satisfactory (<u>see</u> HENRICI [4, pp. 240-241] or HILDEBRAND [7, pp. 132-135]). The O.D.E. (E.22) has the unique exact solution $x(t) = x_0 e^{-t}$, while the finite-difference approximation (E.23) (being a second-order difference equation) possesses a general solution made up of two linearly-independent components, one of which behaves sort of like $e^{-t}$ for small values of $\Delta t$ but the other of which behaves like $e^t$. Consequently, the finite-difference equation (E.23) possesses an extra "spurious" (also "extraneous" or "parasitic") solution that has growth properties contrary to those of the exact solution to the differential equation (E.22) and hence will spoil the numerically computed values $\hat{X}_n$ (<u>see</u> HENRICI [4, pp. 240-241] for further details).

Without going into mathematical details here (e.g. <u>see</u> HILDEBRAND [7, pp. 132-145] or HENRICI [4, pp. 209-288] for such details), the approximation of a differential equation of a given order by a difference equation of higher order has the shortcoming of introducing "spurious" solutions as illustrated by the above example. More precisely, the higher-order difference equation has a larger number of fundamental solutions

(i.e. the building blocks out of which one constructs all solutions) than does the original differential equation, and not all of these may behave like the exact solution of the original differential equation. Consequently, for example, the approximation (E.23) is not a satisfactory one. In general terms, higher-order approximations introduce spurious solutions, and such spurious solutions may cause convergence problems. In plain words, this means that one can pick a finite-difference method such that the exact LANCHESTER-type-model results for, for example, $x(t_n)$ and $y(t_n)$ cannot be reached (sometimes even remotely) by the numerical ones $\hat{X}_n$ and $\hat{Y}_n$ by taking $\Delta t$ small enough. Such troubles may be avoided by investigating the (relative) numerical stability of the finite-difference-approximation solution. Unfortunately, not all higher-order approximations (which do possess better truncation error) are numerically stable. The reader is directed to texts on numerical analysis (e.g [4-11]) where finite-difference-approximation methods with such undesirable properties are identified. Thus, whenever one uses a higher-order method, one must make sure that it possesses the desired numerical stability properties. Moreover, the EULER-CAUCHY method recommended above is both convergent and numerically stable.

Let us finally note here that the important mathematical properties of convergence and stability of finite-difference approximations to O.D.E.s are intimately related. It is a rather far-reaching result in numerical analysis that for consistent finite-difference approximations, stability of the difference equations is equivalent to convergence of the difference equations' solution to that of the original differential equation (see HENRICI [4, pp. 217-287], ISAACSON and KELLER [9, pp. 410-417], or HILDEBRAND [7, pp. 140-145] for further details).

## 5. Some Examples.

In this section we will give several examples of finite-difference approximations by the EULER-CAUCHY method to specific homogeneous-force models. The main reason for giving these examples (particularly the first one) is to indicate how such finite-difference approximations may be recursively solved to generate numerical results.

Example E.1: Constant-Coefficient LANCHESTER-Type Equations for $F|F$ Attrition Process. If we consider a fight to the finish, our differential battle model is

$$\begin{cases} \dfrac{dx}{dt} = \begin{cases} -ay & \text{for } x \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases} & \text{with } x(0) = x_0, \\ \\ \dfrac{dy}{dt} = \begin{cases} -bx & \text{for } x \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases} & \text{with } y(0) = y_0 . \end{cases} \tag{E.24}$$

The finite-difference approximation by the EULER-CAUCHY method then reads

$$\begin{cases} \hat{x}_{n+1} = \hat{x}_n - a\hat{y}_n\Delta t & \text{with } \hat{x}_0 = x_0 , \\ \hat{y}_{n+1} = \hat{y}_n - b\hat{x}_n\Delta t & \text{with } \hat{y}_0 = y_0 , \end{cases} \tag{E.25}$$

which in view of the fight-to-the-finish equations (E.24) should be taken to more precisely mean

$$\begin{cases} \hat{x}_{n+1} = \max(0,\ \hat{x}_n - a\hat{y}_n \Delta t) & \text{with} \quad \hat{x}_0 = x_0 \ , \\ \hat{y}_{n+1} = \max(0,\ \hat{y}_n - b\hat{x}_n \Delta t) & \text{with} \quad \hat{y}_0 = y_0 \ . \end{cases} \tag{E.26}$$

The reader will recognize (E.25) as FISKE's equations for modern warfare, which we have examined in Section 2.10 above. Moreover, it should again be emphasized that when we write (E.25) as being a finite-difference approximation to (E.24), we really mean that the equations (E.26) are to be understood. This previously-agreed-to convention will be followed in the balance of this appendix. We will now consider a specific numerical example to illustrate the recursive solution procedure for the approximating difference equations. Numerical results for the input data shown in Table E.I are given in Table E.II. From considering these numerical results, the reader should have no trouble in understanding how the approximate battle results $\hat{x}_n$ and $\hat{y}_n$ are propagated ahead in time from the old time step to the new time step in a step-by-step fashion (cf. Figure E.1).

Example E.2: Variable-Coefficient LANCHESTER-Type Equations for $F|F$ Attrition Process. In this case the battle model reads

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y & \text{with} \quad x(0) = x_0 \ , \\ \dfrac{dy}{dt} = -b(t)x & \text{with} \quad y(0) = y_0 \ . \end{cases} \tag{E.27}$$

TABLE E.I. Input Data for Numerical Example on EULER-CAUCHY Finite-Difference-Approximation Method Applied to Constant-Coefficient Equations for F|F Attrition Process.

$x_0 = 10.0$

$y_0 = 30.0$

$a = 0.06$ X casualties/minute per Y firer

$b = 0.6$ Y casualties/minute per X firer

$\Delta t = 0.01$ minute

NOTE: For the differential-equation combat model, X will win a fight to the finish, since

$$0.333 = \frac{x_0}{y_0} > \sqrt{\frac{a}{b}} = \sqrt{0.1} = 0.316 .$$

TABLE E.II. Numerical Example for EULER-CAUCHY Method for Input Data Shown in Table E.1.

| $t_n$ (minutes) | time step n | $\hat{x}_n$ | $\hat{y}_n$ | $b\hat{x}_n\Delta t$ | $a\hat{y}_n\Delta t$ |
|---|---|---|---|---|---|
| 0.00 | 0 | 10.0000 | 30.000 | 0.060 | 0.0180 |
| 0.01 | 1 | 9.9820 | 29.940 | 0.060 | 0.0180 |
| 0.02 | 2 | 9.9640 | 29.880 | 0.060 | 0.0179 |
| 0.03 | 3 | 9.9461 | 29.820 | 0.060 | 0.0179 |
| 0.04 | 4 | 9.9282 | 29.760 | 0.060 | 0.0179 |
| 0.05 | 5 | 9.9103 | 29.700 | 0.059 | 0.0178 |
| 0.06 | 6 | 9.8925 | 29.641 | 0.059 | 0.0178 |
| 0.07 | 7 | 9.8747 | 29.582 | 0.059 | 0.0177 |
| 0.08 | 8 | 9.8570 | 29.523 | 0.059 | 0.0177 |
| 0.09 | 9 | 9.8393 | 29.464 | 0.059 | 0.0177 |
| 0.10 | 10 | 9.8216 | 29.405 | 0.059 | 0.0176 |
| ⋮ | etc. | ⋮ | ⋮ | ⋮ | ⋮ |
| 5.00 | 500 | 4.4317 | 9.8276 | 0.0266 | 0.0059 |
| 5.01 | 501 | 4.4258 | 9.8010 | 0.0265 | 0.0059 |
| 5.02 | 502 | 4.4199 | 9.7745 | 0.0265 | 0.0059 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 7.50 | 750 | 3.4081 | 4.0526 | 0.0205 | 0.0024 |
| 7.51 | 751 | 3.4057 | 4.0321 | 0.0204 | 0.0024 |
| 7.52 | 752 | 3.4033 | 4.0117 | 0.0204 | 0.0024 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 9.55 | 955 | 3.15691 | 0.0645 | 0.0190 | 0.00004 |
| 9.56 | 956 | 3.15687 | 0.0455 | 0.0189 | 0.00002 |
| 9.57 | 957 | 3.15685 | 0.0266 | 0.0189 | 0.00002 |
| 9.58 | 958 | 3.15683 | 0.0077 | 0.0189 | 0.00000 |
| 9.59 | 959 | 3.15683 | 0.0000 | ------ | ------- |

Examples of such time-dependent coefficients (together with their origins from physical circumstances) have been given in Section 6.2 above. If we denote $a(t_n)$ as $a_n$ and $b(t_n)$ as $b_n$, then the EULER-CAUCHY finite-difference approximation to (E.27) reads

$$\begin{cases} \hat{x}_{n+1} = \hat{x}_n - a_n \hat{y}_n \Delta t & \text{with } \hat{x}_0 = x_0 , \\ \hat{y}_{n+1} = \hat{y}_n - b_n \hat{x}_n \Delta t & \text{with } \hat{y}_0 = y_0 . \end{cases} \tag{E.28}$$

In this case the reader can readily see that once the time-dependent attrition-rate coefficients $a(t)$ and $b(t)$ have been specified, the step-by-step numerical integration of the variable-coefficient equations (E.27) by means of (E.28) is actually no more difficult than that of the constant-coefficient equations (E.24) by means of (E.25).

<u>Example E.3: Dynamics of a Fire Fight</u>. Consider a "fire fight" between homogeneous X and Y forces. Assume that LANCHESTER-type equations for $F|F$ attrition describe the attrition process. If we further assume that (A1) whether or not a side has "fire superiority" may be measured in terms of whether or not that side is putting out the greater total volume of fire, and (A2) having (not having) fire superiority yields the consequence that individual firers are overwhelming the enemy with their fire (are being overwhelmed by the enemy's fire) and are consequently increasing (decreasing) their rate of fire up to a maximum value (down to a minimum value); then this combat may be modelled by the following equations (<u>see</u> HUGGINS [8] for further details; <u>see</u> also von FABECK [14] for an examination of the phenomenological bases of fire superiority)

$$\frac{dx}{dt} = -a(t)y \ , \qquad \frac{dy}{dt} = -b(t)x, \tag{E.29}$$

$$a(t) = 1/\{t_{a_{XY}} + 1/[\nu_Y(t)\ P_{SSK_{XY}}]\},$$

$$b(t) = 1/\{t_{a_{YX}} + 1/[\nu_X(t)\ P_{SSK_{YX}}]\},$$

$$\frac{d\nu_X}{dt} = \begin{cases} C_X \ \mathrm{sgn}(\nu_X x - \nu_Y y) & \text{for} \quad m_X < \nu_X < M_X \ , \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{d\nu_Y}{dt} = \begin{cases} C_X \ \mathrm{sgn}(\nu_Y y - \nu_X x) & \text{for} \quad m_Y < \nu_Y < M_Y \ , \\ 0 & \text{otherwise,} \end{cases}$$

with initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad \nu_X(0) = \nu_0^X, \qquad \nu_Y(0) = \nu_0^Y \ ,$$

where $t_{a_{XY}}$, $t_{a_{YX}}$, $P_{SSK_{XY}}$, $P_{SSK_{YX}}$, $C_X$, $C_Y$, $m_X$, $m_Y$, $M_X$, and $M_Y$ denote constants. Here we have assumed the simple model for the LANCHESTER attrition-rate coefficient given in Section 5.2 (<u>see</u> also Section 5.10). Also, the symbol sgn $\theta$, read "signum $\theta$," denotes the <u>signum</u> <u>function</u> denoted by

$$\mathrm{sgn}\ \theta = \begin{cases} +1 & \text{for} \quad \theta > 0 \ , \\ 0 & \text{for} \quad \theta = 0 \ , \\ -1 & \text{for} \quad \theta < 0 \ . \end{cases} \tag{E.30}$$

The above model (E.29) incorporates the feature that individuals on the side that is producing the larger total volume of fire (measured in terms of the total number of rounds fired per unit time) can increase (up to a limit) their firing rate by virtue of having fire superiority. Similarly, an individual's firing rate is "chocked off" when his side loses fire superiority. Introducing notation in the obvious way, we may then write the EULER-CAUCHY approximation to (E.29) as

$$\hat{x}_{n+1} = \hat{x}_n - \hat{a}_n\hat{y}_n\Delta t, \qquad \hat{y}_{n+1} = \hat{y}_n - \hat{b}_n\hat{x}_n\Delta t\ , \tag{E.31}$$

$$\hat{a}_n = 1/\{t_{a_{XY}} + 1/(\hat{\nu}_n^Y P_{SSK_{XY}})\},$$

$$\hat{b}_n = 1/\{t_{a_{YX}} + 1/(\hat{\nu}_n^X P_{SSK_{YX}})\},$$

$$\hat{\nu}_{n+1}^X = \begin{cases} \hat{\nu}_n^X + C_X\Delta t\ \mathrm{sgn}(\hat{\nu}_n^X\hat{x}_n - \hat{\nu}_n^Y\hat{y}_n) & \text{for}\quad m_X < \hat{\nu}_n^X < M_X, \\ \hat{\nu}_n^X & \text{otherwise,} \end{cases}$$

$$\hat{\nu}_{n+1}^Y = \begin{cases} \hat{\nu}_n^Y + C_Y\Delta t\ \mathrm{sgn}(\hat{\nu}_n^Y\hat{y}_n - \hat{\nu}_n^X\hat{x}_n) & \text{for}\quad m_Y < \hat{\nu}_n^Y < M_Y, \\ \hat{\nu}_n^Y & \text{otherwise,} \end{cases}$$

with initial conditions

$$\hat{x}_0 = x_0, \qquad \hat{y}_0 = y_0, \qquad \hat{\nu}_0^X = \nu_0^X, \qquad \hat{\nu}_0^Y = \nu_0^Y\ .$$

Although our model of the dynamics of a fire fight (E.29) is fairly complex and it is for sure impossible to conveniently represent its solution in terms of any elementary functions, it is an easy matter to program a digital computer to recursively compute the numerical solution of the approximating finite-difference equations (E.31) and hence to numerically integrate our differential combat model (E.29) in a step-by-step fashion. Such a numerical procedure was indeed quite tedious and essentially not practical before the advent of the high-speed digital computer.

## 6. Advantages and Disadvantages of Both Analytical Solutions and Also Their Numerical Approximations.

In this monograph we have considered both the formulation and also the solution (i.e. extraction of information for analysis purposes) of LANCHESTER-type homogeneous-force combat models. Both analytical and also numerical-approximation solution approaches have now been examined. Some similar investigations (i.e. formulation and solution) have been carried out to a lesser extent for heterogeneous-force models. Based on these investigations, it seems appropriate to compare the advantages and disadvantages of both analytical solutions and also their numerical approximation. As an example of the latter, the reader should keep in mind the first example of the last section, a numerical example of integration of $F|F$ attrition-process equations by finite-difference means.

Advantages and disadvantages of analytical solutions to LANCHESTER-type models are given in Table E.III, which is self-explanatory and does not need any further elaboration except for the following discussion of a few not-so-obvious points. A real advantage of simple analytical models that yield convenient analytical solutions is their <u>behavioral transparency</u>, i.e. one can easily see how model outputs are related to inputs and other model parameters. For example, we know that for LANCHESTER's equations of modern warfare (i.e. constant-coefficient equations for an $F|F$ attrition process) that the $X$ force level $x(t)$ is given by

$$x(t) = x_0 \cosh(\sqrt{ab}\ t) - y_0 \sqrt{\frac{a}{b}}\ \sinh(\sqrt{ab}\ t) \ . \tag{E.32}$$

Thus, the analytical solution reveals the two important model parameters: (1) the intensity of combat, $I = \sqrt{ab}$; and (2) the relative fire effectiveness of individual combatants, $R = a/b$. It also reveals that of these parameters only relative fire effectiveness $R$ helps determine battle outcome, with the intensity of combat $I$ only adjusting the battle's time scale. Another very important aspect of simple LANCHESTER-type models is their ability to provide a framework for understanding and interpreting results from much more complicated operational models. This characteristic is the basic idea behind the fourth advantage given in Table E.III, and in a similar vein we have discussed above in Section 7.1 the coordinated use of a simple auxiliary model with a complex operational model. A further discussion of this important concept within the context of modelling tactical decision making as

TABLE E.III. Advantages and Disadvantages of an Analytical Solution to a LANCHESTER-Type Model.

| ADVANTAGES | DISADVANTAGES |
| --- | --- |
| (1) exact results<br>(2) behavioral transparency (i.e. can easily see relationship between model's parameters and solution behavior)<br>(3) parametric analyses easily performed<br>(4) can generate hypotheses to be tested in higher-resolution studies. | (1) may be quite complicated<br>(2) available only for very simple cases<br>(a) few state variables<br>(b) simple forms for attrition rates and coefficients<br>(3) may require mathematical sophistication to understand, appreciate, and use. |

a rational process and optimizing tactical resource allocation is to be found in Section 8.5 below. Some additional thoughts on the coordinated use of a simplified auxiliary model with a higher-resolution complex operational model (besides a graphical articulation of the basic concept) are portrayed in Figure E.2.

Moreover, it seems appropriate to point out here that one disadvantage of an analytical solution is that advanced mathematical theory may be required just to understand and use it. An example of this unfortunate situation is the solution to variable-coefficient LANCHESTER-type equations for modern warfare with power attrition-rate coefficients (see Section 6.9). Here for cases of no offset the force levels may be represented in terms of LANCHESTER-CLIFFORD-SCHLÄFLI functions (or, equivalently, modified BESSEL functions of the first kind of fractional order). Thus, some knowledge of special mathematical functions is more or less required for analytically analyzing essentially all but simple constant-coefficient LANCHESTER-type models, in particular for variable-coefficient LANCHESTER-type equations of modern warfare (see Chapter 6).

In a similar fashion, advantages and disadvantages of approximate numerical solutions are given in Table E.IV. The only additional comment that seems necessary here is that one disadvantage of them (the last given in Table E.IV) is that some caution must be observed in their use. For example, one cannot indiscriminately choose the time-stop size $\Delta t$ to be used in numerically generating results with the finite-difference approximation. Also, as we have discussed above, not all finite-difference approximations to LANCHESTER-type differential equations are really satisfactory from the standpoint of military OR. Some

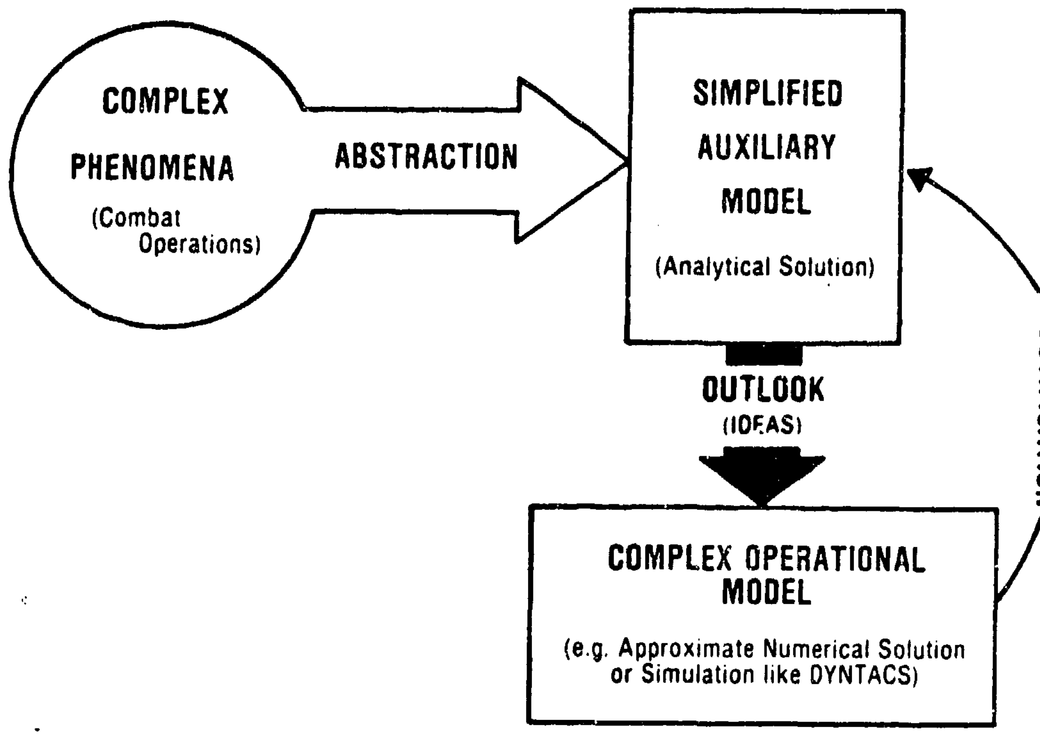


Figure E.2. Coordinated use of simplified auxiliary model and complex operational (i.e. higher-resolution) model.

TABLE E.IV. Advantages and Disadvantages of an Approximate Numerical Solution by Finite-Difference Methods to a LANCHESTER-type Model.

| ADVANTAGES | DISADVANTAGES |
|---|---|
| (1) can always be obtained (i.e. guaranteed answer) | (1) need computer to generate: resources required<br>(a) time<br>(b) money |
| (2) easily generated by recursive algorithm (i.e. finite-difference-equation solution readily computed recursively) | (2) difficult to perceive significant relationships between model parameters and solution behavior |
| (3) no advanced mathematical theory required to understand and use. | (3) might be costly to perform parametric analyses |
| | (4) only obtain approximate solution (beware!) |

knowledge about numerical analysis (such as we have outlined above) is useful for avoiding certain pitfalls of computation.

After comparing the advantages and disadvantages of analytical and approximate numerical solutions shown in Tables E.III and E.IV, the reader should sense that simple analytical solutions and numerical approximations to more complicated models are in some sense complementary. Returning to our theme about considering the information to be extracted from a combat model, we observe that in many cases some information may be obtained from an analytical solution to a simple model, while other complementary information about system performance and effectiveness is probably best obtained from a more complicated model by numerical means (i.e. finite-difference approximation). Again, Figure E.2 portrays some related thoughts along these lines. We feel that much more work is needed on analysis strategies for the coordinated use of simplified auxiliary models with complex operational combat models. In force-on-force combat analysis, no one model can stand alone!

## 7. Suggestions for Further Reading.

In this section we give some selected references for the reader who desires further information about the numerical solution of O.D.E.s. Excellent introductions for the nonspecialist are afforded by HENRICI [5, Chapter 14], McCRACKEN and DORN [10, Chapter 10], HILDEBRAND [7, Chapter 2], and RALSTON and WILF [13, Chapters 8 and 9], with the last reference probably containing the best introduction for the OR worker

(even though the computer material is quite dated). Other good introductory texts are those by MILNE [11] and HILDEBRAND [6], in spite of the fact that they appeared in the relative early days of digital computers. The reader who desires further information about difference equations themselves will find very readable introductions in HENRICI [5, Chapter 3] and HILDEBRAND [7, Chapter 1]. An excellent short summary of the principal finite-difference methods for O.D.E.s appears in DAVIS and POLONSKY [3, pp. 896-897]. More theoretical treatments of the numerical solution of O.D.E.s are to be found in HENRICI [4] and ISAACSON and KELLER [9], with a fairly extensive list of references to the numerical-analysis literature concerning O.D.E.s appearing in HENRICI [4] (see also MILNE [11] for an extensive list of earlier references).

## 8. Final Remarks.

With the information about finite-difference approximations contained in this appendix the reader has the computational means at hand for building operational LANCHESTER-type models of essentially any desired degree of complexity. Such approximation methods allow one with the help of a digital computer to generate numerical results (albeit for particular values of input data) from essentially any kind of LANCHESTER-type model. With such computational support, the military-OR worker can focus on model formulation and, more generally, the iterative process of model building (cf. MORRIS [12]).

For such computational work we have (for a variety of reasons) recommended the use of the EULER-CAUCHY method. In particular, because of the very approximate nature of LANCHESTER-type models in the first place, higher-order finite-difference-approximation schemes hardly seem justified as they have been in, for example, the physical sciences where the differential laws of nature are quite precisely known. Besides the convergence and stability of finite-difference methods discussed above, another important computational consideration is the number of computations required. The EULER-CAUCHY does very well on this criterion because of its simplicity. Moreover, because of the speed of modern digital computers and the simplicity of the EULER-CAUCHY method, the smaller time-step size required by consideration of truncation error in relation to that possible for higher-order schemes is of little consequence. Furthermore, operational LANCHESTER-type combat models in use today such as the BONDER/IUA or any one of the VECTOR series of models use the EULER-CAUCHY method (e.g. see BONDER and HONIG [1, p. 337] or [2, p. 51]).

## EXERCISES FOR APPENDIX E

1. Consider LANCHESTER's constant-coefficient equations of modern warfare for a fight to the finish (E.24) and the EULER-CAUCHY finite-difference approximation (E.25) that we developed for them.

   Part a. Using the finite-difference approximation and the following input data $x_0 = 20$, $y_0 = 70$, $a = 0.1$ X casualties/minute per Y firer, $b = 0.5$ Y casualties/minute per X firer, and $\Delta t = 0.1$ minute; compute by hand the approximate force levels $\hat{x}_n$ and $\hat{y}_n$ for several time steps in order to get a feel for the recursive solution procedure.

   Part b. Based on your computational experience gained in Part a, automate the computational procedure by developing an algorithm and writing a computer program to calculate the approximate force levels $\hat{x}_n$ and $\hat{y}_n$.

   Part c. Using the data of Part a, exercise the computer program developed in Part b. Plot the exact force level values $x(t)$ and $y(t)$ against time $t$, and on these same plots show the values for the approximate force levels $\hat{x}_n$ and $\hat{y}_n$.

   Part d. Using experimental computation (i.e. by trial and error), find a value for the time-step size $\Delta t$ that yields satisfactory approximate results. (Hint: as suggested in this appendix, take several trial values for $\Delta t$ (e.g. $\Delta t = 0.001$ minute, 0.01 minute,

0.1 minute, 1.0 minute, 10.0 minutes), compute the approximate force-level trajectories for each of these different values, and compare results).

Part e. Modify the finite-difference approximation and your computer program to handle the case of a fixed-force-level-breakpoint battle (see Section 2.8). Exercise your computer program with the data of Part a and $f_{BP}^{X} = 0.5$ and $f_{BP}^{Y} = 0.15$, where (as usual) $x_{BP} = f_{BP}^{X} x_0$ and $y_{BP} = f_{BP}^{Y} y_0$.

2. Recall the LANCHESTER-type model that we developed in Section 3.10 and that considers unit deterioration due to attrition with fixed-force-level breakpoints

$$\begin{cases} \dfrac{dx}{dt} = \begin{cases} -a(1-f_I^Y)\left\{1 - \left(\dfrac{y_0-y}{y_0-y_{BP}}\right)^{\nu}\right\} y & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \dfrac{dy}{dt} = \begin{cases} -b(1-f_I^X)\left\{1 - \left(\dfrac{x_0-x}{x_0-x_{BP}}\right)^{\mu}\right\} x & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{E.33}$$

where $f_I^X$ and $f_I^Y$ denote the fractions of the X and Y forces that are permanently ineffective, and $\mu$ and $\nu$ are constant parameters modelling the unit-deterioration process.

Part a. Develop the EULER-CAUCHY finite-difference approximation to (E.33).

Part b. Using the finite-difference approximation developed in Part a, write a computer program to calculate the approximate force levels $\hat{x}_n$ and $\hat{y}_n$.

Part c. Using the data $x_0 = 30$, $y_0 = 80$, $a = 0.05$ X casualties/minute per Y firer, $b = 0.2$ Y casualties/minute per X firer, $f_I^X = 0.1$, $f_I^Y = 0.3$, $\mu = 2.5$, $\nu = 2.5$, $f_{BP}^X = 0.5$, and $f_{BP}^Y = 0.15$; find a satisfactory time-step size $\Delta t$ for this finite-difference approximation by experimental computing. (Hint: as suggested in this appendix, first find a satisfactory time-step size for the finite-difference approximation of Problem 1. Denote this value as $h_S$. Then compute approximate results for the model (E.33) for several values of $\Delta t$, using $h_S$ as a point of departure (e.g. $\Delta t = 0.1h_S$, $0.5h_S$, $h_S$, $2.5h_S$, $5h_S$), and compare results.)

Part d. Using the data of Part c and the time-step value $\Delta t$ developed there, compute the approximate force levels, and plot $\hat{x}$ against time t and also $\hat{y}$ against time. Develop similar plots for cases in which $\mu = 1$ and $\nu = 1$, $\mu = 1$ and $\nu = 2$, $\mu = 1$ and $\nu = 2$, and $\mu = 10$ and $\nu = 10$. Compare these numerical results with those for LANCHESTER's equations of modern warfare with the same fixed-force-level breakpoints.

Part e. What other graphical plots would be of interest to a military-OR analyst?

3. Using the computation aids (i.e. the computer programs) developed above for the combat models (E.24) and (E.33), evaluate the following rule of thumb frequently used by military planners: do not attack unless you possess a three-to-one advantage in combat power.

## REFERENCES FOR APPENDIX E


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## Chapter 8. OPTIMIZING TACTICAL DECISIONS[1]

### 8.1. Introduction.

In this chapter we will briefly examine developing insights into the structure of optimal tactical decisions by combining combat modelling and optimization theories. Our approach is to apply so-called generalized control theory (i.e. optimization theory for dynamic systems[2]) to relatively simple LANCHESTER-type combat models in which the combat strategies of tactical decision makers are represented by "decision variables.[3]" The "best" values for these decision variables are then determined by invoking optimality conditions from generalized control theory.

This chapter, however, is substantially different from the other chapters of this monograph in the sense that its purpose is to provide an introduction to and overview of the quantitative analysis of military strategy and tactics and not to provide complete details on how this is done. The author has felt it to be important to show how LANCHESTER-type models can be used prescriptively for military decision making (at least conceptually), even though circumstances have prevented complete details being given here. Thus, our purpose here is to provide the reader with some indication as to how the LANCHESTER theory of combat can be combined with optimization theory to quantitatively study military strategy and tactics. The author has felt that it would be better to provide a rather sketchy introduction to and overview of this important topic rather than omit it entirely. Thus, complete details will not be given, with the reader

being referred to the literature for them. In partciular, essentially no details from optimization theory (i.e. generalized control theory), no details of procedures for developing solutions, and even no complete solutions will be presented here. However, we will try to establish a framework for the use of such normative models. Consequently, problem formulations and the insights to be gained from such investigations will be stressed.

The structure of optimal time-sequential combat strategies has been studied by the author[4] by considering a sequence of specific problems, and we will examine a few selected representatives from this collection of specific problems. However, these combat models are too simple to be taken literally but should be interpreted as only indicating general principles to serve as hypotheses in subsequent studies with more detailed operations models (e.g. a high-resolution Monte Carlo combat simulation such as DYNTACS, or a complex operational analytical model such as BONDER/IUA or VECTOR-2)[5]. Since these mathematical models are such idealizations of the (rational) decision-making process in combat, probably the only significant result obtained from then is the structure of the optimal combat strategies. Consequently, the author's research has initially concentrated on relating the structure of optimal combat strategies to the conceptualization of the tactical decision problem. Such work may be helpful for understanding optimization results from (and, hence, for making better use of) more complex operational models.

In this chapter we will therefore briefly examine several specific optimization problems for determining optimal time-sequential combat strategies (primarily fire-distribution strategies, i.e. strategies for

distributing fire over enemy target types). We will also consider the optimal initial commitment of forces in battle, and this examination of ours will provide fresh insights into the "principle of concentration," which F. W. LANCHESTER [53] first attempted to quantitatively justify in 1914 (see also Section 2.9 above). On the battlefield, the opposing commanders have conflicting interests, and this basic conflict of interests leads to a so-called game-theoretic or two-sided optimization problem for determining the "best" combat strategy for each side[6], i.e. each side is faced with a tactical choice problem that is in turn affected by the enemy's tactical strategy. Because such two-sided time-sequential optimization problems (i.e. differential games) are generally so difficult to solve and usually have such fantastically complicated solutions, we will accordingly consider some one-sided optimization problems[7] (i.e. one side's strategy is fixed and thus only the other side has a free choice of its combat strategy) in order to illustrate modelling points and study the structure of optimal combat policies[8] (or tactics).

8.2. Quantitative Analysis of Military Strategy and Tactics

From the standpoint of modern operations research (OR), problems of military strategy and tactics[9] may be viewed as being basically resource-allocation problems over time. For example, a military commander of ground forces is frequently faced with the problem of when and where to commit his reserve forces into battle. As another example, the allocation of a specific weapon-system type to an acquired target is an important tactical decision in the fire-support process. Accordingly, the determination of optimal (or even "good") fire-distribution strategies for supporting weapon systems[10] has been a major problem of military OR. In particular, the determination of the optimal allocation of general-purpose aircraft to missions in a multiperiod war with a specified number of periods has been much studied in the past and continues today to be of significant interest to defense planners.

Many people believe that such tactical decisions (quantified in models as behavioral and/or decision variables) are the most significant factors driving the course of combat to its end. Thus, one is faced with the problem of modelling tactical decisions[11]. After such tactical-decision models have been developed, it becomes of interest to find a preferred course of action from among the feasible alternatives.

Optimal strategies for such tactical-allocation problems may be investigated by means of

(I) war gaming[12],

(II) mathematical modelling combined with optimization theory.

These two approaches both share some common dimensions of the tactical decision-making process, but they may also be characterized by their differences. The distinguishing feature of war games is that they use real people playing the roles of the battlefield commanders and their staffs to simulate tactical decision processes, while combat simulations and analytical models use symbols, algorithms, or some other type of logic to represent such decision processes. All such approaches and/or modelling methodologies, however, play the same functional role in combat simulations[13]: they produce requisite tactical decisions (i.e. the outputs of decision processes) at appropriate times during the course of simulated combat. Moreover, war games are descriptive, while optimization problems are prescriptive (or normative).

When we analytically model the tactical choice problem with each of the opposing commanders seeking to use his "best" combat strategy, we are led to a game-theory model for optimizing tactical decisions in which there are at least two players or decision makers (cf. HO [34; 35]). When there are two decision makers, such a normative model is also frequently called a two-sided optimization problem (e.g. see HO, BRYSON, and BARON [36]). Moreover, there is an intimate connection between game theory and war gaming (e.g. see THOMAS and DEEMER [101]). In particular, SHUBIK [72] has stressed that a knowledge of the theory of games provides a useful benchmark and a fundamentally important methodological approach to the study of situations involving potential conflict of interests. Table 8.I presents a brief synopsis of the major assumptions in game-theoretic optimization problems and war gaming (i.e. behavioral model

TABLE 8.I. Brief Synopsis of the Major Assumptions in Game-Theoretic Optimization Problems and War Gaming (i.e. Behavioral Model Building).

| Game Theory | Behavioral Theories |
|---|---|
| Rules of the game | Military doctrine and custom |
| External symmetry | Personal detail |
| No social conditioning | Socialization assumed |
| No role playing | Role playing |
| Fixed well-defined payoffs | Difficult to define and may change |
| Perfect intelligence | Limited intelligence |
| No learning | Learning |
| No coding problems | Coding problems |
| Primarily static | Primarily dynamic |

building). Many of the same comparisons, of course, also apply to the comparison between one-sided optimization (i.e. the combat strategy assumed to be known for one side) and war gaming (see SHUBIK [72, pp. 157-180] for further details).

The author's approach for investigating optimal combat strategies (e.g. see TAYLOR [85-88; 91-97]) has been to develop an analytical model of the tactical engagement, to quantify the tactical choices and/or allocations of the commanders through decision variables, to incorporate these decision variables into the combat model, and finally to determine the "best" values for these decision variables. We have, of course, used LANCHESTER-type models to represent the combat dynamics in these optimization problems.

Thus, the topics covered in this book on LANCHESTER combat theory fall essentially into two categories: namely, material on

(C1) simple LANCHESTER-type models,

and

(C2) determining optimal tactical decisions with such simple models.

Models in the first category may be classified as being descriptive, while those in the second may be classified as being normative. In the latter case, the LANCHESTER-type equations are used to assess the consequences of the decisions made by the commanders and modelled by decision variables. The focus of the author's work has been on understanding the dynamics of combat and optimization of tactical decisions through studying

simplified analytical models, especially those that provide an understanding of the basic nature of more complex operational models. A good analytical model, of course, should simplify, be transparent and easy to understand, be easy to manipulate, and increase our understanding of real-world processes (i.e. yield important insights). For reasons that should be obvious to the reader by now, the combat-optimization problems studied by the author are far too simple to be taken literally but should be interpreted as yielding insights that can provide valuable guidance for subsequent higher-resolution computerized investigations. As we have already stressed above in Section 8.1, probably the only significant result obtained from such combat-optimization problems is the structure of the optimal combat strategies, since these mathematical models are such great idealizations of the (rational) decision-making process in combat.

## 8.3. Information to be Obtained from the Quantitative Analysis of Military Strategy and Tactics.

Thus, as discussed above, the author's research on determining optimal tactical decisions has been based on applying optimization theory to such simple LANCHESTER-type combat models as we have predominantly considered previously in this monograph, with tactical decisions quantified through decision variables. Our work has emphasized understanding the dependence of the structure of optimal time-sequential combat strategies on the basic elements of the combat-optimization problem (see Section 8.4 below). As we have previously stressed for our analytical investigation of simple LANCHESTER-type (descriptive) models (cf. the questions of Section 6.3), we have used judiciously selected questions to guide our research efforts on optimizing tactical decisions. Specifically, we have been guided by trying to answer such questions as shown in Table 8.II. Other such questions may be found posed in TAYLOR [92, p. 2; 94, p. 1; 96, pp. 2-3].

Furthermore, our own research efforts have mainly concerned optimal time-sequential tactics for the distribution of fire over enemy target types, with some idealized looks at optimal fire-support strategies. Our research approach has been to consider a sequence of specific problems, to investigate for each problem such questions as shown in Table 8.II, and to compare the structures of optimal fire-distribution strategies among these different problems. Analytical rather than numerical methods have been stressed. A scenario has been developed for each such specific problem expressed in qualitative terms, and the military operations analyzed. Appropriate LANCHESTER-type models of the combat process have then been

TABLE 8.II. Information to Extract from Combat-Optimization Problem.

(Q1) Do target priorities change over time?

(Q2) How should fire be distributed over enemy targets and how should targets be optimally selected?

(Q3) How do force levels affect the optimal time-sequential fire-distribution policy?

(Q4) How do the number of target types and the nature of combat-attrition processes affect the optimal fire-distribution policy?

(Q5) How does the nature of the planning horizon (i.e. battle-termination conditions) affect the optimal fire-distribution policy?

(Q6) What are the affects of logistics constraints on such policies?

(Q7) How do command-and-control capabilities affect the optimal policy?

developed, with decision variables used to represent the feasible actions of the opposing combatants. An optimization problem (reflecting the tactical allocation problem faced by the combatants) has next been formulated and solved by applying the appropriate optimization theory. Finally, after a sequence of such optimization problems has been solved, their solutions have been studied and compared to gain insights into the structure of optimal fire-distribution strategies. This approach of considering a sequence of specific problems has been repeatedly used to investigate the influence of the following factors on optimal time-sequential fire-distribution strategies:

(F1) combatant objectives (quantification of military objectives),

(F2) dynamics of the combat-attrition process,

(F3) weapon-system-performance characteristics,

(F4) termination conditions of the conflict,

(F5) force strengths and composition,

(F6) effects of resource constraints,

(F7) range capabilities of weapon systems.

8.4. Basic Elements of the Combat-Optimization Problem.

Consider two opposing forces in combat. Each force has a commander who makes decisions that influence the course of combat.

What can each do?

What does each know?

What does each want to do? { What criteria does each base his decisions on? How is what each decides related to what happens? }

In more analytical terms, if we assume that each commander is a so-called rational decision maker and we attempt to model how each makes decisions, then the essential aspects of each commander's decision process may be stated as follows:

(EA1) the feasible courses of action available to each decision maker,

(EA2) the information available to each decision maker,

(EA3) the outcome "yardstick" (decision criterion) used by each decision maker,

(EA4) the relation between the joint course of action and conflict outcome.

However, we can formalize much further our method of inquiry and (as discussed above) investigate optimal strategies for tactical decisions by using mathematical modelling combined with optimization theory. Let us now formally call such an optimization problem that is used for investigating optimal tactical-allocation strategies a combat-optimization problem. For the purposes of military OR it is convenient to consider that there are five fundamental elements[14] of any time-sequential combat-optimization problem:

(E1) the decision criteria (for both commanders),

(E2) the model of conflict termination,

(E3) the model of combat dynamics,

(E4) the feasible actions for each decision maker,

(E5) the information available to each decision maker.

It is intuitively obvious that each and every of the above five factors can have a significant influence on what the "best" course of action will be in combat. Furthermore, partially because of the paucity of real combat data, there are alternative models which are essentially equally plausible for each of these factors.

Modern air-ground combat operations may be characterized both by their diversity and also by the vast scope of the sheer numbers of

weapon systems involved. Consequently, current modelling and computer-system technologies cannot directly reproduce such large-scale operations, and large-scale systems must be considered in much system-evaluation work for various reasons such as resource allocation, the combined-arms nature of operations, etc. Since the resultant combat models representing a tactical-decision problem for such systems (and even smaller) must be highly idealized, probably the only significant aspect is how the structure of optimal combat strategies depends on the above five essential elements of the combat-optimization problem. Thus, an important problem for military OR is to determine how the structure of optimal combat strategies depends on these elements of the combat-optimization problem.

In essentially all optimization-theory application to tactical decision making known to this author, it is assumed that decision makers have essentially "perfect knowledge" about enemy capabilities. Hence, we will not consider the information structure further, although it certainly will play a major role in actual real-world military decisions. Also, in much analysis a relatively simple structure for the feasible actions of each decision maker is assumed. For example, it is frequently assumed that an aircraft can be assigned to just one of a number of different tactical missions, although in reality an aircraft might perform several missions on a particular sortie. Hence, we will also not explicitly consider the feasible actions for each decision maker (E4) further. Moreover, concerning the first three items in the above list of elements of the combat-optimization problem, our knowledge about such topics increases as we go down the list. In other words, more is known about

modelling the dynamics of combat than about modelling conflict termination, and still less is known about the decision criteria actually used by decision makers. The author's research has emphasized relating these three elements of the decision problem to the structure of optimal combat strategies by considering a sequence of specific problems. Thus, the consequences and implications of alternative assumptions about these elements may (hopefully) be better appreciated.

## 8.5. Simple Auxiliary Models and Complex Operational Models.

In this chapter we will present some elements of a theory of optimal tactical allocation by examining a sequence of idealized combat-optimization problems that we have considered in our research. For reasons of mathematical tractability, we have primarily considered one-sided time-sequential optimization problems (i.e. so-called optimal-control problems), but we have also considered some time-sequential combat games. We justify our examination of deterministic optimal-control problems on the following grounds:

(F1) LANCHESTER-type differential games are extremely difficult to solve,

and

(F2) there is a well-known intimate connection between the mathematical theories of optimal control and differential games.

Our idea behind studying such one-sided problems is to discover properties of optimal time-sequential combat policies that will provide guidance for studying two-sided time-sequential tactical decision problems. However, one must be aware of the fact that differential games do possess many subtle mathematical features that do not occur in one-sided optimization problems (see ISAACS [47] for further details).

Our approach for studying the optimization of time-sequential tactical decisions has been to consider a sequence of judiciously-chosen simple problems, to analytically solve each optimization problem to determine the optimal time-sequential combat policy, and to compare the

structures of these optimal policies. Although these problems are too simple to be taken literally, such an analytical investigation of the optimization of combat dynamics may be useful for

(a) guiding higher-resolution studies,

and

(b) identifying cause-effect relations between optimal military tactics and modelling assumptions.

Some of the philosophy behind this type of investigation is shown in Figure 8.1. Thus, we do not claim that the simple combat-optimization problems that we have studied should "stand alone" but rather that they should be viewed as points of departure for more detailed investigations using either simulation (see NOLAN and SOVEREIGN [63]), large-scale optimization (see GEOFFRION [26]), or even war gaming. The basic idea is to coordinate the use of a complex operational model with that of a simple auxiliary model, although in this monograph we will consider only the latter. We have already discussed in Chapter 7 such complementary use of models within the context of descriptive models, and we will now briefly reexamine this important concept for normative models.

GEOFFRION [26] has pointed out that a serious inherent limitation of large-scale optimization models is that they do not explain WHY the optimal policy or strategy is what it is, although they certainly can deliver an optimal solution for a given set of input data. For optimizing tactical decisions, we are more interested in (at least initially) the structure of optimal combat strategies and their dependence on modelling

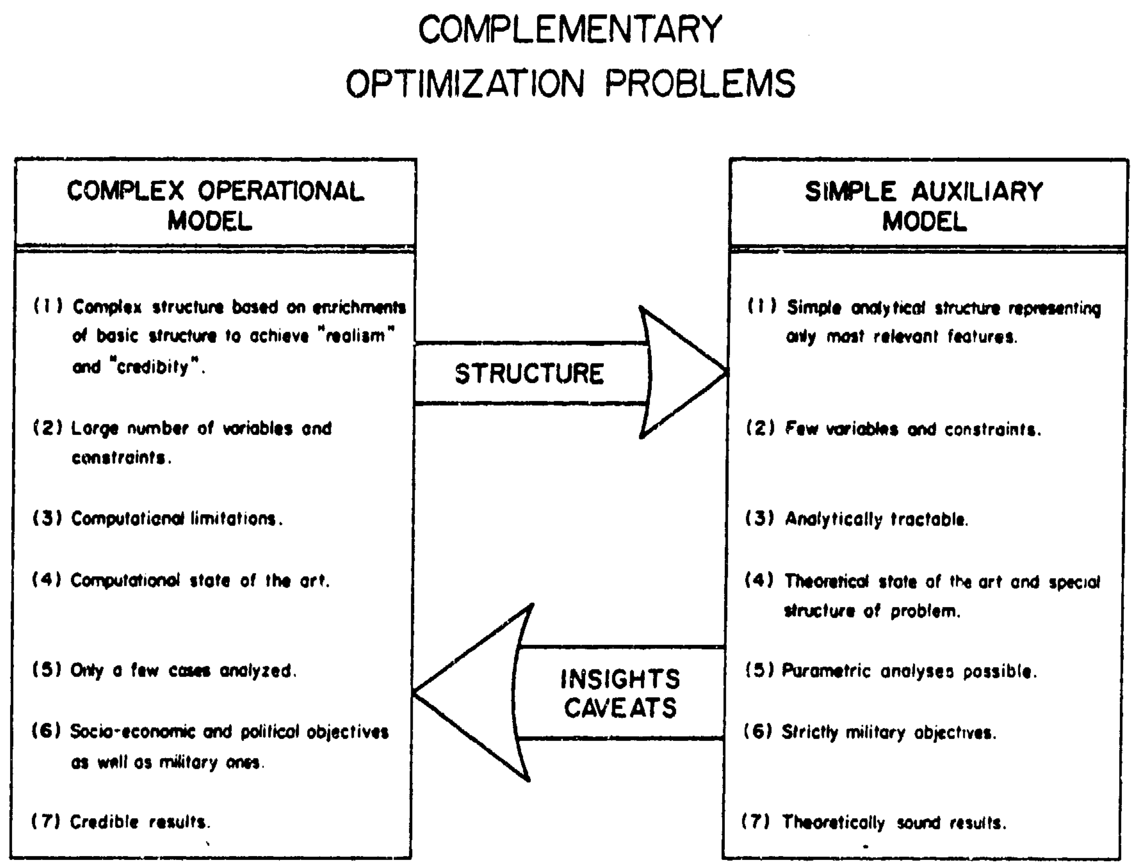


Figure 8.1. Complementary relation between the analytical study of optimizing combat dynamics with a simple auxiliary model and a more detailed examination with a complex operational model.

assumptions and variations in system parameters because of the many uncertainties inherent in combat analyses. Furthermore, few (if any) tactical-decision problems lead to a single perfect numerical model whose solution is directly translatable into practical action. GEOFFRION [26] has stressed that there is rather an entire family of imperfect numerical models reflecting alternative assumptions, objectives, and data estimates. An understanding of solution behavior for the entire family of models is required to fully support the development of an appropriate plan of action.

GEOFFRION [26, pp. 81-82] has further stressed that insights into the determinants of an optimal solution are important because they help to overcome the serious validation/credibility obstacles that are usually present in practical applications (particularly military ones). How can one be convinced a model is a useful representation of the real system? Furthermore, how can the ultimate user of information generated by the model - in DoD applications usually a senior military officer, civilian manager, or politician rather than a technical person - be persuaded to use the model as a decision aid? GEOFFRION [26, p. 82] feels (and so do we) that the answer to both these important questions is that purely numerical results must be supplemented by intuitively reasonable explanations as to why these numerical results have occurred as they have. Otherwise (GEOFFRION has continued) the validity of the model can only be taken on faith, and the decision maker will be inclined to revert to intuition or to some other basis for the decision about which he feels more secure. He has then suggested the use of simple auxiliary models to supplement the use of complex operational models, much as we have depicted in Figure 8.1.

We therefore suggest the following methodological approach for investigating the optimization of tactical decisions (after GEOFFRION [26]):

1. Reduce the level of detail and complexity of the full-scale combat-optimization problem (i.e. the complex operational model) until it can be solved analytically in closed form. Call this a simplified auxiliary model.

2. Derive from the simple auxiliary model a set of tentative hypotheses concerning the general behavior of the solution the full-scale model--the combat-strategy and/or weapon-system trade-offs determining the optimal solution for a given set of data, the nature of the induced change in the optimal solution as certain input data are changed parametrically, and so on.

3. Generate specific predictions from the tentative hypotheses and test these numerically using the full-scale model.

4. To the extent that the numerical tests confirm (actually, do not contradict) the tentative hypotheses about optimal combat strategies, take these hypotheses as a conceptual framework for understanding and interpreting the numerical results provided by the full-scale model.

This approach underlies all our research on optimizing tactical decisions (e.g. see TAYLOR [79, pp. 79-80]). Although GEOFFRION [26] limits his

discussion to optimization models in a nonmilitary context, it is clear that this conceptual approach has much to offer for tactical-decision analysis when used in conjunction with either war gaming, Monte Carlo combat simulations, or complex operational models such as BONDER/IUA or VECTOR-2 that use fixed combat strategies. For example, one could develop a finite number of tentative combat strategies from such a simplified model and then evaluate in more depth each of these strategies by using it in some type of complex operational combat model.

Thus, the relatively simple combat-optimization problems that we will consider in the rest of this chapter should not be taken literally but rather should be interpreted within the framework that we have outlined above.

8.6. Overview of Problems Considered in the Literature.

In this section we will attempt to provide the reader with an overview of the various different types of combat-optimization problems that have appeared in the literature. We will focus here on identifying the principal problem types and on giving references to what work has been done on each type. We will give both a brief overview with a few selective references, and then we will give a more detailed breakdown based on a more comprehensive examination of literature in this field. In subsequent sections we will give detailed mathematical formulations of typical problems from some of these problem-type classes.

First let us give our brief overview. For this purpose, the author perceives that work on optimizing tactical decisions may be classified roughly into the following four categories:

(C1) optimal initial commitment of forces: BACH et al. [6], TAYLOR and PARRY [90], TAYLOR [88];

(C2) optimal distribution of fire (general): ISBELL and MARLOW [50], TAYLOR [76; 78; 79; 82; 84];

(C3) optimal fire-support strategies: WEISS [102; 103], KAWARA [51], TAYLOR [80; 85; 86], TAYLOR and BROWN [89];

(C4) optimal air-war strategies: ISAACS [46], BERKOVITZ and DRESHER [11; 14], BRACKEN et al. [15].

We will give examples of combat-optimization problems from each of these four categories in subsequent sections of this chapter. Except for the first category (C1), all the above work concerns optimizing time-sequential decisions, with both one-sided and also two-sided allocation problems (see HO [34]) being considered. Older work on "static" tactical-allocation problems (i.e. "one-shot" decisions) may be found in DRESHER's book [21]. Further detailed references to the literature may be found in the above papers, particularly BRACKEN et al. [15], TAYLOR and BROWN [89], and TAYLOR [85; 86] (see also TAYLOR [93] and below).

Work in the first category (C1) concerns the same type of problem originally considered by LANCHESTER [53] in 1914 (see Sections 2.1 and 2.9 above) and will be examined in more detail in Section 8.9 below. Work in the second category (C2) concerns the optimal time-sequential distribution of fire over enemy target types in two-sided combat in simple one-sided-decision situations[15] and will be examined in more detail in Sections 8.10 and 8.11 below. In some sense it forms a basis for considering more complicated problems such as those in categories (C3) and (C4). Work in these latter two categories is somewhat similar in mathematical form, with the former category (C3) concerning, for example, artillery allocation and the latter category (C4) concerning the allocation of multipurpose aircraft to different types of tactical missions over time (see Section 8.12 for problem formulations from both these categories). Work in the third category (C3) has been on both one-sided and also two-sided optimization problems, while that in the fourth category (C4) has been both more extensive and also essentially always two-sided.

A much more detailed overview of work done on optimizing tactical decisions with LANCHESTER-type combat models is given in Table 8.III, which can serve the reader as a more detailed guide for further reading. Additional topics have been added here, and the references to the literature are nearly complete. The reader can now see, for example, the large amount of research on optimal air-war strategies over a long period of time. The author has liberally added his own technical reports published by the Naval Postgraduate School (NPS), since such reports are readily available[16] from the National Technical Information Service (NTIS), U. S. Department of Commerce, 5285 Port Royal Road, Srpingfield, Virginia 22151.

The author's own research (see TAYLOR [76-88] and TAYLOR and BROWN [89]; also TAYLOR [91-97] and TAYLOR and POWERS [98]) has mainly concerned optimal time-sequential tactics for the distribution of fire over enemy target types, with some idealized looks at optimal fire-support strategies. Many additional supplemental details such as fairly comprehensive literature reviews, discussions of insights gained, etc. are to be found in the author's NPS reports [91-97]. Our approach has been to consider specific problems and to investigate the influence on optimal fire-distribution strategies of factors[17] such as (F1) through (F7) of Section 8.3.

TABLE 8.III. Detailed Overview of Work on Optimizing Tactical Decisions with LANCHESTER-Type Combat Models.

<u>Optimizing Tactical Decisions (General)</u>

THOMAS and DEEMER (1957)
PUGH and MAYBERRY (1973)
SHUBIK (1975)
TAYLOR (1974Ra, 1979)

<u>Optimal Initial Commitment of Forces</u>

BACH, DOLANSKY, and STUBBS (1962)
TAYLOR and PARRY (1975)
TAYLOR (1979)

<u>Optimal Fire-Distribution Strategies</u>

A. <u>General</u>

ISBELL and MARLOW (1956b)
TAYLOR (1972a, 1973, 1974a, 1974d, 1975, 1977R)
TAYLOR and POWERS (1977R)

B. <u>Optimal Air-War Strategies</u>

MORSE and KIMBALL (1951)
GIAMBONI, MENGEL, and DISHINGTON (1951)
MENGEL (1953, 1954)
ISAACS (1954, 1955, 1965)
ANTOSIEWICZ (1955)
FULKERSON and JOHNSON (1957)
BELLMAN and DREYFUS (1958)
BERKOVITZ and DRESHER (1959, 1960a, 1960b)
BRACKEN (1973)
BRACKEN, FALK, and KARR (1975)
ANDERSON, BRACKEN, and SCHWARTZ (1975)
GOHEEN (1975, 1977)
FISH (1975)
SCHWARTZ (1979)

C. <u>Optimal Fire-Support Strategies</u>

H. K. WEISS (1957, 1959)
KAWARA (1973)
TAYLOR (1974b, 1977, 1978)
TAYLOR and BROWN (1978)

D. <u>Optimal Missile-Warfare Strategies</u>

INTRILIGATOR (1967)
CHATTOPADHYAY (1967, 1969)

E. <u>Other</u>

ISBELL and MARLOW (1956a)
ISAACS (1965)
MOGLEWER and PAYNE (1970)
ETTER (1971)
PUGH (1973)

NOTES: (1) TAYLOR (1974c) = the third paper published by TAYLOR in 1974 as listed in references at end of this chapter.
(2) TAYLOR (1977R) = NPS report published by TAYLOR in 1977.

8.7. Decision Analysis for Tactical Military Decisions.

It is the author's hypothesis that a somewhat different brand of decision analysis (e.g. see HOWARD [37-38] or NORTH [64]) is required for tactical military decision making. The five basic elements of such tactical decisions have been identified in Section 8.4 above. The author's own research has concentrated on investigating the influences of the first three elements (namely, (1) the decision criteria, (2) the model of combat termination, and (3) the combat dynamics) on the structure of optimal combat strategies. Moreover, the author feels that the field of tactical decision-analysis is in its infancy (cf. HOWARD's [37, pp. 56-58] deterministic phase of the decision-analysis procedure) and expects in the future to see a maturing of the embryonic conceptual framework presented here.

In TAYLOR [76; 78-79; 82; 84; 93] a linear utility (see Section 7.18 for methodology for the development of such linear utilities; also HOWES and THRALL [39]) was assumed for the military worth of surviving weapon-system types, and the criterion functional (i.e. payoff) was taken to be the net military worth of survivors. We investigated the sensitivity of the optimal fire-distribution policy (one-sided) to parametric variations in the assigned linear utilities for survivors. It has been shown that the n-versus-one fire-distribution problems studied in TAYLOR [78-39; 82; 84] all have quite simple solutions when enemy survivors are valued in direct proportion to their kill capability against the homogeneous friendly force.

PUGH and MAYBERRY [69] have suggested that an appropriate payoff for the quantitative evaluation of combat strategies is the loss ratio,

with an "almost equivalent" criterion being the loss difference. TAYLOR and BROWN [89] have shown that these criteria are not really equivalent and that the quantification of military objectives may completely change the structure of the optimal combat strategy. Similar results have been obtained by TAYLOR [88], who showed that KAWARA [51] had chosen essentially the only type of payoff that yields optimal fire-support strategies being force-level independent. A general approach was given by TAYLOR [88] for determining the functional form of terminal payoffs that yield state-variable-independent optimal combat strategies.

In TAYLOR [79] we showed that the model of conflict termination may significantly change the optimal fire-distribution policy. For such investigations it has been important to have available complete analytical solutions which are then compared to determine the influence of such a factor.

In TAYLOR [78-79] we have investigated the influences of the nature of the target-type attrition process on the structure of the optimal fire-distribution policies[18]. When target-type attrition (as a rate) is proportional to only the number of firers, we (TAYLOR [79; 82]) have shown that the optimal fire-distribution policy is always to concentrate all fire on a single target type, which may change over the course of battle. We have also studied the nature of such changes in target priorities. However, an optimal fire-distribution policy does not always consist of always concentrating all fire on a single enemy target type. In TAYLOR [78] we have shown that when enemy targets undergo attrition at a rate proportional to the product of the numbers of firers and targets, then an optimal policy may involve firing at several target types to avoid

"overkill." This important result may be best understood in terms of diminishing returns from allocating a unit of weapon system to fire at enemy targets (see TAYLOR [79, pp. 84-85] and below in Section 8.11 for further details). Such a property of optimal fire-distribution strategies (i.e. the splitting of fire between several target types) has been observed by TAYLOR and BROWN [89] and TAYLOR [86] for much more complicated combat dynamics.

## 8.8. Some Combat-Optimization Problems to be Briefly Examined Further.

In the remainder of this chapter we will briefly consider some specific combat-optimization problems concerning (I) otpimal initial commitment of forces, and (II) optimal time-sequential fire-distribution strategies (see Table 8.IV). As we have already indicated in Section 8.1 above, problem formulation will be stressed, with occasional comments being given about insights obtained into the structure of optimal time-sequential combat strategies. The reader is referred to the literature for complete details, including the pertinent optimization theory[19]. A further, more detailed breakout of combat-optimization problems considered in the rest of this chapter is given in Table 8.V, with the section in which each problem is considered being indicated.

TABLE 8.IV. General Types of Combat-Optimization Problems to be Examined in Chapter 8.

(I) Optimal Initial Commitment of Forces

(II) Optimal Time-Sequential Fire-Distribution Strategies

(1) Optimal Fire-Support Strategies

(2) Optimal Air-War Strategies

TABLE 8.V. Detailed Listing of Combat-Optimization Problems to be Briefly Examined in Chapter 8.

(I) Optimal Initial Commitment of Forces (Section 8.9)

(II) Optimal Time-Sequential Fire-Distribution Policies

(1) the simplest fire-distribution problem (Section 8.10)

(2) other battle-termination conditions (Section 8.11)

(3) time-dependent attrition-rate coefficients (Section 8.11)

(4) replacements (Section 8.11)

(5) several enemy target types (Section 8.11)

(6) command and control aspects (Section 8.11)

(7) FT attrition process of enemy target types (Section 8.11)

(8) stochastic LANCHESTER-type attrition processes (Section 8.11)

(9) time-sequential fire-support allocation (Section 8.11)

(III) LANCHESTER-Type Differential Games (Section 8.12)

(1) generalized tactical air-war game (Section 8.12)

(2) modified fire-support differential game (Section 8.12)

## 8.9. Optimal Initial Commitment of Forces.

As we saw in the first section of Chapter 2, LANCHESTER [53] was led to his pioneering mathematical model of combat by his attempt to quantitatively justify the principle of concentration. We subsequently revisited the topic of concentration of forces in Section 2.9, and we analyzed there a commander's decision as to whether or not he should initially commit as many of his forces as possible to battle. We formulated a one-sided[20] combat-optimization problem (2.9.2) and solved it for two special classes of battles (i.e. "square-law" and "linear-law" fixed-force-level-breakpoint battles) for a specific decision criterion (minimizing one's own casualties). We explained how the optimal decision could be very easily understood in terms of the behavior of the instantaneous casualty-exchange ratio, which determined the overall casualty-exchange ratio and related measures of relative casualty-production effectiveness. In the section at hand we will examine this problem more deeply in a more general setting and will justify our contention that many times the optimal initial commitment of forces can be very simply determined by examining how the instantaneous casualty-exchange ratio varies with the victor's force level and time (see TAYLOR [88] for further details).

Let us accordingly consider combat between two homogeneous forces described by the following deterministic LANCHESTER-type equations for $x, y > 0$

$$\begin{cases} \dfrac{dx}{dt} = -G(t,x,y) & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -H(t,x,y) & \text{with } y(0) = y_0 , \end{cases} \tag{8.9.1}$$

where $G$ and $H$ denote force-change rates (with a negative rate signifying a net influx of replacements). For simplicity we will assume that there are no replacements and withdrawals[21], and in this case $G$ and $H > 0$ are simply casualty rates. To insure the existence of partial derivatives needed in subsequent analysis, we assume that $G(t,x,y)$ and $H(t,x,y)$ are each twice continuously differentiable. Let us now consider the decision by the victor[22] (taken to be $X$) in this battle as to how many of his available forces he should initially commit to combat. We will consider the initial-commitment decision by $X$ as a one-sided combat-optimization problem: we assume that the Y-force commander has adopted a known course of action and consider X's initial-commitment decision in this light. This decision is to be made only once, before the battle begins. The decision variable for $X$ in this combat-optimization problem is $x_0$, the initial number of forces committed to battle.

The "best" value of $x_0$ for $X$ to choose may be determined by the following combat-optimization problem (<u>see</u> Section 2.9 for further analysis of the initial-commitment decision):

$$\underset{x_0}{\text{minimize}} \quad C, \tag{8.9.2}$$

$$\text{subject to:} \quad x_0^{min} \le x_0 \le x_0^{max},$$

$$\text{the combat dynamics (8.9.1)},$$

$$\text{and appropriate battle-termination conditions.}$$

Here $C$ denotes the decision criterion ("cost" of doing combat), $x_0^{min} = x_0^{draw} + \varepsilon$, $\varepsilon > 0$, and $x_0^{draw}$ denotes the value[23] of $x_0$ that leads to a "draw." Three possibilities for the decision criterion $C$ are as follows:

(C1) friendly losses, $L_X = x_0 - x_f$ ,

(C2) loss ratio, $R_c = (x_0 - x_f)/(y_0 - y_f)$ ,

and

(C3) loss difference, $D_c = (x_0 - x_f) - (y_0 - y_f)$,

where $x_f$ and $y_f$ denote the final force levels at the end of battle. The battle-termination conditions are taken to correspond to either a fixed-force-level-breakpoint or a fixed-force-ratio-breakpoint battle (<u>see</u> Section 6.6). We will denote the optimal value of $x_0$ as determined by the above optimization problem (8.9.2) as $x_0^*$. Moreover, the above optimization problem (8.9.2) requires calculation of the partial derivative $\partial C/\partial x_0$ and may not always be trivial to solve, since (for example), $\partial C/\partial x_0$ may have multiple zeros in $[x_0^{min}, x_0^{max}]$ and determination of $x_0^*$ could then be tedious. In other cases, however, it may be trivial to solve (e.g. when $\partial C/\partial x_0 < 0$ for all $x_0 \in [x_0^{min}, x_0^{max}]$, then $x_0^* = x_0^{max}$ and $X$ should initially commit as much as possible).

Reparameterizing the course of battle in terms of $y$ by

$$t = t(y) = t(y;x_0,y_0) \quad \text{and} \quad x = x(y) = x(y;x_0,y_0) , \tag{8.9.3}$$

TAYLOR [88] has shown[24] how to express $\partial C/\partial x_0$ in terms of the instantaneous casualty-exchange ratio $dx/dy = G(t,x,y)/H(t,x,y)$ by

$$\left(\frac{\partial C}{\partial x_0}\right)_{y_0} = \left(\frac{\partial C}{\partial x_0}\right)_{x_f,y_0,y_f} + \left(\frac{\partial C}{\partial x_f}\right)_{x_0,y_0,y_f}\left(\frac{\partial x_f}{\partial x_0}\right)_{y_0,y_f}$$

$$+\left\{\left(\frac{\partial C}{\partial x_f}\right)_{x_0,y_0,y_f}\left(\frac{dx}{dy}\right)_f + \left(\frac{\partial C}{\partial y_f}\right)_{x_0,x_f,y_0}\right\}\left(\frac{\partial y_f}{\partial x_0}\right)_{y_0}, \quad (8.9.4)$$

where $(dx/dy)_f$ denotes the final value of the instantaneous casualty-exchange ratio for $t = t_f$, $x = x_f$, and $y = y_f$. TAYLOR [88, pp. 100-101] has also shown how the reparameterization (8.9.3) leads to

$$\frac{\partial x_f}{\partial x_0} = \left(\frac{\partial x_f}{\partial x_0}\right)_{y_0,y_f}$$

$$= \exp\left[-\int_{y_f}^{y_0}\frac{\partial}{\partial x}\left(\frac{dx}{dy}\right)dy\right]$$

$$-\int_{y_f}^{y_0}\left(\frac{\partial t}{\partial x_0}\right)\frac{\partial}{\partial t}\left(\frac{dx}{dy}\right)\exp\left[-\int_{y_f}^{y}\frac{\partial}{\partial x}\left(\frac{dx}{dy_1}\right)dy_1\right]dy, \quad (8.9.5)$$

which relates the instantaneous casualty-exchange ratio $dx/dy = G(t,x,y)/H(t,x,y)$ to changes in the final X force level with variations in X's initial strength. This result (8.9.5) is a key one that TAYLOR has used to develop most of the results of his paper [88]. Through (8.9.4) and (8.9.5) one can many times determine the

sign of $\partial C/\partial x_0$ from only the signs of $\partial(dx/dy)/\partial x$ and $\partial(dx/dy)/\partial t$ without explicit calculation of $\partial C/\partial x_0$. Along these lines, TAYLOR has proved the following results for a <u>fixed-force-level-breakpoint battle</u>.

THEOREM 8.9.1 (TAYLOR [88]): If $\partial(dx/dy)/\partial x < 0$ and $\partial(dx/dy)/\partial t \geq 0$ for all $t \in [0,t_f]$, then $\partial C/\partial x_0 < 0$ for $C = L_X, R_c, D_c$.

THEOREM 8.9.2 (TAYLOR [88]): Assume that $dx/dy = q(t,u)$ where $u = x/y$ and that the LANCHESTER-type equations (8.9.1) are quasi-autonomous, i.e. $\partial/\partial t(dx/dy) \equiv 0$. If $dx/dy = q(u)$ is a strictly convex (concave) function of $u$ on $[0,+\infty)$, then the decision criterion $C$ is a strictly convex (concave) function of $x_0$ for $C = L_X, R_c, D_c$.

The latter theorem tells us that there are decreasing marginal returns from initially committing additional forces to battle when $q(u)$ is convex and $\partial C/\partial x_0 < 0$ for all $x_0 \in [x_0^{min}, x_0^{max}]$.

TAYLOR [88] has also developed corresponding results for fixed-force-ratio-breakpoint battles and has investigated optimality results for both classes of battles when the sign of $\partial(dx/dy)/\partial x$ is always the same. He has shown that the optimal initial-commitment decision is sensitive to the decision criterion for fixed-force-ratio-breakpoint battles but not for fixed-force-level-breakpoint battles. In other words, different optimal initial-commitment actions are possible in these

two types of battles. In particular, the loss ratio and the loss difference may yield different optimal initial-commitment decisions for a fixed-force-ratio-breakpoint battle, although they yield the same optimal decision for a fixed-force-level-breakpoint battle (see TAYLOR [88] for further details and additional results). Similar results on the sensitivity of optimal time-sequential fire-distribution policies to battle-termination conditions have been pointed out by the author (see TAYLOR [79; 93] or Section 8.11 below). Consequently, we feel that more scientific work is required on modelling conflict termination[25] (see Chapter 3 for further information and references).

Thus, the reader has seen that a fairly sophisticated mathematical analysis has been required to justify the simple, intuitively appealing "optimal decision rule" given under more restrictive conditions in Section 2.9 (see TAYLOR [88] for further details): namely, if the instantaneous casualty-exchange ratio (friendly to enemy) always decreases as the force ratio (enemy to friendly) decreases, then additional forces should be committed to battle by the victor (friendly forces). Conversely, a simple principle underlies all this mathematical analysis: the casualty-exchange ratio "in the small" may under the appropriate conditions be projected to "in the large."

8.10. The Simplest Fire-Distribution Problem.

The simplest fire-distribution problem is for a homogeneous Y force (e.g. riflemen only) to determine its "best" time-sequential allocation of fire against a heterogeneous X force consisting of two weapon-system types (e.g. riflemen and grenadiers), denoted as $X_1$ and $X_2$ (see Figure 8.2). Y's distribution of fire may be quantified through the fraction of fire directed at $X_1$, denoted as $\phi$. The problem for the Y commander then is to determine the "best" value over time for $\phi$, denoted as $\phi^*(t)$. For simplicity's sake[26], we will assume that the Y-force commander has perfect information about the battle's current state and also about all parameters in the attrition processes. Before we can determine an optimal fire-distribution policy $\phi^*(t)$ for Y, however, we must complete the formulation of this combat-optimization problem, which as yet lacks the first three basic elements (E1) through (E3) given in Section 8.4. In other words, we must still specify the following basic elements (cf. Section 8.4) of the combat-optimization problem before it can be mathematically solved: namely, (E1) the decision criterion, (E2) the stopping rule for the battle (i.e. the model of conflict termination), and (E3) the model of combat dynamics.

Again for simplicity's sake, we will assume that the objective of the Y force's commander is to maximize the net value of survivors at the battle's end when such survivors are valued according to linear utilities. Following our developments for homogeneous-force models (see Chapters 2 and 6), we will assume that the battle continues until one or the other has been totally annihilated, which is readily recognized as

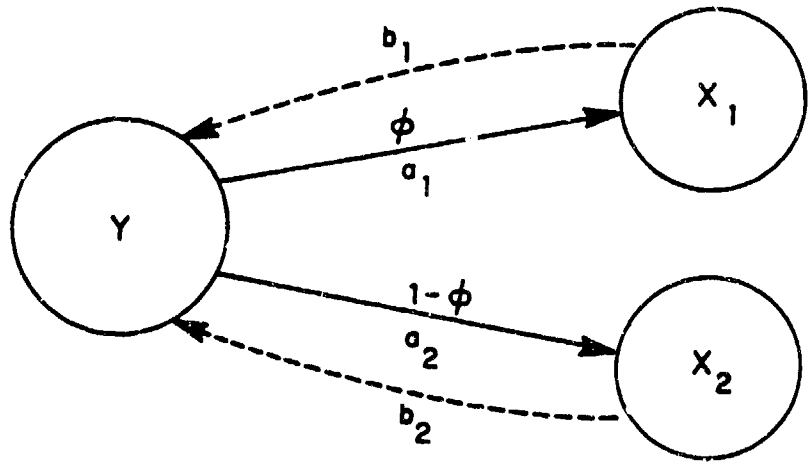


Figure 8.2. The simplest fire-distribution problem. Here $\phi$ denotes the fraction of Y's fire directed at $X_1$. The optimization problem for the Y commander is to determine the "best" time-sequential value for $\phi$, denoted as $\phi^*(t)$.

the simplest conflict-termination model. Later, we will discuss more general breakpoints (<u>see</u> Chapter 3) below. Furthermore, we will assume that all attrition occurs at rates proportional to the numbers of enemy firers and that there are no synergistic effects between the $X$ forces (i.e. the attrition rates of $X_1$ and $X_2$ against $Y$ are additive). For simplicity, we will also assume constant attrition-rate coefficients. Such attrition processes may be thought of as arising when firers engage enemy targets with "aimed fire" and (for example) target-acquisition times are negligible (<u>see</u> Sections 2.2 and 6.1 [also 7.8] for a further discussion of these modelling assumptions). Finally, we will assume that all the $Y$ force's fire may be instantaneously shifted from one X-force target type to the other (i.e. perfect command-and-control capability for the $Y$ force), and we will discuss the relaxing of this last assumption in the next section.

In mathematical terms, the above fire-distribution problem for the $Y$ force may be stated as follows.

$$\underset{\phi(t)}{\text{maximize}}\{ry(T) - px_1(T) - qx_2(t)\} \quad \text{with } T \text{ unspecified}, \tag{8.10.1}$$

with stopping rule: one side or the other annihilated at $t = T$,

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y ,$$

$$\text{(combat dynamics)} \quad \frac{dx_2}{dt} = -(1-\phi)a_2 y ,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2,$$

with initial conditions:

$$x_i(0) = x_i^0 \quad \text{for} \quad i = 1,2, \quad \text{and} \quad y(0) = y_0 ,$$

and

$$0 \leq \phi \leq 1 \quad \text{(Control-Variable-Inequality Constraints)},$$

$$x_1 \text{ and } x_2 \geq 0 \quad \text{(State-Variable-Inequality Constraints)},$$

where

$x_1(t)$, $x_2(t)$, and $y(t)$ denote the numbers of $X_1$, $X_2$, and $Y$ combatants at time $t$,

$a_1$, $a_2$, $b_1$, and $b_2$ denote constant LANCHESTER attrition-rate coefficients (*cf*. Section 7.8),

$T$ denotes the time at which one side or the other is annihilated (i.e. the length of the battle),

$r$, $p$, and $q$ denote the values assigned to single surviving $X_1$, $X_2$, and $Y$ combatants at the end of the battle,

and $\phi$ denotes the fraction of the $Y$ force which fires at the $X_1$ force.

Here we say that $T$ (the time at which one side or the other is annihilated and the battle ends) is unspecified (as opposed to specified in which case the battle ends at $t_f = T$ unless one side or the other is annihilated before this time) because it depends on $Y$'s fire-distribution policy $\phi(t)$.

Such a one-sided combat-optimization problem in which the combat dynamics are modelled by a system of ordinary differential equations is called an <u>optimal control problem</u>[27]. In particular, the above problem (8.10.1) is in many ways the simplest optimal-control problem that arises in the LANCHESTER theory of combat. It has been referred to in the literature (<u>see</u> TAYLOR [76]) as the <u>ISBELL and MARLOW fire-programming problem</u>. Consequently, the development of a complete solution to this problem along with appropriate solution methodology has been essential for guiding extensions to more complex situations. The author has accordingly viewed this problem as a "benchmark case" to which the treatment (both theoretical and computational) of more complicated problems should be related[28]. Moreover, several important insights into the structure of optimal fire-distribution policies in more general cases have been obtained from studying this simple problem (e.g. <u>see</u> TAYLOR [79; 89; 93]).

The optimal time-sequential fire-distribution policy $\phi^*(t)$ for $0 \leq t \leq T$ may be determined by invoking the appropriate optimality conditions from the mathematical theory of optimal control[29]. However, these optimality conditions are only the point of departure for determining an optimal policy. For a problem such as (8.10.1), a solution procedure consisting of the following steps is required (<u>see</u> TAYLOR [76, p. 542] for further details):

(S1) apply the basic necessary conditions of optimality (a key element of which is the so-called maximum principle) to determine an extremal[30] control law,

(S2) synthesize extremals and the corresponding extremal control by working backwards from each terminal state (i.e. determine the time history of the extremal control),

(S3) using the time history of the extremal control, determine the domain of controllability[31] for each terminal state by a forward integration of the state differential equations,

(S4) establish that an optimal policy exists (e.g. see TAYLOR and BROWN [89, pp. 200-201]) and then determine which (if any) domains of controllability overlap; the extremal control is then optimal for those regions of the initial state space covered by only one domain of controllability,

(S5) if certain domains of controllability overlap, then for a point in the initial state space contained in their intersection there is more than one extremal leading to the terminal surface; compute the return associated with each extremal in order to select the optimal control from a finite number of alternatives.

The above solution procedure has been used by us to solve the above simplest fire-distribution problem (see TAYLOR [76; 84]) and other LANCHESTER-type optimal-control problems.

Although this problem (8.10.1) looks quite simple, the development of a complete solution to it (see TAYLOR [76; 84]) has led to a couple of contributions to the control-theory literature on optimality conditions (see TAYLOR [77; 81]). The reason for this is that such combat-optimization problems contain certain mathematical features that are somewhat different than those usually encountered in other dynamic optimization problems arising in the physical sciences, engineering, and other parts of OR. To best appreciate these mathematical difficulties, it is convenient to consider the following generalization of the simplest fire-distribution problem.

$$\underset{\phi(t)}{\text{maximize}}\ J\ , \tag{8.10.2}$$

with stopping rule: one side or the other annihilated,

$$\begin{aligned} \text{subject to:}\quad & \frac{dx_1}{dt} = -\phi a_1 y + r_1\ , \\ \text{(combat dynamics)}\quad & \frac{dx_2}{dt} = -(1-\phi) a_2 y + r_2\ , \\ & \frac{dy}{dt} = -b_1 x_1 - b_2 x_2\ , \end{aligned}$$

with initial conditions:

$$x_i(0) = x_i^0 \quad \text{for } i = 1,2, \quad \text{and} \quad y(0) = y_0 ,$$

and

$$0 \leq \phi \leq 1 \quad \text{(Control-Variable-Inequality Constraints)},$$

$$x_1 \text{ and } x_2 \geq 0 \quad \text{(State-Variable-Inequality Constraints)},$$

where $J$ denotes the criterion functional, $r_i \geq 0$ for $i = 1,2$ denotes a constant replacement rate for $X_i$, and all other symbols are as defined before. A particular difficulty in solving a LANCHESTER-type optimal-control problem such as (8.10.2) has concerned optimality conditions associated with the state-variable-inequality constraints (SVICs). For example, when $r_1$ and $r_2 > 0$, the boundary of the state space is non-absorbing (see TAYLOR [81] for a discussion of the concept of an absorbing state-space boundary), and we have the following boundary condition for the dual variable corresponding to $x_i$

$$p_i(T) = \frac{\partial J}{\partial x_i(T)} + \nu_i , \tag{8.10.3}$$

where $p_i(t)$ denotes the dual variable corresponding to $x_i(t)$ for $i = 1,2$ and $\nu_i \geq 0$. However we need not have $\nu_i \geq 0$ when there are no replacements (i.e. $r_1 = r_2 = 0$) and the boundary of the state space is absorbing (see TAYLOR [84, pp. 632-633]).

Before we consider the optimal policy for the simplest fire-distribution problem (8.10.1), a few general comments seem in order. To solve a LANCHESTER-type optimal-control problem such as (8.10.1) or (8.10.2), one needs to know what regions of initial force levels lead to the various end states of battle (i.e. one needs to know the domain of controllability for each terminal state), and this requirement has partially motivated our work on victory-prediction conditions for LANCHESTER-type combat models (e.g. see Sections 3.5, 3.6, 6.6, and 6.13 above). Moreover, both considerations "in the small" and also considerations "in the large" are required to solve such problems (see TAYLOR [84, pp. 617-618] for further details). Thus, direct computation of the payoff and comparison of such values has been involved in the development of optimal combat strategies in many of the dynamic combat-optimization problems studied by the author (e.g., see TAYLOR [80] or TAYLOR and BROWN [89]).

Using the above solution procedure consisting of steps (S1) through (S5), one can analytically solve the above simplest fire-distribution problem (8.10.1) in so-called "closed form." After much laborious work (see TAYLOR [76; 84]), one can determine the optimal fire-distribution policy. Unfortunately, it is too complicated to be given in its entirety (although we will examine it in a few special cases), but it has been completely given for all parameter values in TAYLOR [76] (with some further refinements given in TAYLOR [84]) as an open-loop control (see Section 8.12 below), i.e. $\phi^* = \phi^*(t; t_0, x_1^0, y_0)$. What is important for us here is that the essential characteristics of an optimal

fire-distribution policy, denoted as $\phi^*$, may be summarized as follows:

(C1)[32] $\phi^*$ is always 0 or 1 (except for at most one point in time),

(C2) parameters on which the optimal policy depends are

(P1) whether Y wins or loses,

(P2) $R = a_1b_1/(a_2b_2)$,

(P3) $\delta = a_1p/(a_2q)$ .

Moreover, there are some important military interpretations of the above parameters: (I) $a_ib_i$ is a measure of the strategic value to Y from firing at $X_i$ (rate of destruction of $X_i$'s kill capability against Y), and (II) $a_1p$ is a measure of short-run return to Y from firing at $X_1$ at the end of battle (rate of destruction of $X_1$ value at the end of battle).

A significant aspect of the[33] optimal fire-distribution policy, expressed as a closed-loop control (see Section 8.12 below), is that it depends on the force levels alone and not on time, i.e. $\phi^* = \phi^*(x_1,x_2,y)$. This result is remarkable because the maximum principle does not directly involve the state variables (i.e the force levels) when the Hamiltonian is maximized for $x_1$ and $x_2 > 0$. Furthermore, the optimal policy for Y may be different for different combat outcomes (i.e. whether Y wins or loses). Assuming that $R = a_1b_1/(a_2b_2) > 1$, then if Y is going to win, $\phi^* = 1$ for $x_1 > 0$. If Y is going to lose, then the optimal fire-distribution policy depends on another parameter, $\delta = a_1p/(a_2q)$, and may be very complicated to express as a closed-loop control.

When Y is going to lose, the general features of Y's optimal fire-distribution policy may be described as follows. Let $p = k(1 + \gamma)b_1$ and $q = kb_2$, where $k$ is a positive constant and $\gamma$ is a parameter that reflects whether Y has valued an individual $X_1$ survivor at the end of battle more $(\gamma > 0)$ or less $(\gamma < 0)$ than in direct proportion to the $X_1$ survivor's kill capability against Y relative to that of an individual $X_2$ survivor. Here kill capability is measured in terms of the kill rate against Y of a single $X_i$ firer, and $\gamma = 0$ yields that $p/q = b_1/b_2$. From the above definition of $\gamma$, it follows that $p = q(b_1/b_2)(1 + \gamma)$ and consequently $\gamma = -1 + \delta/R$. Moreover, the following results are significant to note: (R1) $\gamma = 0$ means that surviving enemy weapon-system types are valued in direct proportion to their kill capabilities; (R2) for $\gamma \geq -(1 - 1/R)$, the optimal policy is very simple: $\phi^* = 1$ for $x_1 > 0$; (R3) for[34] $-(1 - 1/R) > \gamma \geq -\sqrt{1 - 1/R}$, it is complicated to determine the optimal policy; and (R4) for $-\sqrt{1 - 1/R} > \gamma \geq -1$, it is very complicated to determine the optimal policy. In the latter two cases[35], it may be that $\phi^*$ is initially 1 and then changes to 0 later with $x_1 > 0$. When this change occurs is the complicated part (see TAYLOR [84] for further details).

Let us now discuss what important military principles may be deduced from the solution to the ISBELL and MARLOW fire-programming problem. Firstly, from the fact that $\phi^*$ is essentially always 0 or 1, we have a quantitative justification of one of the most significant and oft-quoted of NAPOLEON BONAPARTE's sayings (see LIDDELL HART [54, p. 117])—"The principles of war are the same as those of a siege; fire must be concentrated at one point." Secondly, from the fact that when

Y is going to win (or when he is going to lose with $\delta \geq 1$) the optimal policy is to always concentrate all fire on the available enemy target type with largest $a_i b_i$, we have a quantitative justification of the military principle of attacking "those dangerous enemy targets against which one's fire is most effective." Thirdly, we have a motivation for valuing enemy target types in direct proportion to their kill capability (fire effectiveness) from the fact that the optimal policy is both intuitively appealing and also very simple in this case. The HOWES and THRALL [39] concept of "ideal" linear weights is an extension of this idea to cases of heterogeneous forces on both sides. Thus, we have a motivation for HOWES and THRALL's important military-valuation methodology (see Section 7.18). Fourthly, in battle a commander must use his judgment to ascertain to what ends the course of battle can be steered so that he may devise his strategy accordingly. Computationally this means that to solve such a problem one must know to which extremal end states[36] the battle can be steered (i.e. what force levels are required to drive the LANCHESTER-type battle to a target set such that appropriate necessary conditions of optimality are satisfied at the end). In other words, it turns out that considerations "in the large" dominate obtaining the optimal policy in such problems.

Let us next turn to some important computational aspects of the simplest fire-distribution problem (8.10.1). We will illustrate one of the computational difficulties (multiple extremals) in determining an optimal policy alluded to above [recall steps (S1) through (S5) of the computational procedure given above]. In Table 8.VI are shown the results of applying to (8.10.1) the maximum principle in Step (S1) of

TABLE 8.VI. Extremals for ISBELL-MARLOW PROBLEM FOR $R - \sqrt{R(R-1)} < \delta < 1$.

Nonrestrictive Assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$.

Case (c2): $R - \sqrt{R(R-1)} < \delta < 1$ where $\delta = a_1p/(a_2q)$.

| Terminal State | Extremal Control | Domain of Controllability |
|---|---|---|
| $C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$ | $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$ | $a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^0)^2$<br>$a_1b_1y_0^2 > s^2 - (b_2x_2^0)^2$ |
| $C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$ | $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$ | $a_1b_1y_0^2 > s^2 + (R-1)(b_2x_2^0)^2$ |
| $C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$ | $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$ | $a_1b_1y_0^2 \ge R\{s^2 - (b_1x_1^0)^2\}$<br>$a_1b_1y_0^2 \le s^2 + A(b_2x_2^0)^2$ |
| $C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$ | $\phi^*(t) = 0$ for $0 \le t \le T$ | $a_1b_1y_0^2 \le Rs^2\{1 - 1/z^2\}$<br>$a_1b_1y_0^2 \le R\{s^2 - (b_1x_1^0)^2\}$ |
| $C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$ | $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases}$ | $a_1b_1y_0^2 > Rs^2\{1 - 1/z^2\}$<br>$a_1b_1y_0^2 > s^2 + A(b_2x_2^0)^2$<br>$a_1b_1y_0^2 < s^2 + B(b_2x_2^0)^2$ |

Definition of Times:

(a) $t_1$ is first $t$ such that $x_1(t_1) = 0$.

(b) $t_2$ is first $t$ such that $2b_1x_1(t_2)x_2^0 + b_2(x_2^0)^2 = a_2y^2(t\ )$.

(c) $\tau_1$ is determined by $\cosh \sqrt{a_2b_2}\, \tau_1 = (R-\delta)/(R-1)$.

this solution procedure. Here the parameters A, B, s, and z are defined as

$$A = \frac{z^2(R-1) - R}{(z-1)^2}, \qquad B = A\,\frac{(z-1)^2}{z^2} = \frac{z^2(R-1) - R}{z^2},$$
$$s = b_1x_1^0 + b_2x_2^0, \quad \text{and} \quad z = \frac{R - \delta}{R - 1}. \tag{8.10.4}$$

Thus, we see that A and B have the same sign, and investigation of the dependence of this sign on $\delta$ leads to the following four cases:

(c1) $1 \leq \delta$,

(c2) $R - \sqrt{R(R-1)} < \delta < 1$,

(c3) $\delta = R - \sqrt{R(R-1)}$,

(c4) $0 \leq \delta < R - \sqrt{R(R-1)}$,

where $\delta = a_1p/(a_2q)$. Case (c2) with $A < B < 0$ is the one shown in Table 8.VI, which has been developed by working backwards from each extremal end state of battle. If the initial force levels are such that $P_0 = (x_1^0, x_2^0, y_0)$ belongs to the domain of controllability (see [76]) for the terminal state $C_1$, denoted as $D(C_1)$, then Y can steer the course of battle to this end state with the open-loop extremal control shown in the table. Moreover, it turns cut that several of the domain of controllability shown in Table 8.VI overlap so that for a given set of

initial force levels there may be more than one candidate optimal course of battle. In order to determine which extremal is actually optimal in such cases, one can compute the return associated with each extremal from a given initial point $P_0$ and then determine which of these feasible alternatives (a finite number) yields the greatest return (see TAYLOR [84, pp. 633-634] for further details and justification). This procedure [i.e. steps (S4) and (S5) of the general solution procedure given above] has been followed to obtain the optimal (open-loop) fire-distribution policy shown in Table 8.VII from the information of Table 8.VI. An outline of the determination of the optimal policy for regions of the initial state space with multiple extremals will now be sketched (see TAYLOR [76; 84] for complete details).

We will now indicate how step (S5) is carried out for the simplest fire-distribution problem (8.10.1) for Case (c2), i.e. $R - \sqrt{R(R-1)} < \delta < 1$, which is the one shown in Tables 8.VI and 8.VII. Let $D(C_i)$ denote the domain of controllability for extremals leading to terminal state $C_i$, and let $P_i$ denote the payoff (i.e. return) associated with such an extremaly leading to $C_i$. Then it has been shown (TAYLOR [84]) for $R - \sqrt{R(R-1)} < \delta < 1$ that for terminal state, for example, $C_1$ the domain of controllability is as shown in Table 8.VI and that the return associated with such an extremal is given by

$$P_1 = \left(\frac{-q}{b_2 R}\right) \sqrt{R} \quad \sqrt{s^2 + (R-1)(b_2 x_2^0)^2 - a_1 b_1 y_0^2} \quad . \tag{8.10.5}$$

Using such results, one can show [84, Theorems A1, A2, and A3] by direct computation of the return functional (considerations "in the large")

TABLE 8.VII. Solution to ISBELL-MARLOW Problem for $R - \sqrt{R(R-1)} < \delta < 1$.

Nonrestrictive Assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$

Case (c2): $R - \sqrt{R(R-1)} < \delta < 1$ where $\delta = a_1p/(a_2q)$

<u>Terminal State</u>

$$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases} \quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases} \quad \begin{array}{l} a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^0)^2 \\ a_1b_1y_0^2 \ge s^2 + B(b_2x_2^0)^2 \end{array}$$

$$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases} \quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases} \quad a_1b_1y_0^2 > s^2 + (R-1)(b_2x_2^0)^2$$

$$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases} \quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases} \quad \begin{array}{l} a_1b_1y_0^2 \ge R\{s^2 - (b_1x_1^0)^2\} \\ a_1b_1y_0^2 \le s^2 + A(b_2x_2^0)^2 \end{array}$$

$$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases} \quad \phi^*(t) = 0 \quad \text{for } 0 \le t \le T \quad \begin{array}{l} a_1b_1y_0^2 \le Rs^2\{1 - 1/z^2\} \\ a_1b_1y_0^2 < R\{s^2 - (b_1x_1^0)^2\} \end{array}$$

$$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases} \quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases} \quad \begin{array}{l} a_1b_1y_0^2 > Rs^2\{1 - 1/z^2\} \\ a_1b_1y_0^2 > s^2 + A(b_2x_2^0)^2 \\ a_1b_1y_0^2 < s^2 + B(b_2x_2^0)^2 \end{array}$$

See Table 8.VI for definition of times $t_1$, $t_2$, and $\tau_1$.

that for $R - \sqrt{R(R-1)} < \delta < 1$

$$\text{(a)} \quad P_4(P^0) > P_1(P^0) \quad \text{for all} \quad P^0 \in \{D(C_1) \cap D(C_4)\},$$

$$\text{(b)} \quad P_5(P^0) > P_1(P^0) \quad \text{for all} \quad P^0 \in \{D(C_1) \cap D(C_5)\},$$

$$\text{(c)} \quad P_5^S(p^0) > P_1(P^0) \quad \text{for all} \quad P^0 \in \{D(C_1) \cap D(C_5^S)\}.$$

It may also be shown that $D(C_4) \cap D(C_5) = \Phi$, $D(C_4) \cap D(C_5^S) = \Phi$, and $D(C_5) \cap D(C_5^S) = \Phi$, where $\Phi$ denotes the empty set. The above results (a) through (c) provide the basis for obtaining the optimal fire-distribution policy shown in Table 8.VII from the extremals shown in Table 8.VI.

Next, it seems appropriate to briefly discuss[37] the extension of the above simplest problem (8.10.1) to cases of more realistic breakpoints, in particular, force-level breakpoints (see Section 2.8 and Chapter 3). There are several different ways in which breakpoint considerations can be incorporated into our combat model. The simplest way is to consider $X_1$ and $X_2$ to be two different fighting units. If one considers $X_1$ and $X_2$ as two different military units (each with its own breakpoint), then we could invoke the natural extension of the simple breakpoint model (2.8.12) of Section 2.8 and write for Y's attrition

$$\frac{dy}{dx} = \begin{cases} -b_1x_1 - b_2x_2 & \text{for } x_1 > x_{BP}^1 \text{ and } x_2 > x_{BP}^2 , \\ -b_1x_1 & \text{for } x_1 > x_{BP}^1 \text{ and } x_2 \leq x_{BP}^2 , \\ -b_2x_2 & \text{for } x_1 \leq x_{BP}^1 \text{ and } x_2 > x_{BP}^2 , \end{cases} \tag{8.10.6}$$

where $x_{BP}^1$ denotes the force-level breakpoint for $X_1$. To mathematically solve such a problem [i.e. (8.10.1) with Y's attrition rate replaced by (8.10.6)] and determine the optimal fire-distribution policy, one considers the battle to have different phases in each of which the appropriate right-hand side of (8.10.6) holds. The determination of an optimal policy is now, however, much more complicated than before (cf. TAYLOR [96, Appendix C]) and complete details have not been worked out. If one feels that a more sophisticated breakpoint model is called for [e.g. the natural extension of (3.10.10)], then the problem is analytically even less tractable. However, for either modification, it is conjectured that the basic structure of the optimal fire-distribution policy is not altered. Thus, the incorporation of more realistic breakpoints into the simplest fire-distribution problem leads to a problem that is no longer analytically tractable but that does not yield an optimal fire-distribution policy which is appreciably different in structure than that for the simplest problem. However, the computational solution of ths more complicated problem is facilitated by the insights gained here for the simplest problem (8.10.1).

Finally, let us note that the very striking characteristic (C1) of an optimal fire-distribution policy of always concentrating all fire on one enemy target type depends in an essential way on enemy target-type attrition occurring at a rate proportional to the number of Y firers. If the attrition of enemy target types is modelled by

$$\frac{dx_1}{dt} = -\phi a_1 x_1 y, \qquad \text{and} \qquad \frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y \ , \tag{8.10.7}$$

then $\phi^*$ does not always have to be 0 or 1: it can sometimes be optimal to divide one's fire between enemy target types (i.e. $0 < \phi^* < 1$) for a finite interval of time (see below in the next section for further details; also TAYLOR [78-79]).

## 8.11. Optimal Control of LANCHESTER-Type Attrition Processes.

Based on the intimate relationship between the mathematical theories of optimal control and differential games (e.g. see HO [33-34]), the author's research approach for investigating the optimization of tactical decisions has been to consider one-sided[38] versions of time-sequential tactical-allocation problems before tackling the more realistic (and complex) two-sided tactical-allocation problems themselves. Our intent has been to firmly establish both the theoretical[39] and computational bases for solving such optimal-control problems before attempting to solve the much more complex differential-game versions of these tactical-allocation problems. This does not mean that the author does not recognize that solutions to differential games have many unique aspects not possessed by solutions to optimal-control problems (e.g. see ISAACS [47-48]), but that in order to recognize such unique aspects and attendant special difficulties, one must know and understand the optimization results for these one-sided versions of tactical-allocation problems. As discussed in TAYLOR [79, pp. 102-103], we have used such one-sided combat-optimization results for guiding extensions to LANCHESTER-type differential games (see next section).

A number of variations on the simplest time-sequential fire-distribution problem (8.10.1) have consequently been examined by the author in order to develop an understanding of how various factors (cf. Section 8.4) influence the structure of optimal tactical decision making. These variations are listed in Table 8.VIII, with references being given as to where such investigations have been reported in the literature (see also

TABLE 8.VIII. Variations of the Simplest Fire-Distribution Problem (8.10.1) that Have Been Examined to Provide Insights Into the Optimal Control of LANCHESTER-Type Attrition Processes.

(V1) Prescribed-duration versus fight-to-the-finish battle-termination conditions [79]

(V2) Time-dependent attrition-rate coefficients and replacements [79; 82]

(V3) n-versus-one combat [92, Appendix E; 79; 82]

(V4) Command and control aspects [97]

(V5) Heterogeneous-force $FT|F$ attrition process [78-79]

(V6) Stochastic LANCHESTER-type attrition processes [98; 30]

(V7) Time-sequential fire-support allocation [89; 96]

TAYLOR [92, pp. 59-64]). It should be pointed out that the author's work [76, 84] on the simplest fire-distribution problem (8.10.1) has been essential for guiding these extensions and establishing a framework for interpreting and analyzing results on the structure of optimal time-sequential fire-distribution policies. We will now briefly highlight this work, usually providing a formulation of the optimal-control problem under consideration.

In variation (V1) (see TAYLOR [79]) of the simplest fire-distribution problem (8.10.1), one replaces the stopping rule: "one side or the other annihilated" by

$$\text{with stopping rule:}\quad t_f = T \ \text{ or one side or the other annihilated at } t_f < T\,, \tag{8.11.1}$$

where $t_f$ denotes the final battle time (i.e. the time at which the engagement ends) and $T$ denotes a specified time beyond which the battle cannot last (see also TAYLOR [92, Appendix G]). We will refer to a battle with the stopping rule (8.11.1) as a prescribed-duration battle [as opposed to a terminal-control battle such as (8.10.1) that only ends by the battle being steered to a given end-of-battle state]. In TAYLOR [79] we found it convenient to summarize the variations (on the simplest fire-distribution problem) considered there as shown in Table 8.IX, with the above variation (V1) denoted there as Problem 1 and the simplest problem (8.10.1) as Problem 3. For the fire-distribution problem (8.10.1) with stopping rule (8.11.1), i.e. a prescribed-duration battle, the

TABLE 8.IX. Summary of Problems Considered in TAYLOR [79] to Study the Effects of Model Form on Optimal Fire-Distribution Policy.

| Problem | Number of Target Types | Target-Type Attrition Process | Attrition-Rate Coefficients | Battle-Termination Conditions |
|---|---|---|---|---|
| 1 | 2 | F | C | PD |
| 2 | 2 | F | C | TC |
| 3 | n | F | C | PD |
| 4 | 2 | F | V | PD |
| 5 | 2 | FT | C | PD |

<u>EXPLANATION OF SYMBOLS</u>

Target-Type Attrition Process: F = attrition rate proportional to number of firers only, FT = attrition rate proportional to product of numbers of firers and targets

Attrition-Rate Coefficients: C = constant, V = variable

Battle-Termination Conditions: PD = prescribed-duration battle (special case of $x_1$, $x_2$, $y > 0$), TC = terminal-control battle (fight to the finish)

optimal fire-distribution policy $\phi^*$ again turns out to be 0 or 1 for at most one point in time, but now $\phi^*$ depends on time $t$ in addition to the force levels, i.e. $\phi^*(\text{Problem 1}) = \phi^*(t, x_1, x_2, y)$, and this dependence (see TAYLOR [92, Appendix G] for complete details) is much more complicated than for the terminal-control battle[40]. Again, let us make the nonrestrictive assumption that $R = a_1b_1/(a_2b_2) > 1$. We then have shown [79] that for the special case in which $\delta = a_1p/(a_2q) < 1$ and $x_1(t_f)$, $x_2(t_f)$, and $y(t_f) > 0$, the optimal fire-distribution policy depends on the problem's battle-termination conditions (i.e. it may be different for Problems 1 and 2). On the other hand, when $\delta \geq 1$, the optimal fire-distribution policy is the same for both problems: namely, $\phi^* = 1$ as long as $x_1 > 0$.

In analytical terms, we have for $\delta = a_1p/(a_2q) < 1$ and $x_1(t_f)$ and $x_2(t_f) > 0$

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t < t_f - \tau_1 , \\ 0 & \text{for } t_f - \tau_1 \leq t \leq t_f , \end{cases} \tag{8.11.2}$$

where $\phi^*(t)$ denotes the optimal distribution of fire over time and the backwards switching time $\tau_1$ is given by

$$\tau_1(\alpha) = \frac{1}{\sqrt{a_2b_2}} \ \ell n \left\{ \frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right\} \tag{8.11.3}$$

with $z = (R - \delta)/(R - 1)$. Thus, in this case with $\delta < 1$ an optimal fire-distribution policy involves a switch from all fire concentrated on $X_1$ by $Y$ to all on $X_2$ when the initial force levels are such that

neither enemy target type can be annihilated and the battle is scheduled to last long enough, i.e. $T > \tau_1$. Then the nature of the planning horizon affects the optimal fire-distribution policy in the sense that for the appropriate initial conditions in both problems [i.e. initial force levels such that[41] at the battle's end $x_1(t_f)$ and $x_2(t_f) > 0$, with also $y(t_f) > 0$ in the prescribed-duration battle with $t_f = T$], $R = a_1b_1/(a_2b_2) > 1$, $\delta = a_1p/(a_2q) < 1$, and $r > 0$ (see TAYLOR [79, pp. 86-87] for further details)

$$\tau_1(\text{Problem 1}) < \tau_2(\text{Problem 2}) . \tag{8.11.4}$$

Furthermore, the optimal fire-distribution policy in the prescribed-duration battle depends on an additional parameter

$$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}} \quad ,$$

since the backwards switching time $\tau_1$ may depend on $\alpha$, i.e. for $x_1(t_f)$, $x_2(t_f)$, and $y(t_f) > 0$

$$\tau_1(\text{Problem 1}) = \tau_1\left(\frac{r}{q}\sqrt{\frac{b_2}{a_2}}\right) \tag{8.11.5}$$

with $\tau_1(\alpha)$ given by (8.11.3). Let us also note that for $x_1(T)$ and $x_2(T) > 0$ and $y(T) = 0$

$$\tau_1(\text{Problem 2}) = \tau_1(0) . \tag{8.11.6}$$

The fact that $\partial\tau_1/\partial\alpha < 0$ for $\delta < 1$ and the above results (8.11.5) and (8.11.6) lead[42] to the conclusion (8.11.4). Finally, it should be noted that this case [i.e. $R = a_1b_1/(a_2b_2) > 1$ and $\delta = a_1p/(a_2q) < 1$] only arises when Y values a unit of the $X_2$ force out of proportion to its kill rate against the Y force (i.e. too high) relative to that of one of the $X_1$ force, i.e. $p/q < b_1/b_2$. In other words, the more dangerous weapon-system type is valued less highly, e.g. a rifle is valued more than a machine gun.

A typical problem along the lines of variation (V2) is given by (see TAYLOR [79, pp. 97-99; 82])

$$\underset{\phi(t)}{\text{maximize}}\{ry(t_f) - px_1(t_f) - qx_2(t_f)\}, \tag{8.11.7}$$

with stopping rule: $t_f = T$ or one side of the other annihilated at $t_f < T$,

subject to: (combat dynamics)

$$\frac{dx_1}{dt} = -\phi a_1(t)y + r_1(t) ,$$

$$\frac{dx_2}{dt} = -(1-\phi)\ a_2(t)y + r_2(t) ,$$

$$\frac{dy}{dt} = -b_1(t)x_1 - b_2(t)x_2 + s(t) ,$$

with

$$0 \leq \phi \leq 1 \quad \text{(Control-Variable-Inequality Constraints)},$$

and

$$x_1 \text{ and } x_2 \geq 0 \quad \text{(State-Variable-Inequality Constraints)}.$$

Here (and henceforth) we have omitted statement of the initial conditions for simplicity. Also, nonnegativity of a term like $r_1(t)$, $r_2(t)$, or $s(t)$ signifies a net continuous influx of replacements for the weapon-system type corresponding to the force-level equation in which such a term appears. This problem has been fairly extensively studied [79, Problem 4; 82] under the assumption that

$$b_i(t) = k_{b_i} h(t) \qquad \text{for } i = 1, 2, \tag{8.11.8}$$

which may be considered to have the physical interpretation that both X-force weapon-system types have basically the same type of range capability, but one weapon-system type dominates the other in exactly the same manner at all ranges (cf. the model with range-dependent attrition-rate coefficients in Section 6.2). When there are no replacements or withdrawals (i.e. $r_1(t) \equiv r_2(t) \equiv s(t) \equiv 0$) and the Y commander values enemy survivors of each weapon-system type in direct proportion to their kill rate against the Y force at the end of battle,[43] i.e.

$$p = kb_1(t_f) \qquad \text{and} \qquad q = kb_2(t_f) \ , \tag{8.11.9}$$

then the optimal fire-distribution policy takes a very simple form

when $x_1(t_f)$ and $x_2(t_f) > 0$, namely

$$\phi^*(t) = \begin{cases} 1 & \text{for } a_1(t)\, b_1(t) \geq a_2(t)\, b_2(t) \,, \\ 0 & \text{for } a_1(t)\, b_1(t) \leq a_2(t)\, b_2(t) \,. \end{cases} \tag{8.11.10}$$

Here, the term $a_1(t)\, b_2(t)$ may be interpreted as the rate of destruction of the $X_1$-weapon-system-type kill rate against the $Y$ force (see TAYLOR [79] for further details). Thus, the $Y$ force simply concentrates all its fire on the enemy weapon-system type against which it can destroy the weapon-system type's fire effectiveness (i.e. kill rate against $Y$) more quickly. When there are continuous replacements, however, determination of an optimal policy is much more complicated, and certain multiplier conditions (e.g. see TAYLOR [77; 81]) corresponding to the state-variable-inequality constraints (SVIC's) play an even more prominent role: in particular, the multiplier corresponding to the terminal SVIC, for example, $x_1(t_f) \geq 0$ is restricted in sign only if $r_1(t_f) > 0$ (cf. (8.10.3) and see TAYLOR [81] for further details). Also, many of our results in TAYLOR [82] (also [79, Problem 4]) have been based on our knowledge about the conditions under which variable-coefficient $F|F$ attrition equations possess a simple analytical solution in terms of elementary functions (see Section 6.5 for further details).

Another variation (V3) of the simplest fire-distribution problem (8.10.1) is to consider the $X$ force to be composed of more target types, e.g.

$$\underset{\phi(t)}{\text{maximize}}\{vy(t_f) - \sum_{i=1}^{n} w_i x_i(t_f)\} \quad \text{with } T \text{ specified,} \tag{8.11.11}$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = -\phi_i a_i y \qquad \text{for } i = 1,2,\ldots, n,$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i x_i ,$$

with

$$\sum_{i=1}^{n} \phi_i \leq 1, \quad \phi_i \geq 0, \quad x_i, y \geq 0, \quad \text{and} \quad t_f \leq T .$$

A rather illuminating result (see TAYLOR [92, Appendix E; 82]) is that again when the Y-force commander values surviving enemy weapon-system types in direct proportion to their kill capabilities against the Y force, i.e.

$$w_i = kb_i \qquad \text{for} \quad i = 1, 2, \ldots , n , \tag{8.11.12}$$

then the optimal fire-distribution policy for Y is very simple: always concentrate all fire on the available enemy target type for which $a_i b_i$ is largest. When survivors are not valued in direct proportion to their dangerousness against the Y force [i.e. when (8.11.12) does not hold], then determination of an optimal policy may be quite involved (see TAYLOR [79] for further details; also TAYLOR [92, Appendix G]). A variable-coefficient version of (8.11.11) has been investigated in TAYLOR [82], and results for the optimal distribution of fire shown to resemble the constant-coefficient ones under the appropriate circumstances.

A fourth variation (V4) on the simplest fire-distribution problem (8.10.1) is to consider how command and control limitations on the redistribution of fire influence the structure of optimal time-sequential fire-distribution policies. In all the fire-distribution problems so far considered, it has been assumed that Y's distribution of fire against the heterogeneous X forces can instantaneously change from one value to another, e.g. in the simplest fire-distribution problem (8.10.1) the rate of change of the fraction $\phi$ of Y's fire directed at $X_1$ is unrestricted and consequently can instantaneously change, for example, from 0 to 1. In other words, we have been assuming that the Y force can instantaneously change their distribution of fire against enemy target-types at will. Command and control limitations, however, may cause restrictions on how fast fire can be redistributed, i.e. restrictions on the rate of change of $\phi^*$. Such command and control aspects have been investigated with the following optimal-control problem (see TAYLOR [97] for futher details)

$$\underset{u(\cdot)}{\text{maximize}}\{ry(t_f) - px_1(t_f) - qx_2(t_f)\} \quad \text{with } T \text{ specified}, \qquad (8.11.13)$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y \,,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y \,,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2 \,,$$

$$\frac{d\phi}{dt} = u \,,$$

with

$$0 \le \phi \le 1, \quad x_1, x_2, y \ge 0, \quad -R_L \le u \le R_U, \text{ and } t_f \le T.$$

Here $R_U$ and $R_L > 0$ denote upper and lower bounds on the rate of change of the distribution of fire. It has been shown [97] that such command and control limitations on the redistribution of fire do not essentially change the structure of the optimal fire-distribution policy, although the shifting of fires is initiated earlier when command and control limitations exist than when an entire force can instantaneously shift all its fire from one target type to another. In other words, due to decreased reaction ability a force must begin to change its distribution of fire before target priorities actually change in anticipation of this coming change.

A fifth variation (V5) (see TAYLOR [78; 79]) concerns changing the functional form of the $Y$ force's attrition rate against each enemy target type to the case in which such an attrition rate is proportional to the product of the numbers of firers and targets. For simplicity we have denoted this variation as "heterogeneous-force FT|F attrition." The optimal-control problem corresponding to the prescribed-duration-battle version of the simplest fire-distribution problem then reads

$$\underset{\phi(t)}{\text{maximize}}\{ry(t_f) - px_1(t_f) - qx_2(t_f)\} \quad \text{with } T \text{ specified,} \qquad (8.11.14)$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 x_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 x_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

with

$$0 \le \phi \le 1, \qquad x_1, x_2, y \ge 0, \qquad \text{and} \qquad t_f \le T.$$

There is a fundamental difference between the structure of an optimal fire-distribution policy for the simplest problem (8.10.1) and that for the above problem (8.11.14): when enemy target types undergo an "FT attrition process" (cf. Table 8.VIII), the optimal distribution of fire does not consist of always (except for a finite number of points in time) concentrating all fire on a single enemy target type. In other words (cf. the optimal policy described in Section 8.10 for the simplest problem), $\phi^*(t)$ may be other than 0 or 1 for a finite interval of time (cf. the solutions for Problems 1 and 5 in TAYLOR [79]). The maximum principle is no longer adequate, and the so-called theory of singular extremals (see TAYLOR [78] for further information) is required to solve the above optimal-control problem (8.11.14), with $\phi_S^*$ such that $0 < \phi_S^* = a_2/(a_1 + a_2) < 1$ being the "singular control." In this case the optimal fire-distribution policy depends directly on the force levels (and possibly time). In TAYLOR [78] it was shown for constant attrition-rate coefficients that no change ever occurs in the ranking of target priorities when survivors of each X-force weapon-system type are valued in direct proportion to their kill rate against Y (i.e. $p = kb_1$ and $q = kb_2$), and this important result is independent of whether both X-force target types undergo an "F attrition process" or an "FT attrition process."

We will now briefly examine the above problem's optimal fire-distribution policy expressed as a closed-loop control (see Section 8.12 below) and graphically exhibited in state-space-decision-rule diagrams. The optimal time-sequential fire-distribution policy for (8.11.14) in the case

in which $p/q = b_1/b_2$ (i.e. enemy survivors valued in direct proportion to their kill-rate capabilities) is graphically depicted in Figure 8.3. When $a_1b_1x_1 > a_2b_2x_2$, the optimal policy is for Y to concentrate all fire on $X_1$. The line with equation $a_1b_1x_1 = a_2b_2x_2$ (denoted as L in Figure 8.3) is called a singular "surface" and divides the state space into two different decision regions. When a force-level trajectory reaches L, the optimal policy says that fire should be divided between the two target types in such a way that the trajectory stays on L (i.e. the singular control

$$\phi_S^* = \frac{a_2}{a_1 + a_2} \tag{8.11.15}$$

is used to remain on the singular "surface"). Thus, when $p = kb_1$ and $q = kb_2$, the optimal fire-distribution policy may be expressed very simply as

$$\phi^*(x_1,x_2) = \begin{cases} 1 & \text{for } a_1b_1x_1 > a_2b_2x_2, \\ a_2/(a_1 + a_2) & \text{for } a_1b_1x_1 = a_2b_2x_2, \\ 0 & \text{for } a_1b_1x_1 < a_2b_2x_2. \end{cases} \tag{8.11.16}$$

When enemy survivors of each weapon-system type are not valued in direct proportion to their kill rate against Y (e.g. $p/q > b_1/b_2$), the situation is more complicated, with the battle being divided into two phases as far as describing the optimal fire-distribution policy is concerned. For the case in which $p/q > b_1/b_2$, the optimal policy is graphically depicted in Figure 8.4.

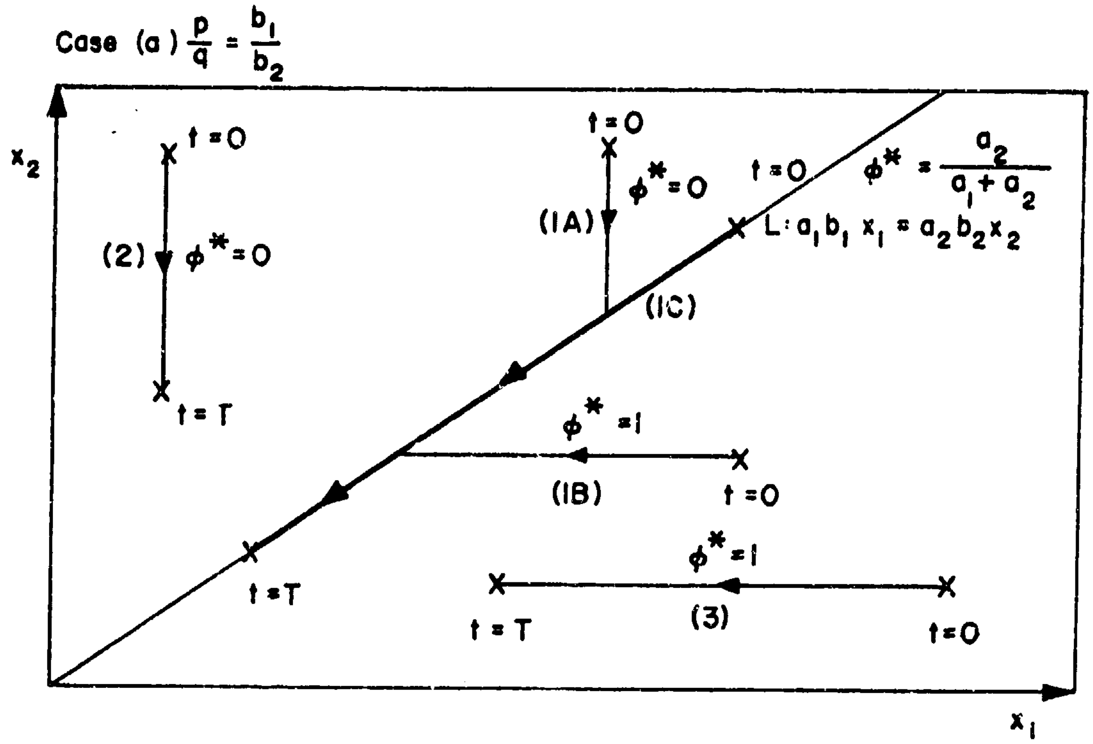


Figure 8.3. Optimal fire-distribution policy and corresponding battle trajectories in the state space for heterogeneous-force FT|F attrition process when surviving weapon-system types are valued in direct proportion to their kill rates. The optimal battle trajectories identified in this figure are discussed in detail in TAYLOR [78, pp. 686-688].

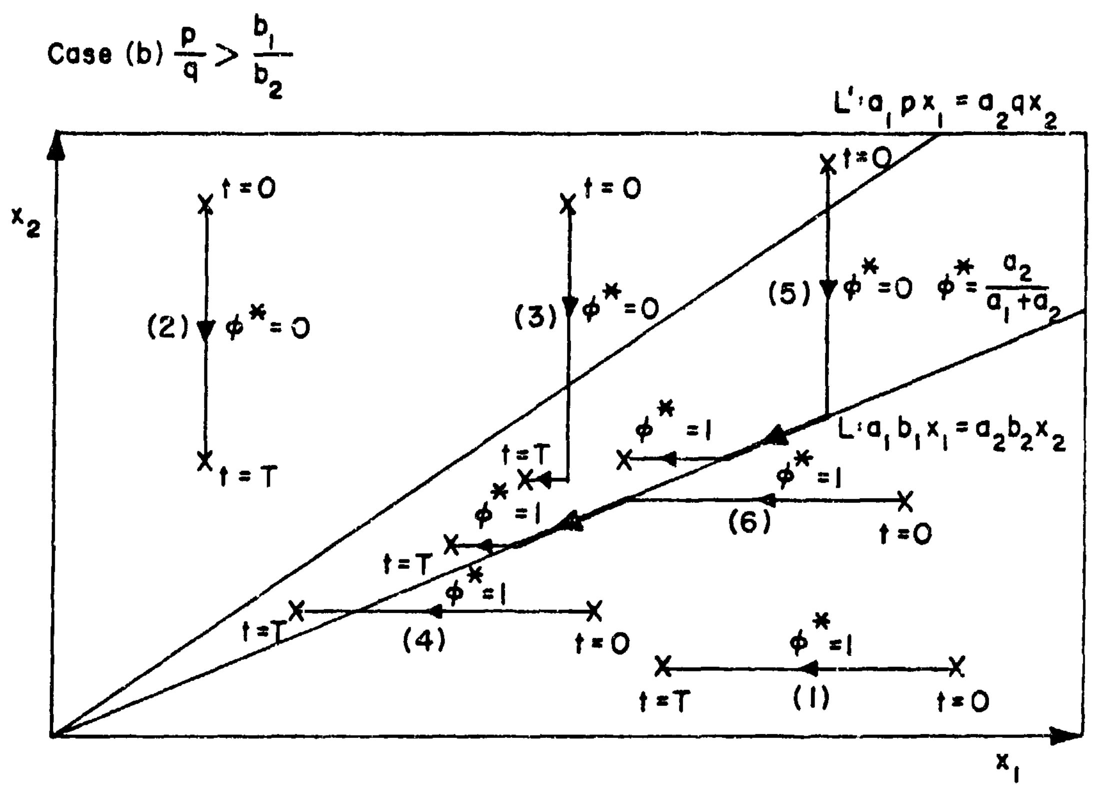


Figure 8.4. Optimal fire-distribution policy and corresponding battle trajectories in the state space for heterogeneous-force FT|F attrition process when surviving weapon-system types are not valued in direct proportion to their kill rates (here case in which $p/q > b_1/b_2$). The optimal battle trajectories identified in this figure are discussed in detail in TAYLOR [78, pp. 688-690].

For describing the optimal fire-distribution policy, we divide the battle into two time phases: Phase I for $0 \le t \le t_I$ and Phase II for $t_I \le t \le T$. During Phase I the optimal fire-distribution policy is again given by (8.11.16), but during Phase II the optimal policy is given by (cf. Figure 8.4)

$$\phi^*(t,x_1,x_2) = \begin{cases} 1 & \text{for } a_1px_1 > a_2qx_2, \\ 0 & \text{for } a_1px_1 < a_2qx_2. \end{cases} \tag{8.11.17}$$

Further details are to be found in TAYLOR [78].

At this juncture it seems appropriate for us to briefly make a few remarks about how the functional form for the attrition rates of enemy target types influcences the structure of an optimal fire-distribution policy. In particular, we will compare the structure of an optimal time-sequential fire-distribution policy when each enemy target type undergoes an "F-type attrition process" (i.e. the attrition rate for each enemy target type is proportional to only the number of friendly firers) to that when each enemy target type undergoes an "FT-type attrition process" (i.e. the attrition rate proportional to the product of the numbers of firers and targets). As we have seen above in both the simplest fire-distribution problem (8.10.1) and also the corresponding prescribed-duration battle (Problems 1 and 2 of [79]), an optimal fire-distribution policy when each enemy target type undergoes an F-type attrition process consists of always concentrating all fire on a single enemy target type,

while an optimal fire-distribution policy when each enemy target type undergoes an FT-type attrition process as in (8.11.14) may (depending on the densities of enemy target types) sometimes involve dividing one's fire between the two enemy target types[44]. In this latter case, an optimal policy basically has the property that one concentrates all fire on one target type until the relative number of enemy target types reaches an equilibrium point, and fire is then divided between the two target types. In essence, one must guard against "overkill" when each enemy target type undergoes an FT-type attrition process (cf. the optimal policies shown in Figures 8.3 and 8.4).

Moreover, there is a very simple principle that underlies all the above results about the dependence of the structure of an optimal fire-distribution policy on the functional form for the attrition rates of enemy target types: an optimal allocation policy involves concentration of all effort on a single alternative when there are constant marginal returns (measured in terms of kill rate) over time from each alternative[45] and the total effort available is limited. Furthermore, constant marginal return over time is a basic property of an F-type target-type attrition process. This important result is readily seen by considering the attrition of, for example, $X_1$ (with $\phi = 1$) in (8.10.1), namely

$$\frac{\left(-\frac{dx_1}{dt}\right)}{y} = a_1 = \begin{pmatrix} \text{rate of enemy casualties produced} \\ \text{per unit of } Y \text{ weapon system} \end{pmatrix}. \quad (8.11.18)$$

Thus, there is the same constant marginal return at any point in the battle from the Y force allocating fire against a particular enemy

target type when each undergoes an F-type attrition process. This situation should be contrasted with the corresponding one for an FT-type enemy-target-type attrition process, i.e.

$$\frac{\left(-\dfrac{dx_1}{dt}\right)}{y} = a_1 x_1 = \left(\begin{array}{l}\text{rate of enemy casualties produced} \\ \text{per unit of } Y \text{ weapon system}\end{array}\right). \qquad (8.11.19)$$

In this latter case, however, the marginal return from allocating fire diminishes over time as the $X_1$ force level decays, and consequently a division of total effort (i.e. allocation of fire) in an optimal policy may be called for when the number of this particular target type is sufficiently reduced. B. O. KOOPMAN's [52] 1953 article on the optimal distribution of effort contains an excellent discussion of such principles that underlie an optimal allocation policy determined by such an optimization problem (see also TAYLOR [79, pp. 84-85]).

Another important variation (V6) considers casualties to occur randomly over time (see Chapter 4). TAYLOR and POWERS [98] have investigated a stochastic version of variation (V1) above (i.e. Problem 1 of [79]) in which casualties are assumed to follow stochastic LANCHESTER-type attrition processes (see Chapter 4). They considered the following problem.

$$\underset{\phi}{\text{maximize}}\ E[rN(t_f) - pM_1(t_f) - qM_2(t_f)] \quad \text{with } t_f \text{ specified,} \tag{8.11.20}$$

subject to: casualties occur randomly as a continuous-time MARKOV chain with stationary transition probabilities corresponding to the deterministic heterogeneous-force F|F attrition process (8.10.1),

with

$$M_1, M_2, N \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1.$$

Here $\phi$ is taken to be a closed-loop control (see Section 8.12 below), the integer-valued random variables $M_1(t)$, $M_2(t)$, and $N(t)$ denote the $X_1$, $X_2$, and $Y$ force levels, and $E[\cdot]$ denotes mathematical expectation. TAYLOR and POWERS [98] have concluded that the deterministic and stochastic versions of this time-sequential fire-distribution problem yield essentially the same optimal policy, although the optimal policy followed by $Y$ in a realization of the stochastic combat process may differ appreciably from that for the deterministic formulation if this realization does not "follow the corresponding deterministic trajectory very closely." Furthermore, HANNA [30] has shown for a fight to the finish that conditions do exist for which the deterministic and stochastic formulations do not yield similar results at all for the optimal fire-distribution policy for very small numbers of combatants.

Further variations [identified as (V7) in Table 8.VIII] on such LANCHESTER-type deterministic optimal-control problems have been investigated by TAYLOR [96] and TAYLOR and BROWN [89] within the context

of time-sequential fire-support allocation. TAYLOR [96] has considered 10 variations on the same theme (i.e. a sequence of 10 closely related fire-support problems), with some of these variants being investigated much more thoroughly than others. This investigation exercises many of the insights into the structure of optimal fire-distribution policies discussed above. TAYLOR and BROWN [89] have shown that the structure of such optimal policies depends not only on the functional form assumed for target-type attrition rates (e.g. F or FT as shown in Table 8.IX) but also on the quantification of military objectives. They have proven the rather remarkable result for a given set of combat dynamics that the splitting of the allocation of supporting fires between two enemy forces in any optimal policy depends on whether the terminal payoff reflects the objective of attaining an "overall" military advantage or a "local" one.

## 8.12. LANCHESTER-Type Differential Games.

Military conflict provides the classical contextual framework for game theory: two or more decision makers with conflicting objectives. Moreover, combat models in general and LANCHESTER-type models in particular provide a natural framework for formulating and analyzing time-sequential games that reflect the antagonistic aspects of military decision making. We will accordingly consider a couple of LANCHESTER-type (as opposed to pursuit-evasion) differential games, which have provided some important insights into normative aspects of the dynamics of combat. A differential game is simply a time-sequential game (i.e. game in extensive form [55]) in which the system dynamics are given by a system of ordinary differential equations. Others have found it to be convenient to think of a differential game as a two-sided optimal-control problem (e.g. see HO[33]). By a LANCHESTER-type differential game we mean a differential game in which the system dynamics are given by LANCHESTER-type equations of warfare. It should be pointed out that essentially all the early differential-game literature has concerned pursuit-evasion problems (however, see ISAACS [46, pp. 96-104 and Chapter 11] for notable exceptions).

More precisely, we will consider LANCHESTER-type differential games that are two-person zero-sum deterministic differential games in which each player uses a closed-loop (or feedback) pure strategy with perfect state information (see HO [34; 35] for a discussion of other possibilities). In other words, each of the two decision makers has his own (scalar) criterion functional which he seeks to maximize but

which is in direct antagonistic conflict with his opponent's in the sense that the two criterion functionals have a constant sum (which may be taken to be zero) so that one person's loss is the other person's gain. Each player (i.e. decision maker) is taken to have perfect information about the system state and combat dynamics, but each does not know the strategy of his opponent. Since a differential game is a game in extensive form, a pure (as opposed to mixed) strategy (within the context of perfect state information) is a decision rule for determining one's action based on the current system state, i.e. a mapping of the state space into the space of feasible actions at time $t$. Such a pure strategy for a game in extensive form is also called a closed-loop (as opposed to open-loop) strategy. Mathematically we may express the concept of a closed-loop (or feedback) control as

$$u_C = k(t,x) , \tag{8.12.1}$$

where $u_C$ denotes the closed-loop control (or strategy), $t$ denotes time, $x$ denotes the state variables, and $k$ denotes the given functional relation (i.e. the decision rule). Equation (8.12.1) shows us that a closed-loop strategy is a function of the current system state. On the other hand, an open-loop control specifies one's action as a function of time $t$ and initial conditions $t_0$, $x_0$. Thus, an open-loop control may be mathematically expressed during the length of the planning horizon for $0 \leq t \leq T$ as

$$u_O = u(t;\ t_0,\ x_0) , \tag{8.12.2}$$

where $u_0$ denotes the open-loop control (or strategy). For one-sided deterministic optimal control problems, it is well known that open-loop control and closed-loop control yield identical results both for the system trajectory and also for the payoff, but this situation is not true for differential games (e.g. see HO [35]). Consequently, one must distinguish between open-loop and closed-loop strategies as we have done here.

We will now give two examples of LANCHESTER-type differential games. Although we will not present any solution details here, the selection of these examples has been influenced both by their analytical tractability and also by the significance of insights that they provide into optimizing time-sequential tactical decisions.

Example 8.12.1: Generalized Tactical Air-War Game. This problem is a generalization of R. ISAACS's [46, pp. 96-104] tactical air-war game, which apparently owes its origin to A. S. MENGEL (see [27]). It considers a war between X and Y, each of which is composed of ground and air forces. The progress of the ground war is measured in terms of the position of the contact zone between the opposing ground forces or FEBA (Forward Edge of the Battle Area) (see Section 7.15 for further details). Both X and Y have a single type of aircraft that can fly two types of missions: (M1) ground-support missions against the enemy's ground forces to influence the outcome of the land war in terms of FEBA position,

and (M2) <u>counter-air missions</u> which result in the shooting down of enemy planes (but not direct help for the ground forces). The problem for each of the two opposing commanders is to find the "best" time-sequential allocation of his aircraft to mission type according to the decision criterion of the sum of the net residual value of surviving aircraft at the end of the campaign (and measured with linear utilities) and the net amount of value obtained from ground-support missions flown (and measured in terms of the return from planes dropping ordnance on the FEBA). These objectives of the opposing commanders are taken to be directly conflicting (i.e. the two payoffs have a constant sum), and thus it suffices to consider a single scalar payoff which one player seeks to maximize and the other to minimize. Also, the air campaign is taken to last for a prescribed length of time, denoted as T, and it is assumed that new aircraft are introduced on both sides at constant rates. This situation is shown diagramatically in Figure 8.5.

Mathematically the above two-sided combat-optimization problem may be stated as follows.

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\{v_X\, x(t_t) - v_Y\, y(t_f) + \int_0^{t_f} [R_X(t)ux - R_Y(t)vy]dt\}, \qquad (8.12.3)$$

with stopping rule: $t_f - T = 0$ ,

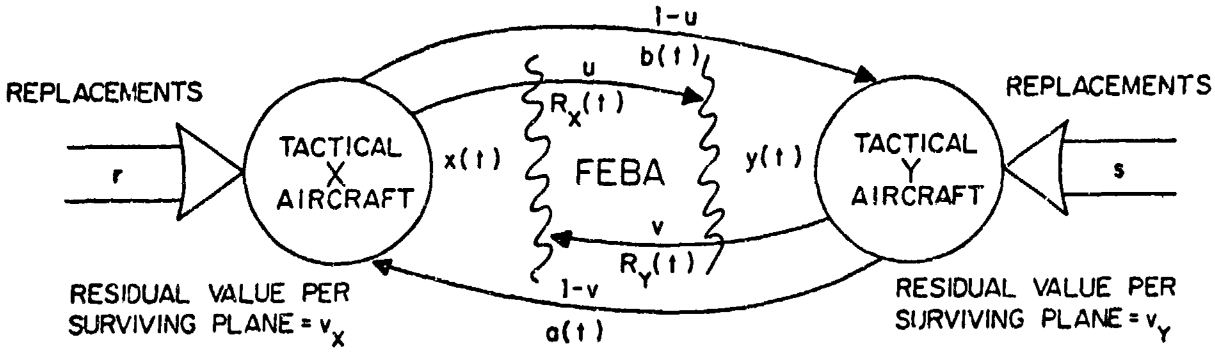


Figure 8.5. Diagram of generalization of tactical air-war game (8.12.3).

subject to: $\frac{dx}{dt} = r - (1 - v)\ a(t)y$ ,

(air-battle dynamics) $\frac{dy}{dt} = s - (1 - u)\ b(t)x$ ,

with initial conditions:

$$x(0) = x_0 \quad \text{and} \quad y(0) = y_0 ,$$

and

$0 \leq u,\ v \leq 1$ (Strategic-Variable-Inequality Constraints),

$x$ and $y \geq 0$ (State-Variable-Inequality Constraints),

where

$x(t)$ and $y(t)$ denote the numbers of X and Y aircraft at time t,

$a(t)$ and $b(t)$ denote time-dependent attrition-rate coefficients representing the effectiveness of aircraft in shooting down enemy aircraft,

$r$ and $s > 0$ denote constant replacement rates for each side's aircraft,

$v_X$ and $v_Y$ denote the values for each surviving X and Y aircraft at the end of the campaign,

$R_X(t)$ and $R_Y(t) > 0$ denote the time-dependent returns per unit time obtained from flying an X and Y ground-support missions,

$u(t)$ and $v(t)$ are strategic variables that denote the fractions of X and Y aircraft allocated to flying ground-support missions at time t,

and $t_f$ denotes the final campaign time.

Here the strategic (or control) variables $u(t)$ and $v(t)$ are taken to represent the outcomes (or realizations) of closed-loop strategies[46], e.g. $u(t) = U(t,x,y)$. A further discussion of this model and its rather long history is to be found in TAYLOR [94, Appendix B], and optimal air-war allocation strategies for the above LANCHESTER-type differential game are developed there, with complete details being worked out for the special case of constant coefficients (see also TAYLOR [83]).

Example 8.12.2: Modified Fire-Support Differential Game. This problem is a variation of Y. KAWARA's [51] fire-support differential game and considers the attack of heterogeneous X forces against the static defense of heterogeneous Y forces. Each side is composed of infantry and artillery. The X infantry (denoted as $X_1$) launches an attack against the position of the Y infantry (denoted as $Y_1$). We will consider only the battle's "approach-to-contact" phase that lasts from the start of the advance of the $X_1$ forces against the $Y_1$ defensive position until contact is made between them. It is assumed that this latter time is fixed and known to both sides. Using "cover and concealment," the $X_1$ forces begin their advance against the $Y_1$ forces from a distance and move towards the Y position. Small-arms fire by the $X_1$ forces is held at a minimum to facilitate their movement, and hence the effectiveness of $X_1$'s fire "on the move" will be assumed to be negligible against $Y_1$. Since the $X_1$ forces are so far away from the defenders, $Y_1$

is assumed to use "area fire" against the attacking $X_1$ forces. During this "approach to contact," the fire-support units (i.e. each side's artillery) remain stationary and deliver either counterbattery fire against enemy artillery or "area fire" against the enemy's infantry. By virtue of its defensive posture, the $Y$ force obtains better information about the location of the $X$ fire-support units, and hence $Y_2$ can deliver "aimed counterbattery fire" against $X_2$, but $X_2$ can only return "area counterbattery fire" against $Y_2$. It is the objective of each side to attain the most favorable infantry force ratio possible at the end of the "approach to contact" at which time "hand-to-hand" combat occurs between the two infantries and consequently artillery fire can no longer be directed at the enemy's infantry for safety reasons. The decision problem facing each side is to determine the "best" time-sequential distribution of artillery fire in order to maximize the infantry force ratio at the time of "hand-to-hand" contact between the two infantries. Again, the objectives of the two opposing commanders are taken to be directly conflicting, and thus it suffices to consider a single scalar payoff which one player seeks to maximize and the other to minimize. This situation is shown diagramatically in Figure 8.6.

Mathematically the above two-sided combat-optimization problem may be stated as follows.

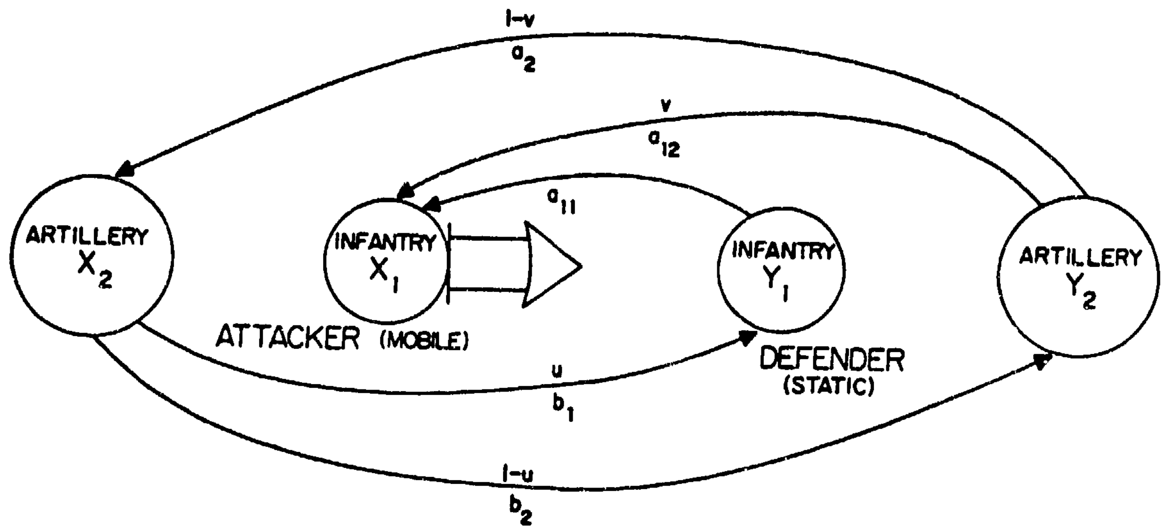


Figure 8.6. Diagram of modified fire-support differential game (8.12.4).

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}} \left\{\frac{x_1(t_f)}{y_1(t_f)}\right\}, \tag{8.12.4}$$

with stopping rule: $t_f - T = 0$

subject to: $\frac{dx_1}{dt} = -a_{11}x_1y_1 - va_{12}x_1y_2$,

(battle dynamics) $\frac{dx_2}{dt} = -(1-v)a_2y_2$,

$$\frac{dy_1}{dt} = -ub_1y_1x_2,$$

$$\frac{dy_2}{dt} = -(1-u)b_2y_2x_2,$$

with initial conditions:

$$x_i(0) = x_i^0 \quad \text{and} \quad y_i(0) = y_i^0 \quad \text{for } i = 1,2,$$

and

$0 \leq u,\ v \leq 1$ (Strategic-Variable-Inequality Constraints),

$x_1$, $x_2$, $y_1$, and $y_2 \geq 0$ (State-Variable-Inequality Constraints),

where

$x_1(t)$ and $y_1(t)$ denote the numbers of X and Y infantry at time t,

$x_2(t)$ and $y_2(t)$ denote the numbers of X and Y artillery at time t,

$a_{11}$, $a_{12}$, $a_2$, $b_1$, and $b_2$ denote constant LANCHESTER attrition-rate coefficients,

$u(t)$ and $v(t)$ are strategic variables that denote the fraction of X and Y artillery fire allocated against opposing infantry forces,

and $t_f$ denotes the final "approach-to-contact" time.

Again, the strategic (or control) variables $u(t)$ and $v(t)$ are realizations of closed-loop strategies, e.g. $u(t) = U(t, x_1, x_2, y_1, y_2)$. A further discussion of this model and its history is to be found in TAYLOR [86], and optimal fire-support strategies for the above LANCHESTER-type differential game are developed there.

Other LANCHESTER-type differential games (besides those found in ISAACS' book [46]) have been studied by WEISS [103], CHATTOPADHYAY [19; 20], INTRILIGATOR [42], MOGLEWER and PAYNE [60], KAWARA [51], STERNBERG [75], and TAYLOR [80; 85]. These differential games are generally only partially solved, with a lot of work usually producing only rather meager results. It should also be finally noted that a number of closely related discrete-time-sequential games have been investigated by both analytical and also computational means (e.g. see FULKERSON and JOHNSON [25], BELLMAN and DREYFUS [8], BERKOVITZ and DRESHER [11-13], PUGH [68], BRACKEN, FALK, and KARR [15], and GOHEEN [29].

## 8.13. Insights Gained

Based on our studies of the optimization of combat dynamics [76-98] using generalized control theory, we have learned the following:

(A) The structure of optimal time-sequential combat strategies depends on all the following five factors:

(1) the decision criteria,

(2) the battle-termination model,

(3) the combat-operations model,

(4) the feasible actions for each decision maker,

and (5) the information available to each decision maker.

The dependence is complex, and future research should concentrate on simplified models of tactical interest to explore further how optimal strategies depend on these factors.

(B) Force levels always effect optimal combat strategies. The dependence may be indirect, however, through who "wins" and "loses."

(C) The quantification of combatant objectives affects optimal combat strategies. The most important planning decision is whether to seek a "local" military advantage or an "overall" one.

(D) The time-sequential nature of target effects has a significant influence on optimal fire-support strategies. Furthermore, optimization of fire-support strategies should be based on ground-support objectives.

(E) It may be quite dangerous to generalize optimal time-sequential combat strategies from specific problems. More research should be done on better understanding the qualifications that should be placed on such specific results.

The above insights are illustrative of those salient features about optimizing tactical decisions that we have uncovered in our work. A further discussion about insights gained into the optimization of combat dynamics may be found in TAYLOR [92, pp. 61-64; 94, pp. 8-9; 96, pp. 12-15], where a discussion about the implications of such results for defense planners is also contained. Although all these insights have been developed within the context of specific problems, most of the properties of the structure of an optimal time-sequential combat strategy appear to be of general applicability. As we have stressed in the introduction to this chapter, such insights into the structure of optimal combat strategies are probably the only significant result obtained from this work, since the underlying mathematical models are such idealizations of the (rational) decision-making process in force-on-force combat operations.

8.14. Role of Optimization in Decision Analysis for Tactical Military Decisions

Here we will make a few final comments about considering such combat-optimization problems in the quantitative study of tactical (as opposed to strategic) decision making. These remarks are meant to stimulate further thought and discussion, rather than providing any final definitive answers.

The author feels that the most important current issue is to determine the role of normative models in tactical decision analysis. What exactly is the role of optimization in tactical military decision making? Optimization problems arising from the modelling of tactical decision making with any degree of realism in the modelling of combat are too large scale for even contemporary computing capabilities. If we cannot optimize the detailed simulated system, what should we do? The interchange of ideas between military gaming (e.g. see SHUBIK [72]; an excellent reference is still THOMAS and DEEMER [101]) and combat optimization (as outlined above) needs to be stimulated. In particular, mathematical programmers involved in such work should become more aware of the analysis and modelling of combat operations, since they give special structure to such optimization problems. The modelling of such complex systems necessarily must precede system optimization, and the author views the latter as but an extension of the former.

As we have stressed in the past (see TAYLOR [79]), more work should concentrate on developing exact optimal solutions to "approximate" models of combat operations in order to develop a better understanding of

how to really improve tactical decision making (both in the model world and also in the real world). After all, the purpose of combat optimization is insight, not numbers.[47]

FOOTNOTES FOR CHAPTER 8

1. This chapter is an expansion upon TAYLOR [87, pp. 778-779 and pp. 793-801]. It is also partially based on portions of the author's unpublished paper "Survey on the Optimal Control of Lanchester-Type Attrition Processes," presented at the Symposium on the State-of-the-Art of Mathematics in Combat Models, June 1973 (available in report from as TAYLOR [93]).

2. More precisely, generalized control theory is the mathematical theory of optimizing the performance of a dynamic system (see Section 1.6 above for a discussion of the concept of a dynamic system). The term "generalized control theory" was apparently first coined by Y. C. HO [34] in 1969 (see also HO [35]). It includes both deterministic and stochastic optimal control, dynamic programming, and differential games (see HO [34] for further details).

3. Actually, these "decision variables" are really decision functions, since they are functions defined on some time interval (e.g. $\phi = \phi(t)$ for $0 \leq t \leq T$). The term decision variable is probably used in analogy with the term state variable, which also evolves dynamically over time.

4. See TAYLOR [76-88], TAYLOR and BROWN [89], TAYLOR [91-97], and TAYLOR and POWERS [98] for documentation of the author's research on the structure of optimal time-sequential combat strategies.

5. These operational combat models have been discussed (including the nature and availability of documentation about them) in Section 1.3 above (see also Section 7.1).

6. For the mathematical modelling of rational choice under conflict of interests, see LUCE and RAIFFA [55] or SHUBIK [72]. For an excellent investigation of methodology for determining how people actually make decisions in a non-conflicting environment (i.e. no conflict of interests), see WILCOX [104].

7. Such a one-sided time-sequential optimization problem is called an optimal-control problem. Relatively recent mathematical interest (and also that of other scientists and technolgists) in optimal-control theory stems from the work of PONTRYAGIN and his associates on the mathematical theory of optimality conditions for such problems (e.g. see PONTRYAGIN et al. [67]; see also HESTENES [32]).

8. We are using here the word strategy to denote a game-theoretic strategy, i.e. a completely specified plan of action which covers all contingencies (e.g. see SHUBIK [72, p. 42]). We then use the word policy to denote a "strategy" in a one-sided optimization (or optimal control) problem, i.e. a control. In military circles, the word strategy has a different meaning, the plans for conducting a war in the widest sense including diplomatic, political, and economic considerations as well as those of a purely military nature [31] (see also LUTTWAK [56, p. 183]). One then uses the word tactics to refer to the method employed by a commander to implement his strategic plan [31] (see also [56, p. 199]).

9. Here we are using the words strategy and tactics as usually used by military planners and not in the game-theoretic sense (<u>see</u> Footnote 8 above).

10. <u>See</u> WEISS [103] for a brief discussion of the distinction between a "primary" weapon system (e.g. infantry) and a "supporting" weapon system (e.g. artillery, tactical aircraft, etc.).

11. For an excellent general discussion of the modelling of tactical decisions for use in combat models, <u>see</u> ANDERSON [1].

12. It is beyond the scope of this monograph to give a detailed treatment of war gaming, but we will attempt here to outline some further reading for those who are interested. Excellent introductions are afforded by PAXSON [66] (a brief introduction) and McHUGH [57] (a longer introduction which includes a historical summary) (<u>see</u> also SHUBIK [72]). For a very readable and informative popular account of war gaming, <u>see</u> WILSON's book [105], which apparently draws heavily on McHUGH's work [57]. A very thorough historical summary (unfortunately, only through the late 1950's) is YOUNG [107]. For other excellent accounts of operational gaming and its role in military OR, <u>see</u> THOMAS and DEEMER [101] and THOMAS [99; 100]. Although somewhat dated, the references [99-101] are still an excellent introduction to gaming, probably still the best technical one in the military field. Other more recent accounts are by SHEPHARD [71], ARCHER and BYRNE [4], SHUBIK [72], and especially [74]. P. BRACKEN [16] has discussed through some very interesting historical case studies some

very subtle difficulties in the use of war-gaming results. SHUBIK's book [72] not only provides an excellent general introduction to gaming but also gives an important comparison between game theory and behavioral theories (see [72, pp. 156-166]), which has had a significant impact on our own thinking (e.g. see TABLE 8.I in Section 8.2). BREWER and SHUBIK [17] have concentrated on the professional and organizational environments for war gaming in the United States and have made a number of critical recommendations for enhancing the effectiveness of war gaming in solving defense problems. However, little attention is given to combat-modelling aspects. For some European accounts of war gaming, the reader should consult SHEPHARD [71], WOLF [106], NIEMEYER [62], and especially HUBER, NIEMEYER, and HOFMANN [41]. The latter book [41] probably provides the best view of modern German thought on this important topic. Other related references on the general topic of operational gaming are to be found in the Notes and References for Chapter 1. Finally, let us note that SHUBIK and BREWER [73, p. 8] (discussing gaming more generally) have stressed that "the amount of publicity given free-form, political-diplomatic-military games has been enormously disproportionate to the financial and intellectual investments in them. Popular accounts aside (such as [105]), research on the intellectual foundations and used of this type of work has been negligible." Unfortunately, these statements are even more true about war gaming.

13. We are using here the term "simulation" in its broadest sense (cf. the simulation types shown in Figure 1.1).

14. An abbreviated version of this list first appeared in TAYLOR [93, p. 3] (and later TAYLOR [94, p. 8; 96, p. 12]), where such a factorization of a time-sequential combat-optimization problem was first discussed (see also TAYLOR [85; p. 507]). In our work we have stressed the importance of this conceptual factorization for tactical decision analysis, but others have not yet apparently appreciated our point of view.

15. Here we mean that fire is exchanged between the two opposing forces ("bullets fly in both directions") but that only one side is faced with a fire-distribution-optimization problem.

16. One simply orders a report from NTIS according to its so-called "AD-number," e.g. TAYLOR [96] would be referred to as AD A033 761.

17. Other such lists of factors influencing opitmal fire-distribution strategies may be found in TAYLOR [92, p. 2; 93, p. 2; 96, p. 3].

18. See Footnote 8 above.

19. See HO [34; 35] for a discussion of generalized control theory (in particular, various generic types of dynamic optimization problems). Further information about optimal-control theory may be found in PONTRYAGIN et al. [67], HESTENES [32], ATHANS and FALB [5], and BRYSON and HO [18], which are standard references (see also BELL and JACOBSON [7]). Further information about differential games, may be found in

ISAACS [46], BERKOVITZ [9; 10], and FRIEDMAN [24] (see also BRYSON and HO [18, Chapter 9] and PARTHASARATHY and RAGHAVAN [65]). A very readable general introduction to all these topics is afforded by INTRILIGATOR [43].

20. Here (as elsewhere in this chapter) one-sided (as opposed to two-sided) optimization problem means that there is only one (as opposed to two with conflicting objectives) decision maker. We may think of such a situation as arising because the combat strategy for one of the two opposing commanders has been previously determined. Hence only one player's combat strategy remains to be optimized.

21. Extension to cases with replacements and/or withdrawals is discussed in TAYLOR [88, p. 112].

22. Since our combat model is deterministic, in principle we can always determine who will win before the battle is actually fought.

23. As we saw for an $F|F$ attrition process in Section 6.6, it is not generally true that such a single unique initial-force-level value exists (cf. also Section 2.9). Consequently, we are implicitly assuming here that the combat dynamics are such that it does.

24. The result (8.9.4) was not explicitly given by TAYLOR [88], but it is implicit in his developments.

25. Here we mean that more effort should be spent on developing scientifically valid (<u>see</u> HUBER, LOW, and TAYLOR [40]) models of conflict termination because of the sensitivity of analysis results to such models.

26. As discussed in Section 8.4 above, such perfect information is usually assumed for combat-optimization problems. Thus, we are well within the current state of the art to assume such perfect information.

27. <u>See</u> HO [34; 35]; also INTRILIGATOR [43, p. xiii]. For an introduction to the literature of optimal-control theory, <u>see</u> Footnote 19 above.

28. For example, one could test the capability of a computational approach like LAGRANGE dynamic programming (<u>see</u> PUGH [68]) on a discrete-time version of this problem.

29. Such optimality conditions may be found in, for example, the references on optimal-control theory mentioned in Footnote 19.

30. By an <u>extremal</u> we mean a trajectory on which the necessary conditions of optimality are satisfied. An <u>extremal control law</u> is then used to denote the policy followed in order to instantaneously satisfy these necessary conditions and is usually determined by considering the maximum principle. An extremal policy, of course, may not turn out to be an optimal policy.

31. By the domain of controllability for a given terminal state we mean that subset of the initial state space from which extremals lead to the terminal state (see TAYLOR [76, pp. 542-543] for further details).

32. This first characteristic is a consequence of $Y$ causing attrition to $X_1$ at a rate proportional to only the number of firers. It is not true in general (see TAYLOR [78; 79] and Section 8.11 below).

33. Except when $\delta = R - \sqrt{R(R-1)}$, the optimal fire-distribution policy is unique.

34. It should be noted that for $R > 1$ we have $0 < 1 - 1/R < 1$.

35. From the relation $\gamma = -1 + \delta/R$, we readily see that $-(1 - 1/R) \leq \gamma$ if and only if $1 \leq \delta$, $-\sqrt{1 - 1/R} \leq \gamma < -(1 - 1/R)$ if and only if $R - \sqrt{R(R-1)} \leq \delta < 1$, and $-1 \leq \gamma < -\sqrt{1 - 1/R}$ if and only if $0 \leq \delta < R - \sqrt{R(R-1)}$.

36. The author has developed theoretical results along this line, i.e. boundary conditions for the dual variables (see TAYLOR [81]).

37. See also the discussion in TAYLOR [93, pp. 22-23].

38. Here (as elsewhere in this chapter) one-sided (as opposed to two-sided) optimization problem means that there is only one (as opposed to two with conflicting objectives) decision maker. A game may then be considered to be a two-sided optimization problem. Such a one-sided time-sequential optimization problem is also frequently called an optimal-control problem (see also Footnotes 7 and 20 above).

39. Some new facets of optimal-control theory have been uncovered by these investigations, and consequently a couple of contributions (TAYLOR [77; 81]) have been made to the control-theory literature (see also TAYLOR [83]).

40. We have already seen above in Section 8.10 that for the fight to the finish (8.10.1) the optimal fire-distribution policy depends on only the force levels and not on time, i.e. $\phi^*(\text{Problem 2}) = \phi^*(x_1, x_2, y)$.

41. In other words, $x_1^0$, $x_2^0$, and $y_0$ are such that $x_1(T)$ and $x_2(T) > 0$ but $y(T) = 0$ in the terminal-control battle (8.10.1), but that they are such that $x_1(t_f)$, $x_2(t_f)$, and $y(t_f) > 0$ with $t_f = T$ in the prescribed duration battle. Such conditions for the initial force levels are given in TAYLOR [92, Appendix G] for the prescribed-duration battle and in TAYLOR [76; 84] for the fight to the finish (8.10.1) (see also Table 8.VII above).

42. Here (as elsewhere) one also makes the physically realistic assumption that p, q, and $r > 0$.

43. By virtue of (8.11.8), at any given point during the battle will suffice.

44. In our discussion here we are assuming two enemy target types. Extension of these remarks to an arbitrary number of enemy target types proceeds in the obvious manner.

45. Here we mean that the marginal return from firing at a particular enemy target type does not change over time due to the decrease in the number of that target type.

46. Such a distinction plays an essential role in the development of the basic necessary conditions of optimality for such a differential game (e.g. see TAYLOR [84; 95, Appendix A]).

47. As we have discussed in Section 8.5, GEOFFRION [26] has suggested a similar conceptual approach of using a simple auxiliary model to generate tentative hypotheses to be tested in a full-scale operational model and thus to provide guidance for further computerized higher-resolution investigations. We also have felt (see TAYLOR [79]) that the use of relatively simple auxiliary models in conjunction with complex operational models has much to offer for the analysis of military operations (see also NOLAN and SOVEREIGN [63]). In fact, this has been the hypothesis upon which all our research has been based.

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## APPENDIX F: COMPREHENSIVE BIBLIOGRAPHY ON THE LANCHESTER THEORY OF COMBAT

### 1. Introduction.

This appendix contains a comprehensive bibliography on the LANCHESTER theory of combat, i.e. organized knowledge concerning some aspect of a LANCHESTER-type paradigm. Its objective is to provide the interested reader with relevant and available information concerning LANCHESTER-type combat models for further independent research. It should be of use to OR researchers and other readers of this monograph who wish further more-detailed information.

It seems appropriate to define our terms a little more precisely here in order to better communicate to the reader exactly what type of information he can expect to find in these references. First of all, the reader should be aware that any theory about military combat is more speculative than scientific because of the essential absence of historical combat data (see Section 7.22 for further details), and the LANCHESTER theory of combat (taken here to mean organized knowledge concerning some aspect of a LANCHESTER-type paradigm) is no exception. By the term LANCHESTER-type paradigm we mean a lucid simple example of the approach of using differential equations to model the force-on-force combat-attrition process. The term theory itself involves a number of subtleties: it turns out that a technically precise definition of the term theory is somewhat complicated and no such definition is apparently universally accepted (e.g. see ACKOFF [1, pp. 22-23], CAMPBELL [5], or

BUNGE [4]). Thus, we will not precisely define the term theory, and all this bibliography promises is further information about some aspect concerning the models and topics studied in this monograph.

## 2. Nature and Scope of This Bibliography.

This bibliography is a comprehensive list of unclassified references on the LANCHESTER theory of combat. It is primarily composed of journal articles to which the author has selectively added some company and agency reports. The author has personally reviewed and has a copy of each entry, particularly of industrial reports. Internal publications that duplicate open literature publications have been specifically not included. To the best of the author's knowledge, the list of open-literature publications is complete. Finally, this bibliography is more than a synthesis and integration of the references cited in the individual chapters of this monograph, since additional references that for one reason or another would have been inconvenient to cite in some chapter have been included here. Thus, this bibliography should be taken as the most up-to-date list of LANCHESTER literature contained in this monograph.

The criteria for inclusion of references that are not journal articles have been relevance and availability. The author has given preference to citing those documents that an interested reader would have a good chance in obtaining. In particular, three good sources of "internal" publications are the National Technical Information Service (NTIS), University Microfilms International, and The RAND Corporation,

for which complete mailing addresses are as follows:

1. National Technical Information Service
   U.S. Department of Commerce
   5285 Port Royal Road
   Springfield, Virginia 22151

2. University Microfilms International
   P.O. Box 1764
   Ann Arbor, Michigan 48106

3. The RAND Corporation
   1700 Main Street
   Santa Monica, California 90406

Documents available through NTIS are identified by their so-called "AD number."

References have been narrowly limited to only those that consider some aspect concerning LANCHESTER-type combat models themselves. The closely related topic (at least from the standpoint of combat modelling) of stochastic duels has been omitted, except for a few papers that show relationships to Lanchester-type combat models. The reader who is interested in stochastic duels is directed to the comprehensive, exhaustive, and fully annotated bibliography on one-on-one stochastic duels by C. ANCKER [3] or his earlier comprehensive review of developments in the theory of stochastic duels in general [2]. Likewise, references pertaining to Monte Carlo simulation of combat and war gaming have been omitted. Finally, literature concerning differential-equation models of conflict (as opposed to combat itself) such as RICHARDSON-type models of arms races (e.g. see ZINNES [17]) has also not been considered here. (The interested reader will find an introduction to this closely allied literature in MOLL and LUEBBERT [10].)

3. <u>Its Origins</u>.

It may be of interest for the reader to know how the present bibliography has evolved, especially since its predecessors have apparently influenced the work of others in ways that may not be readily apparent. The author's 1970 report [12] on applications of differential games to tactical-allocation problems already contained the nucleus of a literature review on the LANCHESTER theory of combat. Further references were subsequently collected, and a M.S. thesis that gave a comprehensive literature review was directed (see HALL [8]). This work took DOLANSKY's [7] 1964 review article as its point of departure. Subsequently, the author prepared in December 1972 a selected bibliography [13] (60 references), which was distributed to students in combat-modelling courses at the Naval Postgraduate School, and any other interested parties upon request. Here the author followed the policy (which he still does) of citing only those references that he had personally reviewed.

It was then the author's good fortune to be invited by the Military Applications Section (MAS) of the Operations Research Society of America (ORSA) to deliver a "tutorial" entitled "LANCHESTER-Type Models of Warfare" at the 46th National ORSA Meeting on Thursday, October 17, 1974 in San Juan, Puerto Rico. A revised selected bibliography [14] (82 references) was consequently prepared in September 1974 and distributed at the "tutorial" and afterwards. This tutorial was repeated at the 35th Military Operations Research Symposium in July 1975, and the bibliography had by this time grown to 89 references. By the time of the appearance of the author's MAS monograph <u>Force-on-Force Attrition Modelling</u> [16] in January 1980, this selected bibliography of primarily journal articles

had evolved into a comprehensive bibliography of 151 references. Subsequent work on the monograph at hand has led to the comprehensive bibliography presented in this appendix.

4. Other Bibliographies.

There are a number of other bibliographies that may be worthwhile for the interested reader to consult. DOLANSKY's [7] 1964 survey paper contains a fairly comprehensive bibliography (51 references) of material published through 1962. In this respect, the Ph.D. theses of CLARK [6] and SPRINGALL [11] are worthwhile to consult, especially concerning stochastic LANCHESTER-type combat models. A comprehensive bibliography (180 references) of material published up to 1980 has recently been published by HAYSMAN and MARTAGY [9]. It should be borne in mind, however, that different criteria have apparently been used for including references in these various bibliographies. It may be of interest for the combat modeller to examine similar material on arms races and other competitive aspects of international relations, especially RICHARDSON-type (i.e. differential-equation) models of arms races. In this respect, the book by ZINNES [17] is very readable and contains a fairly comprehensive bibliography concerning such allied work, and the recent survey article by MOLL and LUEBBERT [10] (containing 127 references) is highly recommended.

5. A Solicitation.

The author would be grateful to receive information concerning any

additions, omissions, or corrections to this bibliography. Such material would be incorporated into any future versions of this work.

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